Approximation and regularization

(one-variable problem)

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Contents

1 Approximations with Quadratic Stabilizers

Consider the problem of approximation of a function $h(x)$ $x \in [a, b]$ by another function $u(x)$ with better smoothness or other favorable properties. For example, we may want to approximate the noisy experimental curve by a smooth one, or approximate a curve with a blocky piece-wise constant curve. The following method is used for approximations: A variational problem is formulated to minimize a norm of the difference $u - h$:

$$D_2 = \frac{1}{2}\int_a^b (u-h)^2 dx \text{ or } D_1 = \int_a^b |u-h| dx$$

plus a penalty of the form

$$P = \int_a^b p(u, u', u'')dx$$

The *stabilizer* P is chosen to penalize the approximation u for being non-smooth or having large variation. The approximate $u(x)$ balances the closeness to the approximating curve and the smoothness properties. The problem of the best approximate is a variational problem for an unknown function $u(x)$ that minimizes the integral functional

$$\min_{u(x)}(D + \gamma P) = \min_{u(x)} \int_a^b \left[(u-h)^q + \gamma p(u, u', u'')\right] dx \tag{1}$$

where γ is a penalty coefficient. Here we consider several problems of the best approximation.

1.1 Penalized magnitude

A simple penalization $p = u^2$ is proportional to the square of the magnitude of the approximate u. The approximation problem has the form

$$\min_{u(x)} J_1(u), \quad J_1(u) = \int_a^b \frac{1}{2}\left(\gamma u^2 + (h-u)^2\right) dx \tag{2}$$

it minimizes the sum of square D_2^2 of D_2 and the penaltyThe Lagrangian does not contain derivative u', therefore the stationarity condition is

$$\frac{d}{du}\left(\gamma u^2 + (h-u)^2\right) = 0$$

We find $u_\gamma(x) = \frac{1}{1+\gamma}h(x)$ - the approximate u is proportional to h. The penalty reduces the magnitude of the function. When $\gamma \to 0$, the approximation coincides with $h(x)$, and when $\gamma \to \infty$, the approximation $u_\gamma = 0$,

$$\lim_{\gamma \to 0} u_\gamma(x) = h(x), \quad \lim_{\gamma \to \infty} u_\gamma(x) = 0$$

1.2 Penalized growth rate

Approximation of the given function $h(x)$ by function $u(x)$ with a limited growth rate can be formulated as a variational problem

$$\min_u J_2(u), \quad J_2(u) = \int_a^b \frac{1}{2}\left(\gamma u'^{\,2} + (h-u)^2\right) dx \tag{3}$$

The first term of the integrant represents the penalty for the growth rate to be non-constant, and the second term describes the closeness of u and h.

The equation for the approximate (Euler equation of (3)) is

$$\gamma u'' - u + h = 0, \quad u'(a) = u'(b) = 0 \tag{4}$$

Here, the natural boundary conditions are assumed since there is no reason to assign special values of the approximation curve at the ends of the interval.

The approximation depends on the parameter γ, $u = u_\gamma$. When $\gamma \to 0$, the approximation coincides with $h(x)$ and when $\gamma \to \infty$, the approximation is a constant curve, equal to the mean value of $h(x)$,

$$\lim_{\gamma \to 0} u_\gamma(x) = h(x), \quad \lim_{\gamma \to \infty} u_\gamma(x) = \text{ constant} = \frac{1}{b-a} \int_a^b h(\xi) d\xi$$

1.3 Penalized smoothness

The problem of smooth approximation is similar. We choose the penalization as integral of the square of the second derivative u''. This functional increases with deviation of u from a straight line. The corresponding variational problem can be formulated as

$$\min_u J(u), \quad J(u) = \int_a^b \frac{1}{2} \left(\gamma (u'')^2 + (h-u)^2 \right) dx \tag{5}$$

The equation for the best approximate (Euler equation of (5)) is

$$\gamma u^{IV} + u - h = 0, \quad u''(a) = u''(b) = 0, \quad u'''(a) = u'''(b) = 0,$$

Here, the natural boundary conditions are assumed since there is no reason to assign special values of the approximation curve at the ends of the interval.

When $\gamma \to 0$, the approximation coincides with $h(x)$ and when $\gamma \to \infty$, the approximation is a straight line closest to h.

Problem 1.1 Find the limit of u_γ when $\gamma \to \infty$

2 Appendix. Methods for linear boundary-value problems

The Euler equation is a linear boundary value problem of the type

$$\mathcal{L}_\gamma(u) = h$$

where $\mathcal{L}$ is the linear operator of the Sturm-Liuville type, such as $\mathcal{L}_\gamma = \gamma u'' - u$. A linear differential equation is defined in the interval $[a, b]$, and the homogeneous boundary conditions $u'(a) = u'(b) = 0$ are posed at the ends. Here we remind an approach based on the expansion of u into series of *eigenfunctions* of the operator $\mathcal{L}_\gamma$.

2.1 Spectral method

Solution of linear boundary value problem We briefly comment on solution of the linear Euler equation by spectral method. To solve the problem, we first find the spectrum of operator $\mathcal{L}_\gamma$ solving the Sturm-Liouville problem:

$$\mathcal{L}_\gamma u = \lambda u, \quad u'(0) = u'(1) = 0.$$

This problem has nontrivial solutions ($u(x) \not\equiv 0$ everywhere) for only some values of λ called the eigenvalues $\lambda_0, \lambda_1, \lambda_2, \ldots$. The solutions called the eigenfunctions $u_n(x)$, satisfy the equations

$$\mathcal{L}_\gamma \, u_n = \lambda_n u_n, \quad n = 1, 2, \ldots, .$$

Eigenfunctions u_n are defined up to a multiplier and are mutually orthogonal,

$$\langle u_k, u_m \rangle = 0, \qquad \text{if } k \neq m$$

where the inner product $\langle f, g \rangle$ is defined as

$$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx.$$

The system of eigenfunctions is complete. This means that a general solution can be represented by series

$$u(x) = \sum_{n=0}^{\infty} \alpha_n u_n(x)$$

where α_n are arbitrary coefficients. With this representation, the Euler equation $\mathcal{L}\,(u) = h$ becomes

$$\mathcal{L}(u) = \mathcal{L}\left(\sum_{n=0}^{\infty} \alpha_n u_n \right) = \sum_{n=0}^{\infty} \alpha_n \lambda_n u_n = h$$

To find the coefficients α_n, we expand h as follows

$$h = \sum_{n=0}^{\infty} \beta_n u_n$$

where parameters β_n are found from the orthogonality property of u_n as

$$\beta_n = \frac{\langle h \, , u_n \rangle}{\langle u_n, u_n \rangle}$$

Comparing the expansions, we obtain the unknown coefficients α_n

$$\lambda_n \alpha_n = \beta_n \quad \text{or } \alpha_n = \frac{\langle u_n, h \rangle}{\lambda_n \, \langle u_n, u_n \rangle}$$

and the solution u

$$u(x) = \sum_{n=0}^{\infty} \frac{\langle u_n, h \rangle}{\lambda_n \, \langle u_n, u_n \rangle} u_n(x).$$

We observe that the approximate linearly depends on h as expected.

Example For the operator (**??**) we have

$$a^2 u'' - u = \lambda u, \quad u'(0) = u'(1) = 0.$$

Particular solutions (the eigenfunctions) are $u_n = \cos(\pi n x)$ and the eigenvalues are $\lambda_n = (1 + \pi n^2 a^2)$. The general solution is

$$u(x) = \sum_{n=0}^{\infty} \gamma_n \cos(\pi n x)$$

where γ_n are arbitrary coefficients. Solving, we obtain

$$u(x) = \sum_{n=0}^{\infty} \frac{c_n}{\lambda_n} \cos(\pi n x), \quad c_n = \int_0^1 h(x) \cos(\pi n x) dx$$

Problem 2.1 Let $h(x)$ be $h(x) = |x|$. Find $u(x)$. Graph the results.

2.2 Green's function

Green's function for approximations with quadratic penalty The solution of a linear boundary value problem is most conveniently done by the Green's function. Here we remind this technique.

Consider the linear differential equation with the differential operator L

$$L(x)u(x) = f(x) \quad x \in [a, b], \quad B_a(u, u')|_{x=a} = 0, \quad B_b(u, u')|_{x=a} = 0. \tag{6}$$

an arbitrary external excitation $f(x)$ and homogeneous boundary conditions $B_a(u, u')|_{x=a} = 0$ and $B_b(u, u')|_{x=a} = 0$. For example, the problem (**??**) corresponds to

$$L(x)\, u = \left(\alpha^2 \frac{d^2}{dx^2} - 1\right) u, \quad B_a(u, u') = u', \quad B_b(u, u') = u'$$

To solve the equation means to invert the dependence between u and f, that is to find the linear operator

$$u = L^{-1} f$$

In order to solve the problem (6) one solves first the problem for a single concentrated load $\delta(x - \xi)$ applied at the point $x = \xi$

$$L(x) g(x, \xi) = \delta(x - \xi), \quad B_a(g, g')|_{x=a} = 0 \quad B_b(g, g')|_{x=a} = 0$$

This problem is usually simpler than (6). The solution $g(x, \xi)$ is called the Green's function, it depends on the point of the applied excitation ξ as well as of the point x where the solution is evaluated. Formally, the Green's function can be expressed as

$$g(x, \xi) = L(x)^{-1} \delta(x - \xi) \tag{7}$$

Then, we use the identity

$$f(x) = \int_a^b f(x)\delta(x-\xi)d\xi$$

(essentially, the definition of the delta-function) to find the solution of (6). We multiply both sides of (7) by $f(\xi)$ and integrate over ξ from a to b, obtaining

$$\int_a^b g(x,\xi)f(\xi)d\xi = L^{-1}\left(\int_a^b f(\xi)\delta(x-\xi)d\xi\right) = L^{-1}f(x) = u(x).$$

Notice that operator $L = L(x)$ is independent of ξ therefore we can move L^{-1} out of the integral over ξ.

Thus, we obtain the solution,

$$u(x) = \int_a^b g(x,\xi)f(\xi)d\xi$$

that expresses $u(x)$ as a linear mapping of $f(x,\xi)$ with the kernel $G(x,\xi)$. The finite-dimentional version of this solution is the matrix equation for the vector u.

Green's function for approximation at an interval For the problem (**??**), the problem for the Green's function is

$$\left(\alpha^2\frac{d^2}{dx^2} - 1\right) g(x,\xi) = \delta(x-\xi), \quad u'(a) = u'(b) = 0$$

At the intervals $x \in [a,\xi)$ and $x \in (\xi, b]$ the solution is

$$g(x,\xi) = \begin{cases} g_-(x,\xi) = A_1 \cosh\left(\frac{x-a}{\alpha}\right) & \text{if } x \in [a,\xi) \\ g_+(x,\xi) = A_2 \cosh\left(\frac{x-b}{\alpha}\right) & \text{if } x \in (\xi,b] \end{cases}$$

This solution satisfies the differential equation for all $x \neq \xi$ and the boundary conditions. At the point of application of the concentrated force $x = \xi$, the conditions hold

$$g_+(\xi,\xi) = g_-(\xi,\xi); \quad \left.\frac{d}{dx}g_+(x,\xi)\right|_{x=\xi} - \left.\frac{d}{dx}g_-(x,\xi)\right|_{x=\xi} = 1$$

that express the continuity of $u(x)$ and the unit jump of the derivative $u'(x)$. These allow for determination of the constants

$$A_1 = \alpha\frac{\cosh\left(\frac{\xi-b}{\alpha}\right)}{\sinh\left(\frac{b-a}{\alpha}\right)} \quad A_2 = \alpha\frac{\cosh\left(\frac{\xi-a}{\alpha}\right)}{\sinh\left(\frac{b-a}{\alpha}\right)}$$

which completes the calculation.

Green's function for approximation in R_1 The formulas for the Green's function are simpler when the approximation of an integrable in R_1 function $f(x)$ is performed over the whole real axes, or when $a \to -\infty$ and $b \to \infty$. In this case, the boundary terms $u'(a) = u'(b) = 0$ are3 replaced by requirement that the approximation u is finite,

$$u(x) < \infty \quad \text{when } x \to \pm\infty$$

In this case, the Green's function is

$$g(x, \xi) = \frac{1}{2\alpha} e^{-\frac{|x-\xi|}{\alpha}}$$

One easily check that it safisfies the differential equation, boundary conditions, and continuity and jump conditions at $x = \xi$.

The best approximation becomes simply an average

$$u(x) = \frac{1}{2\alpha} \int_{-\infty}^{\infty} f(\xi) e^{-\frac{|x-\xi|}{\alpha}} d\xi$$

3 Approximation with nonquadratic stabilizers

3.1 L_1-approxmation with quadratic penalty

The variational problem has the form

$$\min_u J_1(u), \quad J_1(u) = \int_a^b \left(\frac{1}{2} \gamma (u')^2 + |u - h| \right) dx \tag{8}$$

The Euler equation is

$$u'' - \frac{1}{\gamma} \text{sign}\,(u - h) = 0, \quad u'(a) = u'(b) = 0 \tag{9}$$

or, equivalently,

$$u'' = \begin{cases} \frac{1}{\gamma} & \text{if } u > h \\ -\frac{1}{\gamma} & \text{if } u < h \end{cases}$$

The remaining case, $u = h$, can be viewed as infinitely often alteration of the two above regimes. It corresponds to the condition $\frac{1}{\gamma} > |u''|$. This regime exists only if $\frac{1}{\gamma} > |h(x)''|$, but not necessarily for all x that satisfy the inequality. The minimizer $u(x)$ must be combined of these three regimes. The Weierstrass-Erdmann conditions requests that $u(x)$ and $u'(x)$ are continuous functions. To summarize, $u(x)$ satisfies the system of relations:

$$\begin{array}{l} u = h, \ |u''| \leq \frac{1}{\gamma} \\ u < h, \ u'' = -\gamma \, , \quad u \in C_1(a, b) \\ u > h, \ u'' = \gamma \end{array} \tag{10}$$

Example 3.1 (Discontinuity) Let $h(x)$ be $h(x) = \mathsf{sign}\,(x)$. Check that $u(x)$ is

$$u(x) = \begin{cases} h(x) = \mathsf{sign}\,(x) & \text{if } |x| \geq \frac{1}{\gamma} \\ \left(\sqrt{\frac{2}{\gamma}}x - \frac{x^2}{2\gamma}\mathsf{sign}\,(x)\right) & \text{if } |x| < \frac{1}{\gamma} \end{cases}$$

Notice that behavior of $h(x)$ in the proximity of the discontinuity point does not affect $u(x)$ in that region. Similarly, the value of penalty coefficient γ does not affect u away from the discontinuity point. This penalization scheme results in function $u(x)$ that coincides with $h(x)$ away from discountinuities and zones of large values of h''. Near these zones, $h(x)$ is smoothed – it is replaced by parabolas.

Problem 3.1 A. Let $h(x)$ be $h(x) = \exp(-\alpha|x|)$. Find $u(x)$. Graph the results.

Problem 3.2 B. Let $h(x)$ be $h(x) = \cos(\alpha x)$. Find $u(x)$. Consider different cases depending on the value of $\alpha \in R$. Graph the results.

3.2 Approximation with penalized total variation

This approximation penalizes the function for its variation. The total variation $TV(f)$ of a function $u(x)$ is defined as

$$TV(u) = \int_a^b |u'(x)| dx$$

For a monotonic function $u(x)$ one evaluates the integral and finds that

$$TV(u) = \max_{x \in [a,b]} u(x) - \min_{x \in [a,b]} u(x)$$

If $u(x)$ has N intervals L_k of monotonicity $(N < \infty)$, the total variation is

$$TV(u) = \sum_k^N \left(\max_{x \in L_k} u(x) - \min_{x \in L_k} u(x) \right)$$

The variational problem with total-variation penalty has the form

$$\min_u J_3(u), \quad J_3(u) = \int_a^b \frac{1}{2} \left(\gamma |u'| + (u-h)^2 \right) dx \tag{11}$$

Here, $\gamma \geq 0$, the first term of the integrant represents the total-variation penalty and the second term describes the closeness of the original curve and the approximate. When $\gamma \to 0$, the approximation coincides with $h(x)$, and when $\gamma \to \infty$, the approximation becomes constant equal to the mean value of h.

The formal application of the stationarity technique gives:

$$(\gamma \mathrm{sign}\,(u'))' + u = h, \quad \mathrm{sign}\,(u'(a)) = \mathrm{sign}\,(u'(b)) = 0 \tag{12}$$

This formula is not very helpful because it requires the differentiation of a discontinuous function sign; besides, the Lagrangian (11) is not a twice-differential function of u' as it is required in the procedure of derivation of the Euler equation.

To deal with this problem, we introduce a family $W(u', \epsilon)$ such that

$$\lim_{\epsilon \to 0} W(u', 0) = w(u'),$$

and $W(u', \epsilon)$ is twice-differentiable function of u' for all $\epsilon > 0$. Particularly, we can use the family

$$W_1(u', \epsilon) = \sqrt{u'(x)^2 + \epsilon^2}. \tag{13}$$

to approximate $w(u') = |u'|$.

Remark 3.1 We could also choose another family of functions which derivarive converges to $w(u') = \text{sign}\,(u')$ when parameter ϵ of the family goes to zero, $\epsilon \to 0$, for example

$$W_2(u'(x), \epsilon) = \frac{2}{\pi} \int_0^x \arctan\left(\frac{u'(t)}{\epsilon}\right) dt = \frac{2}{\pi}\left(x \arctan \frac{x}{\epsilon} - \frac{1}{\epsilon} \log \frac{x^2 + \epsilon^2}{\epsilon^2}\right). \tag{14}$$

or

$$W_3(u', \epsilon) = \begin{cases} |u'| & \text{if } |u'| > \epsilon \\ \frac{u'^2}{2\epsilon} + \frac{1}{2}\epsilon & \text{if } |u'| < \epsilon \end{cases}. \tag{15}$$

The derivatives of these functions are:

$$\begin{aligned}
\frac{\partial W_1}{\partial u'} &= \frac{u'}{\sqrt{u'(x)^2 + \epsilon^2}} \\
\frac{\partial W_2}{\partial u'} &= \frac{2}{\pi} \arctan\left(\frac{u'(x)}{\epsilon}\right) \\
\frac{\partial W_3}{\partial u'} &= \begin{cases} \text{sign}\,(u') & \text{if } |u'| > \epsilon \\ \frac{u'}{\epsilon} & \text{if } |u'| < \epsilon \end{cases}.
\end{aligned}$$

These are continuous functions that tend to the discontinuous function $sign(u')$ when $\epsilon \to 0$.

It need to be shown that the solutions of minimization problem with function $W(u', \epsilon)$ converge to the solution of the original problem when $\epsilon \to 0$. If this is true, than the problem is *regularized*. This technique is called *regularization method*. It allows for construction of solutions even if the original problem is not well posed. Here, we do not prove but assume the convergence of the family of solutions.

The *regularized* Lagrangian is

$$L(u, u', \epsilon) = \gamma \sqrt{u'(x)^2 + \epsilon^2} + \frac{1}{2}(u - f)^2$$

Now we can compute Euler equation. We have

$$\frac{\partial L(u,u',\epsilon)}{\partial u'} = \gamma \frac{u'}{(|u'|^2+\epsilon^2)^{\frac{1}{2}}}$$

and the Euler equation is

$$R(u',\epsilon)u'' = u - h, \quad \text{where } R(u',\epsilon) = \frac{\epsilon^2}{(u'^2+\epsilon^2)^{\frac{3}{2}}} \tag{16}$$

The second step of procedure is the asymptotic analysis of the stationary approximate when $\epsilon \to 0$. Coefficient $R(u',\epsilon)$ reaches its maximum at $u' = 0$. It is or the order of $\frac{1}{\epsilon}$, $R(0,\epsilon) = \frac{1}{\epsilon}$. $R(u',\epsilon)$ quickly and monotonically decays when u' grows. When $u' = \sqrt{\epsilon}$, the coefficient $R(u',\epsilon)$ approximately equals $\sqrt{\epsilon}$, and it becomes even smaller for larger values of u'. Indeed, one can check that $R(\sqrt{\epsilon},\epsilon) = \sqrt{\epsilon} + o(\sqrt{\epsilon}))$. We conclude that the stationary condition (16) can be satisfied (up to the order of ϵ) in one of two ways.

- When $|h'| \gg \epsilon$ the solution $u = h$ is stationary. Indeed, in this case $|u'| = |h'| > \epsilon$, $R(u',\epsilon) \ll \epsilon$, and first term $R(u',\epsilon)u''$ does not influence the equation.

- When $u(x)$ is approximately constant, $|u'| \le \epsilon$, the first term is extremely sensitive to the variation of u' and it can take any value; in particular, it can compensate the second term $u - h$ of the equality. The (always) constant u is another stationary solution.

This sketch shows that in the limit $\epsilon \to 0$, the stationary condition (16) is satisfied either when $u(x)$ is a constant, $u' = 0$, or when $u(x)$ coincides with $h(x)$.

$$u(x) = h(x) \quad \text{or } u'(x) = 0, \quad \forall x \in [a,b]$$

The approximation cuts the maxima and minima of the approximating function, which agrees with the expected behavior of TV-penalized approximation. Indeed, the value of $TV(u)$ depends only on its local maxima and minima and is invariant to the intermediate values of $u(x)$. Therefore, the equality $u = h$ does not changes the penalty $TV(u)$ but decreases the terms $(u-h)^2$. Near the maxima (minima) points, the cut $u = \text{ constant}(x)$ decreases the penalty $TV(u)$ and slightly increases the norm of difference of u and h.

Let us find the cutting points. For simplicity in notations we assume that the function $h(x)$ monotonically increases on $[a,b]$. The approximation $u(x)$ is also a monotonically increasing function, $u' \ge 0$ that either coincides with $h(x)$ or stays constant cutting the maximum and the minimum of $h(x)$:

$$u(x) = \begin{cases} h(\alpha) & \text{if } x \in [a,\alpha] \\ h(x) & \text{if } x \in [\alpha,\beta] \\ h(\beta) & \text{if } x \in [\beta,b] \end{cases}$$

The cost of the problem

$$J = \frac{\gamma}{2}\left[\int_a^\alpha (h(x)-h(\alpha))^2 dx + \int_\beta^b (h(x)-h(\beta))^2 dx\right] + h(\beta) - h(\alpha)$$

depends on two unknown parameters, α and β, the coordinates on the cuts. They are found by straight differentiation. The equation for α is

$$\frac{dJ}{d\alpha} = h'(\alpha)\left(\gamma \int_a^\alpha (h(x)-h(\alpha))dx - 1\right) = 0$$

or, noticing the cut point α is not a stationary point, $h'(\alpha) \neq 0$

$$\int_a^\alpha [h(x)-h(\alpha)]dx = \frac{1}{\gamma}$$

the equation for β is similar:

$$\int_\beta^b [h(x)-h(\beta)]dx = \frac{1}{\gamma}$$

Notice that the extremal is broken; regular variational method based on the Euler equation is not effective. These irregular problems will be discussed later in Chapter **??**.

Problems

Problem 3.3 Use the regularization (15) instead of (**??**) and find a family of solutions in a regularized problem. Graph the results.

Problem 3.4 Let $h(x)$ be

$$\begin{aligned} h_1(x) &= \cos(\omega x), x \in [-\pi, \pi] \\ h_2(x) &= \exp(-\omega|x|), x \in [-\pi, \pi] \\ h_3(x) &= \mathsf{sign}\,(x),\ \ x \in [-\pi, \pi] \end{aligned}$$

Find all three approximate . Discuss the difference. Graph the results.

Problem 3.5 Replace term $|u-h|$ in (8) by a family of smoother functions (14) or (15) and investigate stationarity conditions. Graph the results.

Problem 3.6 (Project) Consider the problem

$$\min_{u(x)} \int_a^b (\gamma|u'| + |u-f|)dx$$

Investigate the properties of u. Use regularization by different families (13) (14) (15)

Problem 3.7 (Project) Regularize the problem of minimal surface and find both the catenoid and Goldschmidt solution by a regular variational method.

4 Solutions with an unbounded derivative. Regularization

4.1 Lagrangians of linear growth

A minimizing sequence may tend to a discontinuous function if the Lagrangian growth slowly with the increase of u'. Here we investigate discontinuous solutions of Lagrangians of linear growth. Assume that the Lagrangian F satisfies the limiting equality

$$\lim_{|u'|\to\infty} \frac{F(x,u,u')}{|u'|} \leq \beta u \tag{17}$$

where β is a nonnegative constant.

Considering the scalar case (u is a scalar function), we assume that the minimizing sequence tends to a finite discontinuity (jump) and calculate the impact of it for the objective functional. Let a miniming sequence u^ϵ of differentiable functions tend to a discontinuous at the point x_0 function, as follows

$$\begin{aligned} u^\epsilon(x) &= \phi(x) + \psi^\epsilon(x) \\ \psi^\epsilon(x) &\rightharpoondown \alpha H(x - x_0), \quad \beta \neq 0 \end{aligned}$$

where ϕ is a differentiable function with the bounded everywhere derivative, and H is the Heaviside function.

Assume that functions ψ^ϵ that approximate the jump at the point x_0 are piece-wise linear,

$$\psi^\epsilon(x) = \begin{cases} 0 & \text{if } x < x_0 - \epsilon \\ \frac{\alpha}{\epsilon}(x - x_0 + \epsilon) & \text{if } x_0 - \epsilon \leq x \leq x_0 \\ \alpha & \text{if } x > x_0. \end{cases}$$

The derivative $(\psi^\epsilon)'$ is zero outside of the interval $[x_0 - \epsilon, x_0]$ where it is equal to a constant,

$$\psi' = \begin{cases} 0 & \text{if } x \notin [x_0 - \epsilon, x_0] \\ \frac{\alpha}{\epsilon} & \text{if } x \in [x_0 - \epsilon, x_0] \end{cases}$$

The Lagrangian is computed as

$$F(x,u,u') = \begin{cases} F(x,\phi,\phi') & \text{if } x \notin [x_0 - \epsilon, x_0] \\ F\left(x, \phi + \psi^\epsilon, \phi' + \frac{\alpha}{\epsilon}\right) = \frac{\alpha\beta}{\epsilon} + o\left(\frac{1}{\epsilon}\right) & \text{if } x \in [x_0 - \epsilon, x_0] \end{cases}$$

Here, we use the condition (17) of linear growth of F.

The variation of the objective functional is

$$\int_a^b F(x,u,u')dx \leq \int_a^b F(x,\phi,\phi')dx + \alpha\beta.$$

We observe that the contribution $\alpha\beta$ due to the discontinuity of the minimizer is finite when the magnitude $|\alpha|$ of the jump is finite. Therefore, discontinuous solutions are tolerated in the problems with Lagrangian of linear growth: They do not lead to infinitely large values of the objective functionals. To the contrary, the problems with Lagrangians of superlinear growth $\beta = \infty$ do not allow for discontinuous solution because the penalty is infinitely large.

Remark 4.1 The problems of Geometric optics and minimal surface are or linear growth because

$$\lim_{|u'|\to\infty} \frac{\sqrt{1+u'^2}}{u'} = 1.$$

but problems of Lagrange mechanics are of quadratic (superlinear) growth because kinetic energy depends of the speed $\dot{q}$ quadratically.

4.2 Examples of discontinuous solutions

Example 4.1 (Discontinuities in problems of geometrical optics) We have already seen in Section **??** that the minimal surface problem

$$I_0 = \min_{u(x)} I(u), \quad I(u) = \int_o^L u\sqrt{1+(u')^2}dx, \quad u(-1) = 1, \quad u(1) = 1, \tag{18}$$

can lead to a discontinuous solution (Goldschmidt solution)

$$u = -H(x+1) + H(x-1)$$

if L is larger than a threshold.

Particularly, the Goldschmidt solution corresponds to zero smooth component $u(x) = 0$, $x = (a, b)$ and two jumps M_1 and M_2 of the magnitudes $u(a)$ and $u(b)$, respectively. The smooth component gives zero contribution, and the contributions of the jumps are

$$I = \frac{1}{2}\left(u^2(a) + u^2(b)\right)$$

The next example (Gelfand & Fomin) shows that the solution may exhibit discontinuity if the superlinear growth condition is violated even at a single point.

Example 4.2 (Discontinuous extremal and viscosity-type regularization) Consider the minimization problem

$$I_0 = \min_{u(x)} I(u), \quad I(u) = \int_{-1}^1 x^2 u'^2 dx, \quad u(-1) = -1, \quad u(1) = 1, \tag{19}$$

We observe that $I(u) \geq 0\ \forall u$, and therefore $I_0 \geq 0$. The Lagrangian is convex function of u', and the third condition is satisfied. However, the second condition is violated in $x = 0$:

$$\lim_{|u'|\to\infty} \left.\frac{x^2 u'^2}{|u'|}\right|_{x=0} = \lim_{|u'|\to\infty} x^2|u'|\Big|_{x=0} = 0$$

The functional is of sublinear growth at only one point $x = 0$.

Let us show that the solution is discontinuous. Assume the contrary, that the solution satisfies the Euler equation $(x^2u')' = 0$ everywhere. The equation admits the integral

$$\frac{\partial L}{\partial u'} = 2x^2u' = C.$$

If $C \neq 0$, the value of $I(u)$ is infinity, because then $u' = \frac{C}{2x^2}$, the Lagrangian becomes

$$x^2u'^2 = \frac{C^2}{x^2} \quad \text{if } C \neq 0.$$

and the integral of Lagrangian diverges. A finite value of the objective corresponds to $C = 0$ which implies that $u_0'(x) = 0$ if $x \neq 0$. Accounting for the boundary conditions, we find

$$u_0(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

and $u_0(0)$ is not defined.

We arrived at the unexpected result that violates the assumptions used when the Euler equation is derived: $u_0(x)$ is discontinuous at $x = 0$ and u_0' exists only in the sense of distributions:

$$u_0(x) = -1 + 2H(x), \quad u_0'(x) = 2\delta(x)$$

This solution delivers absolute minimum ($I_0 = 0$) to the functional, is not differentiable and satisfies the Euler equation in the sense of distributions,

$$\int_{-1}^{1} \frac{d}{dx} \left.\frac{\partial L}{\partial u'}\right|_{u=u_0(x)} \phi(x)dx = 0 \quad \forall \phi \in L_\infty[-1, 1]$$

Regularization A slight perturbation of the problem (regularization) yields to the problem that has a classical solution and this solution is close to the discontinuous solution of the original problem. This time, regularization is performed by adding to the Lagrangian a stabilizer, a strictly convex function $\epsilon\rho(u')$ of superlinear growth.

Consider the perturbed problem for the Example 19:

$$I_\epsilon = \min_{u(x)} I_\epsilon(u), \quad I_\epsilon(u) = \int_{-1}^{1} \left(x^2u'^2 + \epsilon^2u'^2\right) dx, \quad u(-1) = -1, \quad u(1) = 1, \tag{20}$$

Here, the perturbation ϵ^2u' is added to the original Lagrangian ϵ^2u'; the perturbed Lagrangian is of superlinear growth everywhere.

The first integral of the Euler equation for the perturbed problem becomes

$$(x^2 + \epsilon^2)u' = C, \quad \text{or } du = C\frac{dx}{x^2 + \epsilon^2}$$

Integrating and accounting for the boundary conditions, we obtain

$$u_\epsilon(x) = \left(\arctan\frac{1}{\epsilon}\right)^{-1} \arctan\frac{x}{\epsilon}.$$

When $\epsilon \to 0$, the solution $u_\epsilon(x)$ converges to $u_0(x)$ although the convergence is not uniform at $x = 0$.

Unbounded solutions in constrained problems The discontinuous solution often occurs in the problem where the derivative is unbounded. The problem can be regularized if satisfies additional inequalities $c_1 \le u' \le c_2$ are assumed In such problems, the stationary condition must be satisfied in the points where derivative is not at the constrain $u' \in (c_1, c_2)$.

The next example shows, that the measure of such interval can be infinitesimal.

Example 4.3 (Euler equation is meaningless) Consider the variational problem with an inequality constraint

$$\max_{u(x)} \int_0^\pi u' \sin(x) dx, \quad u(0) = 0,\ u(\pi) = 1,\ u'(x) \ge 0\ \forall x.$$

The minimizer should either corresponds to the limiting value $u' = 0$ of the derivative or satisfy the stationary conditions, if $u' > 0$. Let $[\alpha_i, \beta_i]$ be a sequence of subintervals where $u' = 0$. The stationary conditions must be satisfied in the complementary set of intervals $(\beta_i, \alpha_{i+1}])$ located between the intervals of constancy. The derivative cannot be zero everywhere, because this would correspond to a constant solution $u(x)$ and would violate the boundary conditions.

However, the minimizer cannot correspond to the solution of Euler equation at any interval. Indeed, the Lagrangian L depends only on x and u'. The first integral $\frac{\partial L}{\partial u'} = C$ of the Euler equation yields to an absurd result

$$\sin(x) = \text{ constant } \quad \forall x \in [\beta_i, \alpha_{i+1}]$$

The Euler equation does not produce the minimizer. Something is wrong!

The objective can be immediately bounded by the inequality

$$\int_0^\pi f(x) g(x) dx \le \left(\max_{x\in[0,\pi]} g(x)\right) \int_0^\pi |f(x)| dx.$$

that is valid for all functions f and g if the involved integrals exist. We set $g(x) = \sin(x)$ and $f(x) = |f(x)| = u'$ (because u' is nonnegative), account for the constraints

$$\int_0^\pi |f(x)| dx = u(\pi) - u(0) = 1 \quad \text{ and } \max_{x\in[0,\pi]} \sin(x) = 1,$$

and obtain the upper bound

$$I(u) = \int_0^\pi u' \sin(x) dx \le 1 \quad \forall u.$$

This bound corresponds to the minimizing sequence u_n that tends to a Heaviside function $u_n(x) \to H(x - \pi/2)$. The derivative of such sequence tends to the δ-function, $u'(x) = \delta(x - \pi/2)$. Indeed, immediately check that the bound is realizable, substituting the limit of u_n into the problem

$$\int_0^\pi \delta\left(x - \frac{\pi}{2}\right) \sin(x) dx = \sin\left(\frac{\pi}{2}\right) = 1.$$

The reason for the absence of a stationary solution is the openness of the set of differentiable function. This problem also can be regularized. Here, we show another way to regularization, by imposing an additional pointwise inequality $u'(x) \le \frac{1}{\gamma}\ \forall x$ (Lipschitz constraint). Because the intermediate values of u' are never optimal, optimal u' alternates the limiting values:

$$u'_\gamma(x) = \begin{cases} 0 & \text{if } x \notin \left[\frac{\pi}{2} - \gamma, \frac{\pi}{2} + \gamma\right], \\ \frac{1}{2\gamma} & \text{if } x \in \left[\frac{\pi}{2} - \gamma, \frac{\pi}{2} + \gamma\right], \end{cases}$$

The objective functional is equal to

$$I(u_\gamma) = \frac{1}{2\gamma} \int_{\frac{\pi}{2}-\gamma}^{\frac{\pi}{2}+\gamma} \sin(x) dx = \frac{1}{\gamma} \sin(\gamma)$$

When γ tends to zero, I_M goas to its limit

$$\lim_{\gamma \to 0} I_\gamma = 1,$$

the length γ of the interval where $u' = \frac{1}{2\gamma}$ goes to zero so that $u'_\gamma(t)$ weakly converges to the δ-function for u', $u'_\gamma(t) \rightharpoonup \delta\left(x - \frac{\pi}{2}\right)$.

This example clearly demonstrates the source of irregularity: The absence of the upper bound for the derivative u'. The constrained variational problems are studied in the control theory; they are are discussed later in Section **??**.

4.3 Regularization and penalization

Regularization as smooth approximation The smoothing out feature of regularization is easy demonstrated on the following example of a quadratic approximation of a function by a smoother one.

Approximate a function $f(x)$ where $x \in \mathcal{R}$, by the function $u(x)$, adding a quadratic stabilizer; this problem takes the form

$$\min_u \int_{-\infty}^{\infty} [\epsilon^2 (u')^2 + (u - f)^2] dx$$

The Euler equation

$$\epsilon^2 u'' - u = -f \tag{21}$$

can be easily solved using the Green function

$$G(x, y) = \frac{1}{2\epsilon} \exp\left(-\frac{|x - y|}{\epsilon}\right)$$

of the operator in the left-hand side of (21). We have

$$u(x) = \frac{1}{2\epsilon}\int_{-\infty}^{\infty} \exp\left(-\frac{|x-y|}{\epsilon}\right) f(y)dy$$

that is the expression of the averaged f. The smaller is ϵ the closer is the average to f.

Quadratic stabilizers Besides the stabilizer $\varepsilon u'^2$, other stabilizers can be considered: The added term εu^2 penalizes for large values of the minimizer, $\varepsilon(u'')^2$ penalizes for the curvature of the minimizer and is insensitive to linearly growing solutions. The stabilizers can be inhomogeneous like $\varepsilon(u-u_{\mathrm{target}})^2$; they force the solution stay close to a target value. The choice of a specific stabilizer depends on the physical arguments (see Tikhonov).

For example, solve the problem with the Lagrangian

$$F = \epsilon^4(u'')^2 + (u - f(x))^2$$

Show that $u = f(x)$ if $f(x)$ is any polynomial of the order not higher than three. Find an integral representation for $u(f)$ if the function $f(x)$ is defined at the interval $|x| \leq 1$ and at the axis $x \in R$.

4.4 Complement: Regularization of a finite-dimensional linear problem

As the most of variational methods, the regularization has a finite-dimensional analog. It is applicable to the minimization problem of a convex but not strongly convex function which may have infinitely many solutions. The idea of regularization is to slightly perturb the function by small but a strictly convex term; the perturbed problem has a unique solution to matter how small the perturbation is. The numerical advantage of the regularization is the convergence of minimizing sequences.

Let us illustrate ideas of regularization by studying a finite dimensional problem. Consider a linear system

$$Ax = b \tag{22}$$

where A is a square $n \times b$ matrix and b is a known n-vector.

We know from linear algebra that the Fredholm Alternative holds:

- If $\det A \neq 0$, the problem has a unique solution:

$$x = A^{-1}b \quad \text{if } \det A \neq 0 \tag{23}$$

- If $\det A = 0$ and $Ab \neq 0$, the problem has no solutions.
- If $\det A = 0$ and $Ab = 0$, the problem has infinitely many solutions.

In practice, we also deal with an additional difficulty: The determinant $\det A$ may be a "very small" number and one cannot be sure whether its value is a result of rounding of digits or it has a "physical meaning." In any case, the errors of using the formula (23) can be arbitrary large and the norm of the solution is not bounded.

To address this difficulties, it is helpful to restate linear problem (22) as an extremal problem:

$$\min_{x \in R^n} (Ax - b)^T (Ax - b) \tag{24}$$

This problem does have at least one solution, no matter what the matrix A is. This solution coincides with the solution of the original problem (22) when this problem has a unique solution; in this case the cost of the minimization problem (24) is zero. Otherwise, the minimization problem provides "the best approximation" of the non-existing solution.

If the problem (22) has infinitely many solutions, so does problem (24). Corresponding minimizing sequences $\{x^s\}$ can be unbounded, $\|x^s\| \to \infty$ when $s \to \infty$.

In this case, we may select a solution with minimal norm. We use the *regularization*, passing to the perturbed problem

$$\min_{x \in R^n} (Ax - b)^T (Ax - b) + \epsilon x^T x$$

The solution of the last problem exists and is unique. Indeed, we have by differentiation

$$(A^T A + \epsilon I)x - A^T b = 0$$

and

$$x = (A^T A + \epsilon I)^{-1} A^T b$$

We mention that

1. The inverse exists since the symmetric matrix $A^T A$ is nonnegative defined, and ϵ is positive. The eigenvalues of the inverse matrix $(A^T A + \epsilon I)^{-1}$ are not larger than ϵ^{-1}

2. Suppose that we are dealing with a well-posed problem (22), that is the matrix A is not degenerate. When $\epsilon \to 0$, the solution becomes the solution (23) of the unperturbed problem, $x \to A^{-1} b$.

3. If the problem (22) is ill-posed, the norm of the solution of the perturbed problem is still bounded:

$$\|x\| \leq \frac{1}{\epsilon} \|b\|$$

Remark 4.2 Instead of the regularizing term ϵx^2, we may use any positively define quadratic $\epsilon(x^T P x + p^T x)$ where matrix P is positively defined, $P > 0$, or other strongly convex function of x.