Multiple integrals: Sufficient conditions for a local minimum, Jacobi and Weierstrass-type conditions

March 6, 2013

Contents

1 Weak second variation

1.1 Formulas for variation

We develop the approach for the simplest variational problem (**??**) with the scaler minimizer; the generalization to the vector case is straightforward. For convenience, rewrite this problem

$$\min_{u: u\partial\Omega = u_0} I(u), \quad I(u) = \int_\Omega F(x, u, \nabla u) dx \tag{1}$$

Suppose that the variational problem (1) has a stationary solution u that satisfies the Euler equation (**??**). Similarly to the one-dimensional case (see Section **??**) we will compare the stationary solution with close-by admissible functions. The smallness of the perturbation of the solution permits development of verifiable necessary and sufficient conditions for a local minimum.

Consider a perturbed solution $u_p = u + \delta u$ assuming that the perturbation δu is small in the sense that

$$\|\delta u\|_{H_0^1(\Omega)} < \epsilon, \quad \|z\|_{H_0^1(\Omega)} = \frac{1}{2}\int_\Omega \left(z^2 + (\nabla z)^2\right) dx, \quad z = 0 \text{ on } \partial\Omega. \tag{2}$$

Here ϵ is an infinitesimal positive number. The increment ΔI of the objective functional is

$$\Delta I(\delta u) = \int_\Omega [F(x, u + \delta u, \nabla(u + \delta u)) - F(x, u, \nabla u)] dx$$

and the variation is zero on the boundary, $\delta u = 0$ on $\partial\Omega$, because the value of u is prescribed there. Because of smallness of δu, we expand the perturbed Lagrangian into Taylor series, keeping the linear and quadratic terms of the variation δu of the stationary solution u and neglecting the higher-order terms. For convenience, we will use the notation $v = \delta u$ and $\nabla v = \nabla \delta u$. We obtain

$$\Delta I(v) = \delta_1 I + \delta_2 I + o\left(||v||^2, ||\nabla v||^2\right)$$

where $v|_{\partial\Omega} = 0$,

$$\delta_1 I(u, v) = \int_\Omega \left(\frac{\partial F}{\partial u} v + \frac{\partial F}{\partial \nabla u} \nabla v\right) dx,$$

$$\delta_2 I(u, v) = \int_\Omega \left(\nabla v^T \cdot A(u) \nabla v + 2vB(u) \cdot \nabla v + vC(u)v\right) dx, \tag{3}$$

and

$$A(u) = \frac{\partial^2 F}{\partial \nabla u^2}, \quad B(u) = \frac{\partial^2 F}{\partial u \partial \nabla u} \quad C(u) = \frac{\partial^2 F}{\partial u^2}. \tag{4}$$

All the partial derivatives are calculated at the stationary solution $u(x)$. The linear in $v = \delta u$ term $\delta_1 I$ vanishes if the solution u satisfies Euler equation. For the variations with a small enough H_0^1 norm (see (2)), the sign of the increment

ΔI is determined by the sign of the second variation $\delta_2 I$. The variational problem

$$\min_{v: v=0 \ on \ \partial\Omega} \delta_2 I(v) \tag{5}$$

for the second variation is a homogeneous quadratic variational problem for the variations $\delta u = v$. The tested stationary solution u is considered as a known nonvariable (frozen) function.

1.2 Sufficient condition for weak minimum

We establish the sufficient condition for the weak minimum. Is guarantees that there is no other solutions u_{trial} that are close to u, $\|u - u_{\text{trial}}\|_{H_0^1} \leq \epsilon$ and that delivers a smaller value of the objective. If the problem (3) for the second variation has only trivial minimizer, $v = \delta u = 0$, the sufficient test is satisfied and the stationary solution corresponds to a weak minimum of the objective functional. The problem (5) has a trivial solution if the quadratic form of the coefficients of variation (see (3) is positively defined:

$$\Delta = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} > 0 \tag{6}$$

Here, Δ is a block matrix. It consists of the scalar entry C, a $d \times d$ matrix entry A and $d \times 1$ and $1 \times d$ vector entries B and B^T, respectively.

As in the one-variable case, the sufficient condition simplifies the problem by treating v and ∇v as independent entries neglecting the bond between them. As a result, ∇v is treated as a free vector not bounded by the differential constraints $\nabla \times (\nabla v) = 0$ of integrability. Thus, we end up at the theorem:

Theorem 1.1 (Sufficient condition for local minimum) If the matrix

$$V_2 = \begin{pmatrix} A(u) & B(u) \\ B^T(u) & C(u) \end{pmatrix}$$

is nonnegative at a stationary solution $u(x)$ in each point of Ω, then u is the local minimum of the objective.

Indeed, the condition of the theorem states that

$$\delta_2 I(u, v) = \int_\Omega (v, \nabla v) V_2 (v, \nabla v)^T dx > 0$$

for all variations v that satisfy the boundary condition $v = 0$ on $\partial\Omega$ and are not identically zero. Therefore there is no close-by function $\bar{u}$, that corresponds to lower value of the objective, or the stationary solution u corresponds to the local minimum of the objective. Notice how the locality of the minimum simplifies the test: The Lagrangian becomes quadratic and the Euler equation becomes linear.

Remark 1.1 Besides the smallness of the variation, this sufficient condition uses another rough simplification. It is assumed in the derivation that the variation v and its derivative ∇v are independent on each other, therefore the call of variations is enlarged.

Example 1.1 (Second variation) Compute the second variation for the L_p penalized approximation problem (see Section **??**) that corresponds to the Lagrangian.

$$L = \alpha|\nabla u|^p + \frac{1}{2}(u-f)^2$$

The variational problem for the second variation becomes

$$\min_v \int_\Omega \left(\frac{1}{2}\nabla v^T A(\nabla u)\nabla v + v^2\right) dx, \quad v = 0 \text{ on } \partial\Omega \tag{7}$$

where A is defined in (4) and u is the stationary solution to the approximation problem in Example **??**. Compute the coefficients (4) in the expression (**??**) for the second variation:

$$C = \frac{\partial^2 L}{\partial u^2} = 1, \quad B = \frac{\partial^2 L}{\partial u \partial g} = 0$$

Here, we denote for convenience $g = \nabla u$; g is d-dimensional vector. Compute A:

$$A = A(g) = \alpha\frac{\partial^2 L}{\partial g^2} = p|g|^{p-4}\left[(p-2)g\, g^T + (g^T g)I\right] \tag{8}$$

where I is the unit matrix, or

$$A_{ij} = p|g|^{p-4}\left[(p-2) + \delta_{ij}\right] g_i g_j$$

where δ_{ij} is the Kronecker delta.

The second variation is positive is A is positively defined. Compute the eigenvalues and eigenvectors of A. The eigenvalues are

$$\lambda_1 = p(p-1)|g|^{p-2}, \quad \lambda_2 = \lambda_3 = p(p-2)|g|^{p-2}$$

and the eigenvectors are

$$a_1 = (g_1, g_2, g_3), \quad a_2 = (0, -g_3, g_2)\,, \quad a_3 = \left(-\frac{g_2^2 + g_3^2}{g_1 g_3}, \frac{q_2 g_3}{g_1}, g_3\right)$$

We observe that A is nonnegative if λ_2 and λ_3 are nonnegative that is if $p > 2$.

Notice that in the one-dimensional problem, $x \in R_1$, $g = (g_1, 0, 0)$ only one eigenvalue λ_1 remains, and the nonnegativity condition are weaker, namely $p > 1$.

2 Jacobi-type necessary conditions

2.1 Jacobi condition

In this and next Section, we study necessary conditions for the weak minimum. Those are obtained by specifying the form of the second variations. It is assumed that the sufficient condition for the weak minimum is not satisfied, and the stationarity might not correspond to the local minimum. The necessary condition are design to disprove the assumption that the stationarity delivers a local minimum. The perturbations of the solutions are introduced and it is checked if these perturbations can decrease the stationary value of the objective functional.

Jacobi test The scheme of Jacobi test remains the same for one- and multivariable case. It tests the stationary solution $u(x)$ which is considered as a known function. Consider the problem (5), (3) freezing $u(x)$ and treating $v = \delta u$ as a minimizer. The original problem is simplified when the variation v is chosen that vanishes outside of a domain $\Omega_0 \subset \Omega$. It is assumed that Ω_0 is a "simple" domain and therefore the solution is easier to analyze. The Jacobi condition points to the "most dangerous" variation in Ω_0. To find such variation, one needs to solve the Euler equation for the minimizing variation v,

$$\nabla \cdot A \nabla v + (\nabla \cdot B - C) v = 0 \quad \text{in } \Omega_0 \subset \Omega, \quad v = 0 \quad \text{on } \partial\Omega_0. \tag{9}$$

If this equation has a nontrivial solution $v \neq 0$, the variation exists that improves the stationary solution. Summing up, we arrive at the theorem

Theorem 2.1 (Jacobi condition) Every stationary solution u to the problem (5) satisfies the Jacobi necessary condition: The problem (9) has only trivial solution $v = 0$ in any subdomain Ω_0 of Ω. Otherwise an improving variation v exist and the stationary solution u does not corresponds to the minimum of the objective.

Practically, one can try to disprove the optimality of the minimizer by considering equation (9) in a canonic domain $\Omega_0 \subset \Omega$, such as a circle, ellipse, rectangle, or even torus. The problem for the second variation $\delta_2 I_J(v)$ still has the form (3) but the integration is performed over Ω_0 instrad of Ω.

$$\delta_2 I_J(v) = \int_{\Omega_0} \left(\nabla v^T \cdot A(u) \nabla v + 2vB(u) \cdot \nabla v + vC(u)v \right) dx, \tag{10}$$

Unlike the one-dimensional case, the Jacobi condition for the multivariable variational problems includes the nontrivial choice of the domain of the variation. The choice of the domain is characteristic for several multidimensional variational techniques.

Example 2.1 Consider the problem in a two-dimensional domain $\Omega \subset R^2$

$$J = \min_u \int_\Omega \left(\frac{1}{2} (\nabla u)^2 - \frac{c^2}{2} u^2 \right) dx, \quad u = u_0 \text{ on } \partial\Omega$$

We prove

Theorem 2.2 The stationary solution

$$u(x): \quad \nabla^2 u + c^2 u = 0 \text{ in } \Omega, \quad u = u_0 \text{ on } \partial\Omega$$

does not correspond to the minimum of J if the domain Ω contains either

(i) a rectangle $\Omega_0(\alpha, \beta)$ with the sides α and β, such that

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{c^2}{\pi^2} \tag{11}$$

or

(ii) a circle Ω_{00} of the radius $\frac{\rho}{c}$, where $\rho = 2.404825..$ is the minimal positive root of the Bessel equation $J_0(\rho) = 0$.

First, derive the equation (9) for the second variation of Lagrangian. It has the form

$$\nabla^2 v + c^2 v = 0 \quad \text{ in } \Omega_0, \quad v = 0 \quad \text{ on } \partial\Omega_0 \tag{12}$$

Here, $A = 1$, $B = 0$, $C = 1$. Notice, that this equation coincides with the stationary condition, but unlike the stationarity, it satisfies the homogeneous boundary condition at the chosen contour.

Proof of the condition (i) One may choose Ω_0 to be a rectangle with the unknown sides α and β and the improving variation $v(x_1, x_2)$ to be function of the both coordinates that vanishes if $x = \pm\frac{\alpha}{2}$ or if $y = \pm\frac{\beta}{2}$. The equation (12) in a rectangle $|x| \leq \frac{\phi}{2}$, $y \leq \frac{\beta}{2}$ has the form

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} + c^2 v = 0, \quad u = 0 \text{ if either } |x| = \frac{\phi}{2}, \text{ or } |y| = \frac{\beta}{2},$$

it permits the separation of variables, and has the solution

$$v_{\alpha\beta} = \cos\left(\frac{\pi x_1}{\alpha}\right) \cos\left(\frac{\pi x_2}{\beta}\right), \quad \text{where } \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{c^2}{\pi^2} \tag{13}$$

Substituting $v = v_{\alpha\beta}$ into (10) and computing the second variation

$$\delta^2 I_J(v) = \frac{1}{\alpha\,\beta}\left(\pi(\alpha^2 + \alpha^2) - c^2\alpha^2\beta^2\right) \tag{14}$$

one easily checks that $\delta^2 I_J(v) = 0$ if the rectangle with the sides (11) is contained in Ω.

Proof of the condition (ii) One can also build a nonpositive second variation in a larger rectangular $\alpha' \times \beta'$ if $\alpha' > \alpha$ and/or $\beta' > \beta$. The variation $v_{\alpha'\beta'}$ has the form (13). One checks by the direct calculation using (14) that this variation corresponds to the negative second variation.

One may also choose Ω_0 to be a circle with the unknown radius ρ, and the improving variation v to be function only of the radius, $v = v(r)$. The equation (12) becomes an ordinary differential equation

$$\frac{1}{r}\frac{d}{dr}r\frac{dv}{dr} + c^2 v = 0, \quad v(\rho) = 0, \quad v(r) \text{ is bounded if } 0 \le r \le \rho$$

The general solution that is bounded at $r = 0$ is $v = CJ_0(c\,r)$. It satisfies the boundary condition $v(r_0) = 0$ inside the circle of the radius ρ, if $c\,\rho$ is larger than the minimal positive root of the Bessel equation $J_0(c\,\rho) = 0$ which gives $\rho > 2.404825/c$. By the Jacobi condition, if the domain Ω contains a circle of this radius, the cost of the stationary solution (which is unique for this problem) can be improved; therefore the stationary solution does not correspond to a minimum of the objective.

2.2 Simplified nonlocal variation

Lagrangians of more general form result in the quadratic problem for the second variation with variable coefficients. This time, it is more difficult to find the exact solution to the problem (9). However, it may be still possible to find an improving variation assigning its type and estimating the second variation, in addition to the assigning of the subdomain Ω_0

Example 2.2 Consider the problem with the Lagrangian

$$L = \kappa(x)(\nabla u)^2 - c^2(x)u^2$$

where

$$0 < \kappa(x) \le \kappa_+ \quad \text{and } c(x) \ge c_- \ge 0$$

are bounded variable coefficients, which dependence on x might be unknown. The second variation is

$$\delta^2 I(v) = \int_\Omega (\kappa(x)(\nabla v)^2 - c^2(x)v^2)dx$$

where v is a variation. The Jacobi condition asks for the nontrivial solution to the partial differential equation

$$\nabla \cdot \kappa(x)\nabla v + c(x)v = 0, \quad v = 0 \text{ on } \partial\Omega_0$$

which is usually impossible to obtain in a closed analytic form. Instead, we may assign the form of the variation v. For example, let v be as follows

$$v = (x_1^2 - a^2)(x_2^2 - b^2), \text{ if } (x_1, x_2) \in \Omega_0, \quad v = 0 \text{ otherwise}$$

where $\Omega_0(a, b)$ is a rectangle

$$\Omega_0(a, b) = \{(x_1, x_2) : \quad |x_1| \le a, \ |x_2| \le b\},$$

and a and b are two positive parameters. This variation vanishes at the boundary of Ω_0 that is if $x = \pm a$ or $y = \pm b$. Assume also that $\Omega_0 \subset \Omega$. The expression for the second variation becomes an explicit function of κ and c

$$\delta^2 I(v) = \int_{\Omega_0} \big(4\kappa(x) \left[x_1^2(x_2^2 - b^2)^2 + x_2^2(x_1^2 - a^2)^2\right] \\ -c^2(x)(x_1^2 - a^2)^2(x_2^2 - b^2)2\big)\, dx$$

If the sign of the second variation is negative, the stationary condition does not correspond the minimum.

Even if the functions $\kappa(x)$ and $c(x)$ are unknown, we may bound the expression of $\delta^2 I$ from above, replacing $\kappa(x)$ with κ_+ and $c(x)$ with c_- and computing the integrals. We obtain

$$\delta^2 I(v) \le \frac{128}{225} a^5 b^5 \left[5\kappa_+ \left(\frac{1}{a^2} + \frac{1}{b^2}\right) - 2c_-^2\right]$$

If this upper estimate is nonpositive, the so is the second variation. We conclude that the stationary solution does not correspond to minimum, if Ω contains a rectangle $\Omega_0(a, b)$ such that

$$\frac{1}{a^2} + \frac{1}{b^2} \le \frac{2}{5} \frac{c_-^2}{\kappa_+}$$

Notice that the obtained condition is not the most restrictive, but it is simple enough to use.

3 Weierstrass-type Test

3.1 Localized variations

Convexity of Lagrangians and Stability of Solutions We have shown in Chapter **??** that a solution to a one-dimensional variational problem is stable against fine-scale perturbations if its Lagrangian is convex. The lack of convexity of the Lagrangian leads to the appearance of rapidly alternating functions in the optimal solution. Here we develop a similar approach for multidimensional variational problems.

Strong Local Variations Weierstrass Type The Weierstrass-type condition checks that a sharp localized perturbation of the extremal does not decrease the functional. Failure to satisfy the Weierstrass test proves that the checked stationary solution is not optimal because it can be improved by adding an oscillatory component to it. We define the local strong perturbation (or the strong local variation or Weierstrass variation) as follows.

Definition 3.1 By a strong local variation (Weierstrass-type variation) of a multidimensional variational problem we understand a localized perturbation δu of the potential u that

1. is differentiable almost everywhere,
2. has an arbitrarily small magnitude $|\delta u| < \varepsilon$;
3. has a finite magnitude of the gradient $|\nabla u| = O(1)$; and
4. is localized in a small neighborhood $\omega_\varepsilon(x_0)$ of an inner point x_0 in the domain Ω: $\delta u(x) = 0 \, \forall \, x \notin \omega_\varepsilon(x_0)$, where $\omega_\varepsilon(x_0)$ is a domain in Ω with the following properties: $x_0 \in \omega_\varepsilon(x_0)$, $\text{diam}(\omega_\varepsilon) \to 0$ as $\varepsilon \to 0$.

Shape of variation domain There is a freedom in choosing the type of Weierstrass variation in a multidimensional problem. First, we choose the shape of ω_ε. It is important that δw is continuous and vanishes on the boundary $\partial\omega_\varepsilon$ of ω_ε. For example, we may choose ω_ε as a circular domain and consider the trial perturbation δw shaped like a cone, or a symmetric paraboloid. For a polygonal domain ω_ε the variation δw can be shaped like a pyramid.

The increment of the functional due to this variation is of the order of the size of the domain of variation ω_ε. The main term of the increment depends on the perturbation ∇u is the Lagrangian is Lipschitz with respect to u and x and coercive with respect to ∇u (which we will assume from now),

$$F(x, u + \delta u, \nabla(u + \delta u)) = F(x_0, u, \nabla u + \delta v)) + o(\varepsilon), \quad u = u(x_0), \quad \forall x \in \omega_\varepsilon$$

where x_0 is the coordinate of the center of ω_ε,

$$\delta I_W(\omega_\varepsilon) = \int_{\omega_\varepsilon} [F(x_0, u, \nabla u + \nabla v) - F(x_0, u, \nabla u)] \, dx + o\|\omega_\varepsilon\|, \tag{15}$$

Notice that the only variable in ω_ε argument is ∇v, but the slow variable u is "frozen" to be equal to be the values of the checked stationary solution. The independent variable x is replaced by x_0 without change in the main term of the increment. The integral of ∇v over ω_ε is zero,

$$\int_{\omega_\varepsilon} \nabla v \, dx = V \otimes \int_{\omega_\varepsilon} \nabla s \, dx = 0 \tag{16}$$

Consider δu in a shape of a truncated pyramid (a frustum) with m sides B_i with normals n_i The gradient of variation is

$$\nabla(\delta u) = \begin{cases} 0 & \text{if } u \in \text{ out of the pyramid} \\ c_i n_i & \text{if } u \in B_i \\ 0 & \text{if } u \in \text{ in the truncated center region of the pyramid} \end{cases} \tag{17}$$

The resulting necessary condition depends on the chosen shape of the variation. We will call the corresponding inequalities the necessary conditions of Weierstrass type or the *Weierstrass conditions*. The Weierstrass condition depends on magnitude of the variation as well as on the shape of the domain ω_ε.

3.2 Rank-One Convexity

Variation in a symmetric cone or strip The variation (**??**) can be simplified when ad additional assumption of the symmetry of ω_ε is made. Assume that its boundary $b(\phi)$ is symmetric to the rotation on 180°, $b(\phi) = b(\phi + 180°)$. Then $P(\phi) = -P(\phi + 180°)$. The variation (**??**) becomes

$$\delta I_W(\omega_\varepsilon) = \frac{1}{2}\int_0^\pi D_I(\phi) b^2(\phi) d\phi \tag{18}$$

where

$$D_I(\phi) = F(u, \nabla u + V \otimes P(\phi)) + F(u, \nabla u - V \otimes P(\phi)) - 2F(u, \nabla u) \tag{19}$$

The necessary condition (15) is satisfied is

$$D_I(\phi) \geq 0 \quad \forall \phi \in [0, Pi). \tag{20}$$

Notice that the last condition is equivalent to the convexity of $F(u, \nabla u)$ in the "direction" $V \otimes P$. This "direction" is an arbitrary $n \times 2$ dyad because both vectors V and P are arbitrary.

Variation in the parallel strips The opposite type of variation corresponds to the extremely elongated rectangular domain S_ε of the size $(\varepsilon, \varepsilon^2)$ that consists of several thin strips parallel to the longer side. The variation of the potential depends on the normal n to the strips everywhere except in the end domains S_{ε^2} of the size $\varepsilon^4 = \varepsilon^2 \times \varepsilon^2$ at the ends of the strips where it monotonically decays to zero. Assume that the potential is piece-wise linear continuous function of n. Its magnitude is of the order ε^2 of the thickness c_k of the layers. The gradient ∇v, is a piece-wise constant vector function with the values V_k of the finite magnitude everywhere except of the end domains S_{ε^2} where it is bounded. The contribution of these domains is arbitrary small comparing with the contribution of the much larger middle section $S_\varepsilon - S_{\varepsilon^2}$ of the domain.

The main term in the increment comes from the variation the middle part of the strip. Here, the gradient $\nabla v_i = \nu_i(n) n$ of each potential v_i is directed along the normal n to the strips. Function ν is piece-wise constant and takes a constant value V_{ki} in each strip. The variation of the vector potential $v = \{v_1, \ldots, v_n\}$ has the form

$$\nabla v(n) = V(n) \otimes n, \qquad \text{in } S_\varepsilon - S_{\varepsilon^2}$$

where $V(n)$ is a piece-wise constant function with the values $V_k = (v_{k1}, \ldots, v_{kn}$ in the kth strip.

The perturbation of the potential v is zero outside of the ω_ε and is continuous inside the domain; this leads to the constraint on the magnitudes V_k

$$\sum_k c_k V_k = 0, \quad \sum_k c_k = 1, \quad c_k \geq 0 \tag{21}$$

here c_k is the relative thickness of the kth strip.

The increment ΔI of the cost of the variational problem (17) due to the variation in the strip is

$$\Delta I = \sum_k c_k F(u, \nabla u + V_k \otimes n) - F(u, \nabla u) \tag{22}$$

Solution u is stable to the perturbation in a strip if

$$\Delta I > 0 \quad \forall, V_k, c_k \text{ as in (21) }, \quad \forall n$$

Rank-One convexity The condition (**??**) states that the Lagrangian $\alpha F(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{A})$ is convex with respect to some special trial matrices of the type $\boldsymbol{R} = \boldsymbol{\alpha} \otimes \boldsymbol{n}$ but not with respect to arbitrary matrices. The corresponding property is called the rank-one convexity.

Definition 3.2 The scalar function F of an $n \times m$ matrix argument $\boldsymbol{A}$ is called *rank-one convex* at a point $\boldsymbol{A}_0$ if

$$F(\boldsymbol{A}_0) \leq \sum_{i=1}^{N} \alpha_i F\left(\boldsymbol{A}_0 + \alpha_i \xi_i \boldsymbol{R}\right) \tag{23}$$

for any $\alpha_i,\ \xi_i,\ \boldsymbol{R},\ N$ that

$$\sum_{i=1}^{N} \alpha_i = 1, \quad \alpha_i \geq 0, \quad \sum_{i=1}^{N} \alpha_i \xi_i = 0, \quad \boldsymbol{R} = \boldsymbol{a} \otimes \boldsymbol{b}.$$

Here $\boldsymbol{a}$ and $\boldsymbol{b}$ are n-dimensional and m-dimensional vectors, respectively, and α_i are scalars.

Rank-one convexity requires convexity in some matrix "directions," namely, in the "directions" of the rank-one matrices. Obviously, the usual convexity implies rank-one convexity.

There are two cases in which rank-one convexity coincides with convexity:

1. The Lagrangian depends on one independent variable: $\boldsymbol{x}$ is a scalar.
2. The Lagrangian depends on one dependent variable: $\boldsymbol{w}$ is a scalar.

In both cases, the matrix $\boldsymbol{A}_0 = \nabla \boldsymbol{w}$ degenerates into a rank-one matrix.

Example 3.1 (Non-convex but rank-one convex function) Let A be a 2×2 matrix and $F(A)$ be

$$F(A) = [\ \mathrm{Tr}\,(A)]^2 + 2C \det A \tag{24}$$

We show that $F(A)$ is nonconvex, if $C \neq 0$, but it is rank-one convex for all real C. Indeed, F is a quadratic form, $F(A) = A_v^T M\, A_v$, of the elements of A that form the four-dimensional vector $A_v = (a_{11}, a_{22}, a_{12}, a_{21})$. Matrix M of this form is

$$M = \begin{pmatrix} 1 & 1+C & 0 & 0 \\ 1+C & 1 & 0 & 0 \\ 0 & 0 & 0 & -C \\ 0 & 0 & -C & 0 \end{pmatrix}$$

Its eigenvalues are $C, C+2, \pm C$. At least one of the eigenvalues is negative if $C \neq 0$, which proves that $F(A)$ is not convex.

Compute the rank-one perturbation of F. We check by the direct calculation that

$$\sum_k c_k \det\left(A + \alpha_k d \otimes b\right) = \det A,$$

if

$$\sum_k c_k \alpha_k = 0 \tag{25}$$

Indeed, all quadratic in the elements of $d \otimes b$ terms in the left-hand side cancel, and the linear terms sum to zero because of (25). We also have

$$\left(\sum_k c_k \ \mathrm{Tr}\, F(A + \alpha_k d \otimes b) \right)^2 = (\ \mathrm{Tr}\, A)^2 + \left(\sum_k c_k \alpha_k \ \mathrm{Tr}\,(d \otimes b) \right)^2$$

(linear in $d \otimes b$ terms cancel because of (25)).

Substituting these two equalities into F in (24), we find that

$$\sum_k c_k F(A + \alpha_k d \otimes b) = F(A) + \left(\sum_k c_k \ \mathrm{Tr}\,(\alpha_k d \otimes b) \right)^2$$

if if (25) holds. The variation is independent of the value of C. The inequality (23) follows; therefore F is rank-one convex.

Stability of the stationary solution The rank-one convexity of the Lagrangian is a necessary condition for the stability of the minimizer. If this condition is violated on a tested solution, then the special fine-scale perturbations (like the one described earlier) improve the cost; hence the classical solution is not optimal.

Theorem 3.1 (Stability to Weierstrass-type variation in a strip) Every stationary solution that corresponds to minimum of the functional (**??**) corresponds to rank-one convex Lagrangian. Otherwise the stationary solution u can be improved by adding a perturbation in a strip to the solution.

Remark 3.1 Rank-one trial perturbation is consistent with the classical form $L(\boldsymbol{x}, \boldsymbol{w}, \nabla \boldsymbol{w})$ of Lagrangian. This form implies the special differential constraints $\nabla \times (\boldsymbol{v}) = 0$ that require the continuity of all but one component of the field $\nabla \boldsymbol{w}$. The definition of this necessary condition for the stability of the solution can be obviously generalized to the case where the differential constraints are given by the tensor $\mathcal{A}$.

3.3 Legendre-type condition

A particular case of the Weierstrass-type condition is especially easy to check. If we assume in addition that the magnitude V of the variation is infinitesimal, the rank-one condition becomes the requirement of positivity of the second derivative in a rank-one ”direction”