Multiple integrals: Lagrangian Dependent on several minimizers

February 1, 2013

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1 Euler-Lagrange equations

In this chapter, we study stationarity conditions for multivariable variational problems with vector minimizer. Assume that $x \in R^d$, Ω is a domain in R^d, and a vector function $u : R^d \to R^n$, has components $u_k, k = 1, ..., n$. Consider the following variational problem

$$I = \min_{u(x)} J, \quad J = \int_\Omega F(x, u, \nabla u) dx \tag{1}$$

assuming that the Lagrangian F depends on components of $u = (u_1, \ldots, u_n)$ and their derivatives – $d \times n$ Jacobian matrix

$$\nabla \boldsymbol{u} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial u_1}{\partial x_d} & \cdots & \frac{\partial u_n}{\partial x_d} \end{pmatrix}.$$

The stationarity conditions are derived in the same way as in the case of one unknown function u. The first variation $\delta_k J$ of J with respect to u_k – the k-th component of the vector minimizer u is

$$\delta_k J = \int_\Omega \left(\sum_{k=1}^n S_F(u_k) \delta u_k \right) dx + \int_{\partial\Omega} \left(\sum_{k=1}^n S_F^\partial(u_k, n) \delta u_k \right) ds$$

where $S_F(u_k)$ and $S_F^\partial(u_k, n)$ (see (**??**), (**??**), and (**??**)) are the bulk and boundary parts of the functional derivative, respectively. Stationarity conditions correspond to the system of Euler equations

$$S_F(u_k) = 0 \ \text{ in } \Omega, \quad S_F^\partial(u_k, n) \delta u_k = 0 \ \text{ on } \partial\Omega, \quad k = 1, \ldots, n$$

which expresses independency of the variation of each potential u_k. The stationarity conditions form a system of n second-order equations for n unknown potentials. Equations are of the same form as the Euler equations for the scalar case: scalar operations are simply replaced by vectorial ones and vectorial operations become matrix ones:

$$-\nabla \cdot \frac{\partial F}{\partial (\nabla u)} + \frac{\partial F}{\partial u} = 0 \tag{2}$$

where u is a vector $u = (u_1, \ldots, u_n)$. The coordinate form of this system of equations is

$$-\sum_{k=1}^d \frac{\partial}{\partial x_k} \frac{\partial F}{\partial g_{ik}} + \frac{\partial F}{\partial u_i} = 0, \quad g_{ik} = \frac{\partial u_i}{\partial x_k}, \quad i = 1, \ldots, n \tag{3}$$

Obviously, the system degenerates into (**??**) when $n = 1$ and into (**??**) when $d = 1$.

When the boundary values of u are prescribed, the variation δu_k is zero, and we deal with the main boundary conditions. The natural boundary conditions

result from the stationarity requirement when the boundary variation δu_k is not constrained. They are

$$S_\partial(u_i, F) = \frac{\partial F}{\partial(\nabla u_i)} n = 0, \quad i = 1, \ldots, n$$

or, in the coordinate form,

$$\frac{\partial F}{\partial g_{ik}} n_k = 0, \quad g_{ik} = \frac{\partial u_i}{\partial x_k}, \quad i = 1, \ldots, n$$

where n_k is the projection of the normal to the axis x_k. The case of constrained boundary conditions is analogous to one-variable problem discussed above in Chapter **??**. Hence, the equations in (3) can have main or natural boundary conditions.

Example 1.1 (An optimal design problem) Consider the following problem: Suppose that the values of the potential u are given on the boundary of a domain Ω. Minimize the energy E of the domain Ω,

$$E = \min_{u \in \mathcal{U}} \frac{1}{2} \int_\Omega k(x) |\nabla u|^2 dx,$$

by choosing optimal conductivity $k(x) > 0$ in each point of the domain. Here u is the potential that belongs to the set $\mathcal{U}$:

$$\mathcal{U} = \{u : u|_{\partial\Omega} = u_{\text{bound}}\}$$

and u_{bound} is the given boundary function.

The cost c of the material depends on its conductivity. For definiteness, let us assume that it is inverse proportional to conductivity or it is proportional to resistivity, $c = \frac{\alpha}{k}$ where $\alpha > 0$ is a parameter. The total cost C must be fixed,

$$C = \alpha \int_\Omega k^{-1} dx \tag{4}$$

By assumption, the cost is inverse proportional to conductivity or it is proportional to resistivity.

The design problem asks to distribute different conducting materials in the domain in order to minimize the total energy plus the cost of materials. The problem for the minimizers k and u is the following:

$$I = \inf_{k(x)>0} [E + C] = \inf_{k(x)>0} \min_{u \in \mathcal{U}} \int_\Omega L(u, \nabla u, k) dx$$

$$\text{where } L(u, \nabla u, k) = k|\nabla u|^2 + \frac{\alpha}{k} \tag{5}$$

This Lagrangian depends on two minimizers, u and k. It does not depends on ∇k, therefore the optimal k can be expressed through ∇u and excluded. The stationarity with respect to k corresponds to the algebraic relation

$$\frac{\partial L}{\partial k} = |\nabla u|^2 - \frac{\alpha}{k^2} = 0$$

Solving it for k, we obtain dependence of the optimal conductivity k_{opt} on the intensity of the gradient field:

$$k_{\mathrm{opt}} = \frac{\sqrt{\alpha}}{|\nabla u|}.$$

To minimizer the total energy, one places more resistant and more expensed material in the places where the intensity of the field is greater. Substitution of this value into L results in the Lagrangian which depends only on potential u and constant α and represents the energy density of an optimal domain:

$$L(u, \nabla u, k_{\mathrm{opt}}) = 2\sqrt{\alpha}|\nabla u|$$

The Euler equation for this Lagrangian (see (??)) is

$$\nabla \cdot \frac{\nabla u}{|\nabla u|} = 0$$

Notice that the problem of the optimal distribution of linear conducting materials with quadratic energy $k\,|\nabla u|^2$ turns out to be equivalent to a conductivity problem for a nonlinear material. The nonlinearity occurs because the optimal conductivity k_{opt} depends on the field intensity $|\nabla u|$. Local increase of the field intensity changes the energy $k_{\mathrm{opt}}|\nabla u|^2$ both directly and through the change in the optimal material's conductivity k_{opt} following from the optimality requirements.

2 Null-Lagrangians

Example 2.1 (Coupled conductivity) Find the Euler-Lagrange equation for minimization of quadratic Lagrangian

$$F = \frac{1}{2}\nabla u_1 \cdot K_1 \nabla u_1 + \frac{1}{2}\nabla u_2 \cdot K_2 \nabla u_2 + \nabla u_1 \cdot K_{12}\nabla u_2 + \phi(u_1, u_2) \qquad (6)$$

where $u = (u_1, u_2)$, $x \in R^3$, ϕ is a continuously differentiable function. The Euler equations are obtained by varying separately u_1 and u_2

$$(\delta u_1): \quad -\nabla \cdot (K_1 \nabla u_1 + K_{12}\nabla u_2) + \frac{\partial \phi}{\partial u_1} = 0$$

$$(\delta u_2): \quad -\nabla \cdot \left(K_{12}^T \nabla u_1 + K_2 \nabla u_2\right) + \frac{\partial \phi}{\partial u_2} = 0$$

They are partial differential equations of second order. The natural boundary conditions are the coefficients by the boundary variations:

$$(\delta u_1|_{\partial\Omega}): \quad n \cdot (K_1 \nabla u_1 + K_{12}\nabla u_2) = 0$$

$$(\delta u_2|_{\partial\Omega}): \quad n \cdot (K_{12}\nabla u_1 + K_2 \nabla u_2) = 0$$

These equations described a coupled diffusion. They may describe diffusion of two groups of particles that can transform into each other. For example, the model

describes diffusion of groups of fast and slow neutrons in a nuclear reactor; in this example ϕ accounts the recombination. Or u_1 and u_2 may describe the diffusion of two species in the habitat, ϕ describes its interaction.

Nonlinear Null-Lagrangians   The vector problem admits a new type of nonlinear null-Lagrangians. A quadratic nonlinear null-Lagrangian has the form

$$L_0 = \nabla u_1^T A_{12} \nabla u_2$$

where A_{12} is antisymmetric $a_{ij} = -a_{ji}$. We compute the Euler equation varying u_1. The coefficient by δu_1 is:

$$\nabla \cdot A_{12} \nabla u_2 = \sum_{i,j} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u_2}{\partial x_j} = \sum_{i,j} (a_{ij} + a_{ij}) \frac{\partial^2 u_2}{\partial x_i \partial x_j} \equiv 0 \quad \forall u_2$$

(the equation for u_1 is similar). If A is antisymmetric, the coefficient of the Eular equation identically vanish.

   As in the one-dimensional case, the null-Lagrangian is a divergence of a function $L_0 = \nabla \cdot (F(u\nabla u)$, For example, consider a two-dimensional case.

$$\det(\nabla u) = \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} - \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} = \nabla u_1 A_{12} \nabla u_2, \quad A_{12} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We find that the $\det \nabla u$ is a null-Lagrangian. If is also a divergence:

$$\det \nabla u = \nabla \cdot Z, \quad Z = u_1 \begin{pmatrix} \frac{\partial u_2}{\partial x_2} \\ -\frac{\partial u_2}{\partial x_1} \end{pmatrix}$$

The integral of F_0 is expressed through the boundary integral by Green's theorem.

$$\int_\Omega \nabla \cdot Z = \int_{\partial \Omega} Z \cdot n \, ds = \int_{\partial \Omega} u_1 \frac{\partial u_2}{\partial s} ds$$

Clearly, a variation of u inside Ω does not change the functional that depends only on the boundary data. It does, however, influence the natural boundary conditions.

Remark 2.1 Euler equation is identically satisfied thanks to the integrability conditions (the equality of the mixed derivatives). This null-Lagrangian does not have a one-variable analogs. L_0 is polylinear: it depends linearly on ∇u_1 for a fixed ∇u_2, and vice versa.

Example 2.2 (Null-Lagrangians in the coupled diffusion equation) Consider again the problem in Example . If a term $\beta \nabla u_1 A \nabla u_2$ is added to the Lagrangian (6), the Euler equations remain the same, but the main boundary conditions change to

$$(\delta u_1|_{\partial \Omega}): \quad n \cdot (K_1 \nabla u_1 + (K_{12} + A) \nabla u_2) = 0$$
$$(\delta u_2|_{\partial \Omega}): \quad n \cdot (K_2 \nabla u_2 + (K_{12} - A) \nabla u_1) = 0$$

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Null-Lagrangians in the problems with higher derivatives Similarly, nonlinear Null-Lagrangian arrive in the problem with higher derivatives. Consider, for example, minimization of an elastic energy W_{bend} of a bended plate. Its energy is a rotation invariant quadratic function of the Hessian $\nabla\nabla u$ of the vertical deflection $u(x_1, x_2)$ of a point of a plate. A general form of such function is:

$$W_{bend} = a \operatorname{Tr} \nabla\nabla u + b \det(\nabla\nabla u) =$$
$$a \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \right)^2 + b \left[\frac{\partial^2 u}{\partial x_1^1} \frac{\partial^2 u}{\partial x_2^2} - \left(\frac{\partial^2 u}{\partial x \partial x_2} \right)^2 \right]$$

The coefficient by b is a null-Lagrangian, as one can prove straightforward.

3   Elasticity

The problem Here we derive equation of elasticity (Lamé system) by minimizing an elastic energy of the body. Consider a linear elastic body which occupies a bounded three-dimensional domain Ω. A part $\partial\Omega_1$ of the boundary surface is fixed. Some external forces f are applied to the other part $\partial\Omega_2$ of the boundary. Assume that the forces and the reaction of supports are in equilibrium so that the body does not rotate, and the center of mass does not move.

Being deformed by the forces f, the points of the body are displaced: A point with coordinates $x = (x_1, x_2, x_3)$ moves to the point $x + u = (x_1 + u_1, x_2 + u_2, x_3 + u_3)$. The vector $u = (u_1, u_2, u_3)$ is called the vector of *displacement*. The displacement can be found by solving a variational problem of the minimization of the *elastic energy* of the domain Ω.

3.1   Elastic energy

First, we argue on the form of elastic energy. Assume that the elastic equilibrium is defined by the vector u and its gradient – the 3×3 matrix $\nabla u = (\nabla u_1 | \nabla u_2 | \nabla u_3)$. We postulate that the equilibrium corresponds to the minimization of an integral of a Lagrangian (the elastic energy) $W_a(u, \nabla u)$ plus the work $f \cdot u$ of the applied force f:

$$\min_{u \in \mathcal{U}} \int_\Omega W(u, \nabla u) dx + \int_{\partial\Omega_2} f \cdot u \, ds, \quad \mathcal{U} = \{u : \ u|_{\partial_1} = u_0\} \tag{7}$$

We require that the elastic energy $W(u, \nabla u)$ possesses several properties:

1. The energy is invariant to a translation of a body as a solid in space. The translation replaces u with $u + C$ where C is an arbitrary constant vector, and keeps the value of ∇u,

$$u \to u + c, \quad \nabla u \to \nabla u$$

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The invariance reads as $W(u + C, \nabla u) = W(u, \nabla u)$ for any C. Therefore, the energy depend only on ∇u and not on u, $W(u, \nabla u) = W(\nabla u)$.

2. The energy depends only on the symmetric part of the displacement gradient. Indeed, the energy must be invariant to rotations ϕ of the labor coordinate system. This invariance, however, cannot be consistent with the linearity of the Euler equations, see []. Instead, we request the invariance of the energy to *infinitesimal rotations* ϕ. One can see that the symmetric $(\nabla u)^S$ and antisymmetric $(\nabla u)^A$ parts of ∇u

$$\nabla u = (\nabla u)^S + (\nabla u)^A$$
$$(\nabla u)^S = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right), \quad (\nabla u)^A = \frac{1}{2}\left(\nabla u - (\nabla u)^T\right)$$

differently respond to such a rotation. The antisymmetric part of ∇u varies on the order of $\|\phi\|$, while the symmetric part - of the order of $\|\phi\|^2$, see Problem ??. Therefore, the elastic energy depends only on the symmetric part $(\nabla u)^S$ that is called *strain*, $W(\nabla u) = W(\epsilon)$.

$$\epsilon = \frac{1}{2}\left[\nabla u + (\nabla u)^T\right].$$

The coordinate form of the strain is

$$\epsilon_{ii} = \frac{\partial u_i}{\partial x_i} \quad \text{and } \epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{if } i \neq j$$

The eigenvalues of symmetric the 3×3 strain matrix ϵ are real. The strain matrix depends on six entries.

The supplementary antisymmetric part of the gradient represents an infinitesimal rotation or the domain as a whole; such a motion which does not affect the elastic energy.

3. When the external force f is zero, $f = 0$, the body is undeformed. The strain in the undeformed body is identically zero, and energy is zero as well, which implies

$$W(0) = 0, \quad \left.\frac{\partial W}{\partial \epsilon}\right|_{\epsilon=0} = 0$$

4. We want to derive equation for linear elasticity. This means that we consider the displacement u and its gradient to be small and neglect higher order terms in the equations. The linearity of the equations implies that the energy can be expanded into Taylor series, and we keep only first terms neglecting the terms higher in order than quadratic

$$W(\epsilon) = W_0 + \sum_{i,j=1}^{3} \psi_{ij}\epsilon_{ij} + \frac{1}{2}\sum_{i,j,k,l=1}^{3} \epsilon_{ij} : C_{ijkl} : \epsilon_{kl} + o(\|\nabla u\|^2)$$

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Here, ψ is a symmetric tensor called the *self-stress*, C is the fourth-order *stiffness* tensor: $C = \{c_{ijkl}\}$, and : means the contraction by two indices. The previous assumption requires that $\psi = 0$ and $W_0 = 0$ (we always can put $W_0 = 0$ because the energy is defined up to a constant).

Thus, we come to a energy of *anisotropic linear elastic* body

$$W_{\text{anis}}(\epsilon) = \frac{1}{2} \sum_{i,j,k,l=1}^{3} C_{ijkl} \epsilon_{ij} \epsilon_{kl}$$

A quadratic form $x^T A x$ for a vector is defined by a symmetric matrix A; similarly, the quadratic form for a symmetric matrix $\epsilon = \{\epsilon_{ij}\}$, $\epsilon_{ij} = \epsilon_{ji}$ is defined by a four-indexed array C_{ijkl} (fourth-rank tensor C) with special symmetries:

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}$$

Remark 3.1 The three-dimensional stiffness tensor C is determined by 21 constants. Indeed, an arbitrary array C_{ijkl} is defined by $3^4 = 81$ constant because each index can take one of three values. However, C defines a quadratic form and therefore is symmetric $C_{ijkl} = C_{jikl}$. The components ϵ_{ij} are also symmetric, $\epsilon_{ij} = \epsilon_{ji}$, which yields to the symmetries $C_{ijkl} = C_{ijlk}$ and $C_{ijkl} = C_{ijkl}$. These symmetries reduce the number of independent components of ϵ to six. The quadratic form over six-dimensional vector is defined by a 6×6 symmetric matrix is defined by $7 \cdot 6/2 = 21$ constants. Out of these constants, three are the angles of orientation of the labor system in space and eighteen are material characteristics that are invariant to the orientation of the labor system.

A similar consideration for the two-dimensional stiffness tensor shows that the two-dimensional strain corresponds to three components, C is a symmertic 3×3 matrix depending on six entries. One constant defines the angle of orientation of the labor system and five others define the material properties.

Isotropy The coefficients C_{ijkl} determine the material's elastic properties. Here we derive the energy of an *isotropic* material. It corresponds to the isotropic (rotationally invariant) quadratic form $W(\epsilon)$ of ϵ. The rotationally invariant energy is a function only of the eigenvalues of symmetric matrix ϵ or of its *main invariants*: trace $\mathrm{Tr}\,\epsilon$, trace $\mathrm{Tr}\,(\epsilon^2)$ of ϵ^2, and determinant $\det \epsilon$. An the same time, the energy is a quadratic function of the entries of the matrix ϵ. Thus the energy must linearly depend on $(\mathrm{Tr}\,\epsilon)^2$ and $\mathrm{Tr}\,(\epsilon^2)$ (the terms of $\det \epsilon$ cubically depend on the entries, therefore, energy of a linear material cannot depend on $\det \epsilon$). We conclude that a general form of an isotropic quadratic function of a symmetric matrix ϵ is

$$W_{\text{is}} = \mu\,\mathrm{Tr}\,(\epsilon^2) + \frac{1}{2}\lambda(\,\mathrm{Tr}\,\epsilon)^2 \tag{8}$$

Here λ and μ are some real constants called elastic moduli of the material or *Lamé moduli*. This derivation shows that the elastic properties of an arbitrary isotropic elastic material are defined by only two constants.

The coordinate form of the energy is

$$W_{\text{is}} = \mu \left[\frac{1}{4} \sum_{i,j}^{d} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 \right] + \frac{1}{2}\lambda \left[\sum_{i}^{d} \frac{\partial u_i}{\partial x_i} \right]^2 \tag{9}$$

Notice that natural assumptions on the energy form based on expected linearity of the stationary condition and the symmetry principles, lead to a unique form of the energy of a linear isotropic elastic material.

Remark 3.2 Two elastic constants correspond to two different response of an isotropic material. The resistance to the volume change may be different than the resistance to any deformation that leaves the volume constant. For example, a gas has a finite resistance (pressure) to the volume change, but zero resistance to an orthogonal deformation.

Notice, that a gradient ∇u of a scalar u has one rotational invariant, the Euclidian norm of this vector $|\nabla u|$. Correspondingly, the Lagrangian for the conductivity of any isotropic material depends only on $|\nabla u|$ and u, $F = F(x, u, |\nabla u|)$.

A symmetric 3×3 matrix ϵ has 3 invariants, the eigenvalues or the invariants

$$\text{Tr}\,\epsilon, \quad \text{Tr}\,\epsilon^2, \quad \text{Tr}\,\epsilon^3,$$

therefore the Lagrangian for an isotropic material can depend on them. The quadratic Lagrangian is independent of cubic term $\text{Tr}\,\epsilon^3$ and has the form (8).

3.2 Lamé equations

Let us find the stationarity conditions for the energy minimization. We consider domain Ω filled with the linear isotropic elastic material. Assume again that the displacements u are fixed on a component $\partial\Omega_1$ of the boundary, and the boundary force f is applied on the other boundary component $\partial\Omega_2$. The corresponding variational problem is

$$\min_{u \in \mathcal{U}} \int_{\Omega} W_{\text{is}}(\epsilon) dx + \int_{\partial\Omega_2} f \cdot u\, ds, \quad \mathcal{U} = \{u: \ u|_{\partial\Omega_1} = u_0\} \tag{10}$$

Here, the second term represents the *work of boundary force* $f(s)$ on the boundary displacement $u(s)$.

We introduce a matrix $\sigma = \frac{\partial F}{\partial \epsilon}$ with the entries

$$\sigma_{ij} = \frac{\partial W_{\text{is}}}{\partial \epsilon_{ij}}$$

called the *stress* tensor. Notice that σ is a symmetric matrix

$$\sigma = \sigma^T$$

because the energy depends only on the symmetric part ϵ of ∇u.

(CHECK COEFF 2μ is correct!) The Euler-Lagrange equation can be written as a system

$$\nabla \cdot \sigma = 0, \quad \sigma = \sigma^T \tag{11}$$

$$\sigma = \frac{\partial W}{\partial \epsilon} = 2\mu\epsilon + \lambda \operatorname{Tr} \epsilon I \quad \text{in } \Omega \tag{12}$$

where I is the unit matrix. In elasticity, equations (10) are called equilibrium conditions, and equations (12) – the generalized Hook's law or the constitutive relations. Their coordinate forms are, respectively,

$$\sum_i^d \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \; j = 1, \ldots d, \quad \sigma_{ij} = \sigma_{ji},$$

– equilibrium conditions, and

$$\sigma_{ij} = 2\mu\,\epsilon_{ij}, \quad i \neq j, \quad i, j = 1, \ldots d,$$
$$\sigma_{ii} = 2\mu\,\epsilon_{ii} + \lambda \sum_k \epsilon_{kk}, \quad i = 1, \ldots d,$$

– Hooke's law.

The elasticity equations can be rewritten as a system of second-order equations called *Lamé equations*

$$\frac{\partial}{\partial x_i} \left[(\lambda + 2\mu)\frac{\partial u_i}{\partial x_i} + 2\mu \sum_{j \neq i}^3 \frac{\partial u_i}{\partial x_j} \right] = 0, \quad i = 1, 2, 3.$$

for the displacement vector u.

The natural boundary conditions are

$$\sigma \cdot n = f \quad \text{or } \sigma_{nn} = f_n, \; \sigma_{nt} = f_t, \quad \text{on } \partial\Omega_2 \tag{13}$$

where the boundary force $f = (f_n, f_t)$ is applied on the other boundary component $\partial\Omega_2$, n and t is the normal and tangent, respectively.

Recall that by assumption, a main boundary condition is given on $\partial\Omega_1$,

$$u = u_0 \quad \text{on } \partial\Omega_1,$$

is prescribed on $\partial\Omega_1$, and the boundary value problem is complete. The main boundary conditions prescribe the displacement on $\partial\Omega_1$. Generally, they have the form $u(x) = u_0$ if $x \in \partial\Omega_1$. The natural boundary conditions

$$(\delta u)^T \sigma \cdot n = f$$

define a column (and, by the symmetry of σ, the raw) of the stress tensor.

Problem 3.1 Derive the elasticity equations and natural boundary conditions for a linear anisotropic elastic body.

4 Complex conductivity

In this part, we describe an inverse problem: Finding a Lagrangian from the given differential equations. We assume that the equations are Euler-Lagrange equations for a variational problem and recover its form.

4.1 Equations of Complex Conductivity

The Process Consider conductivity in a dissipative medium with inductance and capacity along with resistivity. The current j and the electric field e are now functions of time and space coordinates. The current is divergencefree, and the field is curlfree (see Chapter 2):

$$\nabla \cdot \boldsymbol{j} = 0, \quad \nabla \times \boldsymbol{e} = 0. \tag{14}$$

These constraints allow us to introduce a vector potential $\boldsymbol{a}$ of the current field $\boldsymbol{j}$ and a scalar potential ϕ of the electrical field $\boldsymbol{e}$ through the relations

$$\boldsymbol{j} = \nabla \times \boldsymbol{a}, \quad \boldsymbol{e} = -\nabla \phi. \tag{15}$$

Consider a body Ω occupied by a conducting material and suppose that this body is loaded on the boundary $S = \partial \Omega$. The boundary conditions are similar to those for a conducting material (see Chapter 4)

$$\phi = \phi_0 \text{ on } S_1, \quad \boldsymbol{n} \cdot \boldsymbol{j} = j_0 \text{ on } S_2, \quad S_1 \cup S_2 = S, \tag{16}$$

where $\boldsymbol{n}$ is the normal.

Assume that the properties of the material are local in space and in time: The current field and its derivatives at a point $\boldsymbol{x} \in \Omega$ at the moment t depend only on the electrical field and its derivatives at the same point at the same moment of time. Assume that the material is linear in the following sense: A linear combination of the current and its time derivatives linearly depends on a linear combination of the field and its time derivatives:

$$\sum_k a_k \frac{\partial^k \boldsymbol{j}}{\partial t^k} = \sum_k b_k \frac{\partial^k \boldsymbol{e}}{\partial t^k}. \tag{17}$$

Here $a_k = a_k(\boldsymbol{x})$ and $b_k = b_k(\boldsymbol{x})$ are some time-independent coefficients, which are scalars (for the isotropic conductors) or symmetric matrices (for the anisotropic ones). The properties of the material (i.e., the scalar or matrix parameters a_k, b_k) do not depend on time.

Monochromatic Excitation Consider steady-state oscillations in a dissipative medium caused by a monochromatic excitation. The electrical field and current in the material are also monochromatic, i.e.,

$$\boldsymbol{j}^s(\boldsymbol{x}, t) = (\boldsymbol{J}(\boldsymbol{x})e^{i\omega t})' = \boldsymbol{J}'(\boldsymbol{x}) \cos \omega t + \boldsymbol{J}''(\boldsymbol{x}) \sin \omega t,$$
$$\boldsymbol{e}^s(\boldsymbol{x}, t) = (\boldsymbol{E}(\boldsymbol{x})e^{i\omega t})' = \boldsymbol{E}'(\boldsymbol{x}) \cos \omega t + \boldsymbol{E}''(\boldsymbol{x}) \sin \omega t,$$

where $\Phi_0(s)$, $J_0(s)$, $\boldsymbol{J}(\boldsymbol{x})$, and $\boldsymbol{E}(\boldsymbol{x})$ are the complex-valued Fourier coefficients of corresponding functions, and s is the coordinate along the boundary. Here, the real and imaginary parts of variables are denoted by the superscripts $'$ and $''$, i.e., $\boldsymbol{c} = \boldsymbol{c}' + i\boldsymbol{c}''$.

The Complex-Valued Conductivity Equations The linearity of the constitutive relations (17) leads to a linear relationship between the vectors $\boldsymbol{J}(\boldsymbol{x})$ and $\boldsymbol{E}(\boldsymbol{x})$:

$$\boldsymbol{J} = \boldsymbol{\sigma}\boldsymbol{E}, \tag{18}$$

where $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\omega) = \boldsymbol{\sigma}'(\omega) + i\boldsymbol{\sigma}''(\omega)$ is a complex conductivity tensor that depends on the frequency of oscillations [**?**]. For an isotropic material with state law (17), the tensor $\boldsymbol{\sigma}$ is defined by

$$\boldsymbol{\sigma} = \frac{\sum_k (-i\omega)^k a_k}{\sum_k (-i\omega)^k b_k} I, \tag{19}$$

where I is a unit matrix.

The divergencefree nature of the current field and the curlfree nature of the electrical field means that the Fourier coefficients of these fields satisfy relations similar to (14)

$$\nabla \cdot \boldsymbol{J} = 0, \quad \nabla \times \boldsymbol{E} = 0. \tag{20}$$

Therefore, they allow the representation (see (15))

$$\boldsymbol{J} = \nabla \times \boldsymbol{A}, \quad \boldsymbol{E} = -\nabla\Phi, \tag{21}$$

where $\boldsymbol{A}$ and Φ are the Fourier coefficients of the potentials $\boldsymbol{a}$ and ϕ.

The boundary conditions (16) lead to the relations

$$\Phi = \Phi_0 \text{ on } S_1, \; \boldsymbol{n} \cdot \boldsymbol{J} = J_0 \text{ on } S_2, \; S_1 \cup S_2 = S, \tag{22}$$

where Φ_0 and J_0 are the Fourier coefficients of the functions ϕ_0 and j_0.

A harmonic oscillation in the conducting media is described by the constitutive relations (18) and differential equations (20), (21) in conjunction with the boundary conditions (22).

The System of Real First-Order Equations The complex-valued equations (18), (20), (21), and (22) describe the conductance of the medium. They look exactly like the equations for the real conductivity; however, they correspond to more complicated processes. Indeed, the complex-valued differential equations (20) and (21) form a fourth-order system of differential equations for the real and imaginary parts of the variables $\boldsymbol{J}'$ and $\boldsymbol{E}$,

$$\nabla \cdot \boldsymbol{J}' = 0, \quad \nabla \cdot \boldsymbol{J}'' = 0, \quad \nabla \times \boldsymbol{E}' = 0, \quad \nabla \times \boldsymbol{E}'' = 0. \tag{23}$$

These equations are identically satisfied if the following potentials are introduced:

$$\boldsymbol{J}' = \nabla \times \boldsymbol{A}', \quad \boldsymbol{J}'' = \nabla \times \boldsymbol{A}'', \quad \boldsymbol{E}' = -\nabla\Phi', \quad \boldsymbol{E}'' = -\nabla\Phi''. \tag{24}$$

The currents and electrical fields are connected by the constitutive relations (18)

$$
\begin{aligned}
-\boldsymbol{J}' &= -\boldsymbol{\sigma}'\boldsymbol{E}' + \boldsymbol{\sigma}''\boldsymbol{E}'', \\
\boldsymbol{J}'' &= \boldsymbol{\sigma}''\boldsymbol{E}' + \boldsymbol{\sigma}'\boldsymbol{E}''.
\end{aligned}
\tag{25}
$$

The vector form of the last equations is

$$
\begin{pmatrix} -\boldsymbol{J}' \\ \boldsymbol{J}'' \end{pmatrix} = \boldsymbol{D}_{EE} \begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{E}'' \end{pmatrix},
\tag{26}
$$

where

$$
\boldsymbol{D}_{EE} = \begin{pmatrix} -\boldsymbol{\sigma}' & \boldsymbol{\sigma}'' \\ \boldsymbol{\sigma}'' & \boldsymbol{\sigma}' \end{pmatrix}
\tag{27}
$$

is the conductivity matrix of the medium.

The boundary conditions (22) can be rewritten as

$$
\begin{aligned}
\Phi' &= \Phi'_0 \text{ on } S_1, & (28) \\
\Phi'' &= \Phi''_0 \text{ on } S_1, & (29) \\
\boldsymbol{n} \cdot \boldsymbol{J}' &= J'_0 \text{ on } S_2, & (30) \\
\boldsymbol{n} \cdot \boldsymbol{J}'' &= J''_0 \text{ on } S_2. & (31)
\end{aligned}
$$

The formulated system of the real-valued differential equations and boundary conditions describes the conductivity of the complex conducting medium. Notice that it has double dimensions compared to the real conductivity problem.

The conductivity is defined by two tensors $\boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}'$. The real part $\boldsymbol{\sigma}'$ is nonnegative,

$$
\boldsymbol{\sigma}' \geq 0,
\tag{32}
$$

because the dissipation rate is nonnegative. Indeed, the energy dissipation averaged over the period of oscillations is equal to:

$$
\frac{\omega}{2\pi} \int_t^{t+\frac{2\pi}{\omega}} \boldsymbol{j}^s \cdot \boldsymbol{e}^s dt = \frac{1}{2}(\boldsymbol{J}' \cdot \boldsymbol{E}' + \boldsymbol{J}'' \cdot \boldsymbol{E}'') = \frac{1}{2}(\boldsymbol{E}' \cdot \boldsymbol{\sigma}'\boldsymbol{E}' + \boldsymbol{E}'' \cdot \boldsymbol{\sigma}'\boldsymbol{E}'') \quad (33)
$$

(see [**?**]). The condition (32) expresses the positiveness of the dissipation rate.

Real Second-Order Equations   The system (24), (25) of four first-order differential equations can be rewritten as a system of two second-order equations. We do it in four different ways, and we end up with four equivalent systems. Each of them turns out to be Euler–Lagrange equations for a variational problem.

First, we express the fields though scalar potentials Φ' and Φ'' and take the divergence ($\nabla\cdot$) of the right- and left-hand sides of (25). The left-hand-side terms $\nabla \cdot \boldsymbol{j}', \nabla \cdot \boldsymbol{j}''$ vanish and we obtain:

$$
\begin{aligned}
0 &= \nabla \cdot [-\boldsymbol{\sigma}'\nabla\Phi' + \boldsymbol{\sigma}''\nabla\Phi''], \\
0 &= \nabla \cdot [\boldsymbol{\sigma}''\nabla\Phi' + \boldsymbol{\sigma}'\nabla\Phi''].
\end{aligned}
$$

Thus we obtain two second-order equations for two potentials Φ' and Φ''. The vector form of this system is

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \nabla \cdot & 0 \\ 0 & \nabla \cdot \end{pmatrix} \boldsymbol{D}_{EE} \begin{pmatrix} \nabla \Phi' \\ \nabla \Phi'' \end{pmatrix}. \tag{34}$$

We may also rewrite this system of equations taking any other pair of four scalar and vector potentials (24) and excluding the other two. For example, let us exclude the fields $\boldsymbol{E}'$ and $\boldsymbol{E}''$. First, we solve equations (25) for $\boldsymbol{E}'$ and $\boldsymbol{E}''$:

$$\begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{E}'' \end{pmatrix} = \boldsymbol{D}_{JJ} \begin{pmatrix} -\boldsymbol{J}' \\ \boldsymbol{J}'' \end{pmatrix}, \tag{35}$$

where

$$\boldsymbol{D}_{JJ} = \begin{pmatrix} -(\boldsymbol{\sigma}' + \boldsymbol{\sigma}''\boldsymbol{\sigma}'^{-1}\boldsymbol{\sigma}'')^{-1} & (\boldsymbol{\sigma}'' + \boldsymbol{\sigma}'\boldsymbol{\sigma}''^{-1}\boldsymbol{\sigma}')^{-1} \\ (\boldsymbol{\sigma}'' + \boldsymbol{\sigma}'\boldsymbol{\sigma}''^{-1}\boldsymbol{\sigma}')^{-1} & (\boldsymbol{\sigma}' + \boldsymbol{\sigma}''\boldsymbol{\sigma}'^{-1}\boldsymbol{\sigma}'')^{-1} \end{pmatrix}. \tag{36}$$

(Note that $\boldsymbol{D}_{EE} = \boldsymbol{D}_{JJ}^{-1}$.)

Take the curl ($\nabla \times$) of the right- and left-hand sides of both equations (35). The left-hand-side terms identically vanish, and we obtain two vector equations:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \nabla \times & 0 \\ 0 & \nabla \times \end{pmatrix} \boldsymbol{D}_{JJ} \begin{pmatrix} -\nabla \times \boldsymbol{A}' \\ \nabla \times \boldsymbol{A}'' \end{pmatrix}. \tag{37}$$

Here we use the representation (24) of current fields $\boldsymbol{J}'$ and $\boldsymbol{J}''$ through the vector potentials $\boldsymbol{A}'$ and $\boldsymbol{A}''$.

We may as well solve (25) for the fields $\boldsymbol{E}'$ and $\boldsymbol{J}''$ and obtain

$$\begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{J}' \end{pmatrix} = \boldsymbol{D}_{JE} \begin{pmatrix} \boldsymbol{J}'' \\ \boldsymbol{E}'' \end{pmatrix}, \tag{38}$$

where

$$\boldsymbol{D}_{JE} = \begin{pmatrix} (\boldsymbol{\sigma}')^{-1} & (\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}'' \\ \boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1} & \boldsymbol{\sigma}' + \boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}'' \end{pmatrix}. \tag{39}$$

Recall that $\boldsymbol{E}'$ is curlfree and $\boldsymbol{J}''$ is divergencefree. Therefore, by using (23) and (24) we arrive at the following system of second-order equations:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \nabla \times & 0 \\ 0 & \nabla \cdot \end{pmatrix} \boldsymbol{D}_{JE} \begin{pmatrix} \nabla \times \boldsymbol{A}' \\ \nabla \Phi'' \end{pmatrix}. \tag{40}$$

Similarly, we solve (25) for $\boldsymbol{J}'$ and $\boldsymbol{E}''$ and obtain

$$\begin{pmatrix} \boldsymbol{J}' \\ \boldsymbol{E}'' \end{pmatrix} = \boldsymbol{D}_{EJ} \begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{J}'' \end{pmatrix}, \tag{41}$$

where

$$\boldsymbol{D}_{EJ} = \begin{pmatrix} \boldsymbol{\sigma}' + \boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}'' & -(\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}'' \\ -\boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1} & (\boldsymbol{\sigma}')^{-1} \end{pmatrix}. \tag{42}$$

(Note that $\boldsymbol{D}_{JE}^{-1} = \boldsymbol{D}_{EJ}$.)

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Again, the operations $(\nabla\cdot)$ and $(\nabla\times)$ eliminate the corresponding terms on the left-hand side in equations (41). Applying these operators, we obtain the second-order system

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \nabla\cdot & 0 \\ 0 & \nabla\times \end{pmatrix} \boldsymbol{D}_{JE} \begin{pmatrix} \nabla\Phi' \\ \nabla\times\boldsymbol{A}'' \end{pmatrix}. \tag{43}$$

We have written four different forms of the same equations. The systems (34), (37), (40), and (43) are equivalent to each other and to the original system (23). Each of them in conjunction with the boundary conditions (28)–(31) allows us to find the solution that describes the processes in the conducting medium. We now show that each of them represents the Euler equations for a corresponding variational problem.

4.2 Quartet of variational principles

Let us establish variational principles for the problem of complex conductivity. There is no direct complex analogue to the variational principles for the real-valued problem because the inequalities cannot be considered for complex variables. However, the real-valued differential equations just described are the stationary conditions for some real-valued functionals. These functionals lead to variational principles that describe the complex conductivity processes.

First, we formulate two minimax variational principles. They follow naturally from the equations in the form (34) and (37). Then we obtain two minimal variational principles based on the equations of the problem in the form (40) and (43). Finally, we discuss the relation between these four principles, referring to the procedure of Legendre transform.

The Minimax Variational Principle for the Fields   Consider the following variational minimax problem:

$$\min_{\boldsymbol{E}''} \max_{\boldsymbol{E}'} U_{EE}, \tag{44}$$

where the fields $\boldsymbol{E}'$, $\boldsymbol{E}''$ are subject to the constraints

$$\begin{aligned} \boldsymbol{E}'' &= -\nabla\Phi'', & \Phi'' &= \Phi_0'' \text{ on } S_1, \\ \boldsymbol{E}' &= -\nabla\Phi', & \Phi' &= \Phi_0' \text{ on } S_1; \end{aligned}$$

the functional U_{EE} is

$$U_{EE} = \int_\Omega W_{EE}(\boldsymbol{E}', \boldsymbol{E}'') + \int_{S_2} [\Phi'' J_0'' - \Phi' J_0']; \tag{45}$$

and

$$W_{EE}(\boldsymbol{E}', \boldsymbol{E}'') = \frac{1}{2} \begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{E}'' \end{pmatrix}^T \boldsymbol{D}_{EE} \begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{E}'' \end{pmatrix}. \tag{46}$$

The matrix $\boldsymbol{D}_{EE}$ is defined in (27).

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The vanishing of the first variation with respect to $\boldsymbol{E'}$, $\boldsymbol{E''}$ of the functional U_{EE} (see (45)) leads to two Euler–Lagrange equations that coincide with (25). One can check that they coincide with the original system of equations in the form (34) and with the boundary conditions (30), (31). The boundary conditions (28), (29) must be assumed at all admissible fields.

To check the sense of optimality of the stationary solution we examine the sign of the second variation of the functional; see, for example, [?]. The second variation is the main term of the increment of the functional at the perturbed solution of the Euler–Lagrange equation. Whereas the first variation is zero at the solution, the second variation of the cost is proportional to the quadratic form

$$(\delta\boldsymbol{E}, \delta\boldsymbol{E''})^T \boldsymbol{D}_{EE}(\delta\boldsymbol{E}, \delta\boldsymbol{E''}).$$

The functional has a local minimum at the stationary solution if the second variation is positive, and it has a local maximum at the stationary solution if the second variation is negative. The sign of the variation is determined by the matrix $\boldsymbol{D}_{EE}$.

Here the second variation is neither positive nor negative, because the matrix $\boldsymbol{D}_{EE}$ is neither positive nor negative definite. The stationary solution corresponds to the saddle point of the functional. The variational problem is of the minimax type.

The Minimax Variational Principle for the Currents   Similarly, one can derive the Euler–Lagrange equations of the variational problem

$$\max_{\boldsymbol{J'}} \min_{\boldsymbol{J''}} \; U_{JJ}, \tag{47}$$

where the fields $\boldsymbol{J'}, \boldsymbol{J''}$ are

$$\{\boldsymbol{J'} : \; \boldsymbol{J'} = \nabla \times \boldsymbol{A'}, \quad \boldsymbol{n} \cdot \boldsymbol{J'} = J_0' \text{ on } S_2\},$$
$$\{\boldsymbol{J''} : \; \boldsymbol{J''} = \nabla \times \boldsymbol{A}, \quad \boldsymbol{n} \cdot \boldsymbol{J''} = J_0'' \text{ on } S_2\};$$

the functional U_{JJ} is

$$U_{JJ} = \int_\Omega W_{JJ}(\boldsymbol{J'}, \; \boldsymbol{J''}) + \int_{S_1} [\Phi_0'' \, \boldsymbol{n} \cdot \boldsymbol{J''} - \Phi_0' \, \boldsymbol{n} \cdot \boldsymbol{J'}]; \tag{48}$$

and

$$W_{JJ}(\boldsymbol{J'}, \; \boldsymbol{J''}) = \frac{1}{2} \left(\begin{array}{c} -\boldsymbol{J'} \\ \boldsymbol{J''} \end{array} \right)^T \boldsymbol{D}_{JJ} \left(\begin{array}{c} -\boldsymbol{J'} \\ \boldsymbol{J''} \end{array} \right). \tag{49}$$

The matrix $\boldsymbol{D}_{JJ}$ is defined by (36).

We check that the Euler equations for the functional (48) coincide with equations (35) that describe the same problem in different notation.

The matrix $\boldsymbol{D}_{JJ}$ is neither positive nor negative definite, hence the second variation of the functional U_{JJ} is again neither positive nor negative. We conclude that the variational problem (48) is of the minimax type.

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Remark 4.1 The minimax nature of the variational principles (44) and (47) does not allow us to apply the technique developed to the bounds. This technique uses the fact that the energy (i.e., the value of the functional) on any trial field should exceed the actual energy stored in the material. Therefore the energy on any trial field provides an upper bound on the actual energy. For the minimax principles (44), (47), however, the situation is different. Consider, for example, the problem (44) and let us calculate the energy on trial fields of two potentials Φ' and Φ''. The actual energy is increased if the trial field $\nabla\Phi''$ differs from the optimal one and is decreased if the other trial field $\nabla\Phi'$ is not optimal. The value of the functional (45) on the trial fields can be lower or higher than the actual energy and cannot bound the functional (45) from either side.

The First Minimal Variational Principle Consider the following variational problem for the variables $\boldsymbol{J}'$ and $\boldsymbol{E}''$:

$$\min_{\boldsymbol{J}'} \min_{\boldsymbol{E}''} U_{JE},\tag{50}$$

where the fields $\boldsymbol{J}', \boldsymbol{E}''$ are

$$\{\boldsymbol{J}': \ \boldsymbol{J}' = \nabla \times \boldsymbol{A}', \quad \boldsymbol{n} \cdot \boldsymbol{J}' = J'_0 \text{ on } S_2\},$$
$$\{\boldsymbol{E}'': \ \boldsymbol{E}'' = -\nabla\Phi'', \qquad \Phi'' = \Phi''_0 \text{ on } S_1\};$$

the functional U_{JE} is

$$U_{JE} = \int_\Omega W_{JE}(\boldsymbol{J}', \boldsymbol{E}'') - \int_{S_1} \boldsymbol{n} \cdot \boldsymbol{J}'\Phi'_0 + \int_{S_2} \Phi'' J''_0;\tag{51}$$

and

$$W_{JE}(\boldsymbol{J}', \boldsymbol{E}'') = \frac{1}{2}\begin{pmatrix}\boldsymbol{J}'\\\boldsymbol{E}''\end{pmatrix}^T \boldsymbol{D}_{JE}\begin{pmatrix}\boldsymbol{J}'\\\boldsymbol{E}''\end{pmatrix}.\tag{52}$$

The matrix $\boldsymbol{D}_{JE}$ is defined in (55). The first variation of (50) with respect to $\boldsymbol{J}'$ and $\boldsymbol{E}''$ coincides with the system of original equations in the form (40) and the boundary conditions (29), (31).

Note that this time the quadratic form (52) is positive. This follows from the physically clear condition (32). As we see, this functional is equal to the whole energy dissipated in the body Ω during one period of oscillation (see (33)).

The second variation $\delta^2 U_{JE}$ of the functional (51) is positive due to the positivity of the matrix $\boldsymbol{D}_{JE}$ (for physical reasons we always suppose that $\boldsymbol{\sigma}' \geq 0$ or the dissipation rate is positive). For the quadratic functional (51) the positivity of the second variation is sufficient to guarantee the global minimum at a stationary point [**?**].

The Second Minimal Variational Principle Similarly, we consider the variational problem

$$\min_{\boldsymbol{J}''} \min_{\boldsymbol{E}'} U_{EJ},\tag{53}$$

where the fields $\boldsymbol{J}'', \boldsymbol{E}'$ are

$$\{\boldsymbol{J}'' : \ \boldsymbol{J}'' = \nabla \times \boldsymbol{A}'', \quad \boldsymbol{n} \cdot \boldsymbol{J}'' = J_0'' \text{ on } S_2\},$$
$$\{\boldsymbol{E}' : \ \boldsymbol{E}' = -\nabla \Phi', \qquad \Phi' = \Phi_0' \text{ on } S_1\};$$

the functional U_{EJ} is:

$$U_{EJ} = \int_\Omega W_{EJ}(\boldsymbol{E}', \boldsymbol{J}'') + \int_{S_1} \boldsymbol{n} \cdot \boldsymbol{J}'' \Phi_0'' - \int_{S_2} \Phi' J_0'; \tag{54}$$

and

$$W_{EJ}(\boldsymbol{E}', \boldsymbol{J}'') = \frac{1}{2} \begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{J}'' \end{pmatrix}^T \boldsymbol{D}_{EJ} \begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{J}'' \end{pmatrix}. \tag{55}$$

The matrix $\boldsymbol{D}_{EJ}$ is defined in (**??**).

In considering the first variation of the functional (53), we conclude again that the Euler equations for the functional (54) coincide with the system of original equations in the form (43) and the boundary conditions (28), (30).

One could also see that the second variation of this functional is positive if $\boldsymbol{\sigma}' \geq 0$.

Remark 4.2 Note that the two variational principles are equivalent:

$$W_{JE}(\boldsymbol{J}, \boldsymbol{E}) = W_{EJ}(\boldsymbol{E}, \boldsymbol{J}), \tag{56}$$

This feature is specific for this problem; usually we meet two different variational principles of minimization of the potential energy and the complementary energy (for example, the Dirichlet and Thomson principles).

5 Variational principles for nonconservative systems