WAITING ELEMENT STRUCTURES AND STABILITY UNDER EXTENSION

by Andrej Cherkaev
Department of Mathematics, The University of Utah,
Salt Lake City, UT 84112, U.S.A.
and
Leonid Slepyan
Department of Engineering, Tel Aviv University,
Ramat Aviv 69978 Tel Aviv, Israel

April 27, 1995

Contents

Abstract

In this paper the following questions are considered: What are the phenomena which limit the total fracture energy of a structure under the extension before is breaks? When is the limiting energy level more important than the stress limit? What are possible ways to increase the required fracture energy of a sample before it breaks? It is shown that the required features of a material or of a construction can be achieved by using special structures of ordinary elements. The possibilities are discussed for increasing the fracture energy density in a sample, and increasing the total fracture energy in a construction. The dynamic process of damaging discussed as well.

1 Preliminaries

1.1 The focus

The total fracture energy is the main factor in the resistance of a construction to a dynamic action such as a collision, an earthquake, or an explosion. In these situations a designer is to build a construction that accumulate as much energy as possible before it is broken. This is a well known fact, and many papers have been devoted to investigation of the dynamic resistivity (see, for instance, Chou, Tsu-Wei, 1992, and Jones, N. and Wierzbicki, T. (Ed.), 1993).

However, the main attention has been paid to the dynamic compression action, as one can see in these books. Here we want to concentrate on the extension loading which often plays the main role in dynamic processes. Such loadings are much more unstable then compression. Indeed, the extension of a sample leads to the descreasing of its thickness and to decreasing of the ability to carry the load which is the cause of instability.

The construction resistivity depends on its structure and on the materials it is made of. The material features responsible for the resistivity include a stress and strain limits and the limiting strain energy of the material. The limiting strain energy may be considered as the maximal energy stored in a sample before in breaks down; roughly it can be approximated by the product of the limiting stress and the limiting strain. The stresses are limited by the so-called theoretical strength. The theoretical strength decreases by defects of the material structure and by plasticity of the material. On the other hand, the strain limit is much greater for the plastic materials, especially for so-called superplastic ones (see, for instance, Poluchin, Gorelik and Vorontsov, 1983). However, for these materials, the strain limit strongly depends on conditions of the extension, in particular, on the temperature. For an adiabatic extension, the energy needed for melting down the material may be considered as the strain energy limit.

The mentioned local characteristics of material give some natural bounds of the ability of construction to resist the dynamic loading. However, in practice the destroying of construction is observed long before the limits of material resistance is achieved. A real limiting mechanism is *the instability* of the strains which manifests as the localization of strains. Therefore the obvious way to significantly increase the quality of construction is to stabilize, somehow, the process of damaging. This means that one should increase the energy density consumed in the material deformation and to distribute the damage (the large strains) throughout the large part of the construction.

The process of localization (instability) of strains occurs in different levels. These levels could be illustrated by comparing the process of the strain localization in a small laboratory sample and in a large construction. In the first case, the concentration is a consequence of the instability of the process of material extension under uniformly distributed stresses. The second case usually corresponds to highly non-uniform distribution of stresses throughout the construction. As a result of it, only a small part of the construction is involved in large deformations, and the total fracture energy used before breaking of a construction turns out to be very small in comparison with the energy limit of the construction. These two instabilities could be improved by assining different requirements to the material and to the material distribution. However, as it is shown in this paper, the more stable material is used the higher the limiting strain energy becomes in both the sample and the real construction. The stabilization of strains can be achieved by manufacturing of some non-traditional materials into a special structure, as it is shown here.

The stabilization of the fracture energy distribution may be accomplished due to a proper improvement in the stress - strain response of the material or construction. We discuss here two methods to achieve such an improvement.
(i) The first way is to use a special structure that contains the stabilizing non-active 'waiting' parallel elements. They are involved in the strain process only after the initial strain is large enough. Therefore the stiffness of the construction increases in the range of large strains, which prevents the localization of these strains and helps to distribute strains more uniformly. The action of such elements is described in the paper.
(ii) The second way is to transform the extension strain on the 'macro-level' to more stable types of strains on the 'micro-level'. The plastic helix considered below is an example of an element which performs such transformation.

The additional energy modes are involved in the process of damaging such as energy of the micro-level (high frequence) oscillations which are generated under the dynamic extension of a material of a special structure. In a sense, the phenomenon is similar to the heat energy in a shock wave. This 'dynamic' energy (mainly, the kinetic energy) is especially important for materials with high-strength brittle fibres which cannot absorb much energy in strain.

The effective surface energy creation by crack propagation is to be noted as an important factor in the total energy of fracture formation. On the one hand, the greater this energy is the less brittle is the material, and hence more limiting strain energy may be expected. On the other hand, this energy is a significant part of the total accumulated energy especially in the case of failure due to many micro-cracks; it could be viewed then as a stabilizing factor. The condition can be created to increase the influence of the process of stabilization by means of a special material structure, but we do not discuss the details of those conditions here.

In the next two section we describe some examples that provide some insight into the main concept.

1.2 What is more important, the limiting strain energy or the strength?

Here we will concentrate on the ability of a construction to resist an expension. We want to show that the ability to consume a high level of energy in tension turns out to be very important in many practical situations; such as a moving tank with a fluid under collision with a stationary wall, or a protective structure under a point impact.

Consider, as an example, a cylindrical tank with a liquid of density ρ and sound velocity c under a longitudinal collision with a rigid unmoving plate (Fig. 1). Let v be the velocity of the impact. We assume that $v << c$. To answer the above question let us make simple estimates of two limiting cases.

In the first one the tank is assumed to be rigid. Its role is only to prevent a radial strain of the liquid, and the mass of the tank is neglected. In this case, the liquid becomes the same as a simple elastic beam with zero Poisson's ratio. Let the velocity be $v = v_1$. The collision time is $2L/c$, where L be a length of the tank. After the collision, the tank jumps back with the same speed. Under these assumptions, during the collision time the pressure p in the pressure wave in the liquid is equal to ρvc, and the corresponding normal stresses in a longitudinal section of the shell are equal to

$$\sigma_1 = \frac{pr}{h} = \frac{\rho v_1 cr}{h} \tag{1}$$

where h is the shell thickness.

In the second case, the tank is assumed to be extensible so that all the kinetic energy of the liquid is absorbed as the strain energy of the shell of the tank. Assume, for a rough estimation, that the strain is uniformly distributed over the cylindrical shell of the tank. Assume also that the material is rigid-plastic, and the radius of the tank increases under a constant stress $\sigma = \sigma_2$. Under the collision velocity $v = v_2$, one has the energy relation

$$\frac{\pi r^2 L \rho v_2^2}{2} = 2\pi r L h \sigma_2 \epsilon \tag{2}$$

Here and below $\epsilon = \ln(l/l_0)$, l is the material element length, and l_0 is the initial length.

Limiting velocities of the impact are determined by the limiting stresses. Let us compare these velocities. One can find from (1) and (2) the ratio

$$\frac{v_2}{v_1} = 4\frac{c}{v_2}\frac{\sigma_2}{\sigma_1}\epsilon \tag{3}$$

One can see that the ratio becomes very large for a moderate velocity of the impact, $v_2 << c$, if the limiting strain ϵ, and the ratio σ_2/σ_1 are not too small. It means that the shell which can absorb the kinetic energy by a elongation under a internal pressure has much better resistance than the rigid shell. The latter cannot absorb energy, and, because of the purely elastic impact, it can bear an impact of a comparatively low velocity. Thus, for a moderate velocity of the impact, the level of the total fracture energy under extension is more important than the strength as the limiting stress. It is important to emphasize that this conclusion is based on the assumption that the strains are distributed uniformly, and the problem remains how to prevent the localization of the strain.

1.3 What are the limits of the energy accumulation in quasi-static and dynamic extension ?

We consider first a sample under quasi-static extension. We assume that the strain is distributed uniformly, the material is incompressible, and the tensile stresses σ, are constant. The incompressibility means that the length L and the cross-section area F of the sample are connected by the relation $LF = L_0 F_0$, where L_0 and F_0 are the initial values, and the work of extension per unit volume may be written as

$$A = \frac{1}{L_0 F_0} \int_{L_0}^{\infty} \sigma F \, dL = \sigma \int_{L_0}^{\infty} \frac{dL}{L} = \infty \tag{4}$$

However, this dramatic result cannot be achieved: the tension force decreases from the very beginning of the extension, thanks to the decrease of the cross-section area, and this means that the uniform extension of the sample is unstable. As a result of this instability the process of necking begins immediately with the extension of the sample.

The stress - strain diagram of a material normally satisfies the stability condition $d(\sigma F)/dL > 0$ (Kachanov, 1974) only for small strains. It means that one may not expect a reasonable hardening of the material, so we can presume for these estimations that the stresses are constant. From this, a more realistic limit can be found by assuming that the extension force is on the verge of instability, i.e. it is a constant as well as the stress. These assumptions are reasonable when the sample is under broaching (Fig. 2). In this case, we face a deformation of the system but not the strain throughout the full sample.

The accumulated energy is determined by the limiting stresses and by volume of the material involved in the broaching. If the whole sample is under a broaching then the energy density limit turns out to be equal to

$$A = \sigma \tag{5}$$

A dynamic system considered below shows similar behavior and gives similar results. Let us assume that a body of mass m is exposed under flexible rigid bonds as it is shown on Fig. 3. Being exposed under horizontal forces P =const, the body would acquire kinetic energy, which maximum value corresponds to the horizontal position of the bonds. The maximum work of the bonds which goes into the kinetic energy is

$$A = 2(\sqrt{L^2 + u^2} - L)P \tag{6}$$

where L is the length of the bond and u is the initial lateral coordinate of the mass (Fig. 3). Hence the energy of the bonds per unit volume is equal to

$$A = \sigma \left(1 - \frac{L}{\sqrt{L^2 + u^2}} \right) \to \sigma \quad (\frac{u}{L} \to \infty) \tag{7}$$

where σ is the above tension stress. It may be pointed that the needed mass for such a process is actually the mass of the bonds and not the mass of an additional material.

In this example, the limiting stress is important in the limiting strain energy density. However, the strain energy limit is usually far below the stress limit that is either the strength limit or the plastic limit. The problem of improving the construction become the problem of making the limit of the averaged energy density closer to the stress limit. This is again the problem of increasing stability of the system.

2 Stress – strain diagram and stability conditions

2.1 The less resistance the more limiting strain energy

Here we want to show how the weakening of a material can actually increase its resistivity which is associated with the fracture energy consumed. Here and below we use the definition of stresses $\tilde{\sigma}$ as the tension force per unit of the *initial* cross-section area, and corresponding elongation $\tilde{\epsilon} = l - l_0$, as well as the above introduced true stresses σ as the force per unit cross-section area of the deformed body, and logarithmic strain ϵ as $\epsilon = \ln(l/l_0)$. The first two definitions give us an expression of the strain work per unit volume of the material:

$$\frac{1}{l_0}\int_0^{\tilde{\epsilon}} \tilde{\sigma}\ d\tilde{\epsilon}$$

and the last two give us the strain work per unit volume for an inextensible material as

$$\frac{1}{l_0}\int_0^{\tilde{\epsilon}} \frac{F}{F_0}\sigma\ d\ ep\tilde{s}ilon = \int_0^{\epsilon} \sigma\ d\epsilon \quad (Fl = F_0 l_0)$$

We assume that $l_0 = 1$.

The stability condition can be written as

$$\frac{d\tilde{\sigma}}{d\tilde{\epsilon}} > 0 \ \text{ or } \ \frac{d(\sigma F)}{d\epsilon} > 0.$$

Let us consider a material with a piecewise linear stress – strain diagram (Fig. 4):

$$\tilde{\sigma}(\tilde{\epsilon}) = E\tilde{\epsilon} \ \text{ if } \ \tilde{\epsilon} \leq \tilde{\epsilon}_1$$

$$\tilde{\sigma}(\tilde{\epsilon}) = \tilde{\sigma}_1 - E_t(\tilde{\epsilon} - \tilde{\epsilon}_1) \text{ if } \ \tilde{\epsilon}_1 \leq \tilde{\epsilon} \leq \tilde{\epsilon}_*, \tag{8}$$

where $\tilde{\sigma}_1 = E\tilde{\epsilon}_1$, $\tilde{\sigma}(\tilde{\epsilon}) = 0$ if $(\tilde{\epsilon} \geq \tilde{\epsilon}_*)$, E is the elastic modulus, $-E_t$ is the tangent modulus for the region $\tilde{\epsilon}_1 < \tilde{\epsilon} < \tilde{\epsilon}_*$.

This diagram has two branches: the stable one ($\tilde{\epsilon} < \tilde{\epsilon}_1$) and the unstable one ($\tilde{\epsilon} > \tilde{\epsilon}_1$). The real fracture energy density corresponds to the stable branch. It is equal to

$$A_1 = \frac{E\tilde{\epsilon}_1^2}{2} \tag{9}$$

Let suppose that $E_t < E$. Consider a new, more stable diagram which differs from the first one by a decrease of the stresses in an initial portion of the strain (Fig. 4).

$$\tilde{\sigma} = E_2\tilde{\epsilon} \ \ (\tilde{\epsilon} \leq \tilde{\epsilon}_2, \ E_2 < E) \tag{10}$$

where $\tilde{\epsilon}_2$ is defined by the intersection of this line with the unstable branch of the former diagram:

$$\tilde{\sigma}_1 - E_t(\tilde{\epsilon}_2 - \tilde{\epsilon}_1) = E_2\tilde{\epsilon}_2 \tag{11}$$

The stress – strain relation is assumed to be the same as above for $\tilde{\epsilon} \geq \tilde{\epsilon}_2$.

The fracture energy density is again the strain energy in the stable branch. It is equal to

$$A_2 = \frac{E_2\tilde{\epsilon}_2^2}{2} = \frac{E_2}{2}\left(\frac{\tilde{\sigma}_1 + E_t\tilde{\epsilon}_1}{E_2 + E_t}\right)^2 \tag{12}$$

The ratio between the energies of the two described materials is

$$\frac{A_2}{A_1} = \frac{E_2}{E}\left(\frac{E + E_t}{E_2 + E_t}\right)^2; \tag{13}$$

it increases when E_2 decreases until $E_2 = E_t$, and maximum ratio is

$$\left(\frac{A_2}{A_1}\right)_{max} = \frac{(E + E_t)^2}{4EE_t} \geq 1 \tag{14}$$

One can see that the statement in the title of this section is valid for the range $E_t < E_2 < E$. Thus, to increase the limiting strain energy density, it is useful to decrease the resistance to an extension in an initial range of strain. Clearly, an even greater effect can be achieved by increasing the resistance under a large strain at the cost of a decrease of the resistance at the beginning of the extension. This can be done by using 'waiting elements' as it is shown in section 3.

2.2 Visco-plastic material

Above, we have discussed the behavior of elastic-plastic material. At the same time, the known superplastic alloys are visco-plastic ones which are characterized by a stress-strain relation of the type

$$\frac{d\epsilon}{dt} = f(\epsilon, \sigma) \tag{15}$$

(see Poluchin, Gorelik and Vorontsov, 1983; stability conditions for plastic and visco-plastic materials are considered in the books by Kachanov, 1974, 1986).

Under a one-dimensional extension, the force P is found as $P = \sigma F$, where F is the cross-section area. An instability of a uniform distribution of visco-plastic strain may manifest itself as a necking or as a slip line. Let us consider the necking condition under the one-dimensional extension. Roughly, this condition may be formulated as 'The smaller the cross-section area, the bigger is its negative rate, $-dF/dt$'. For the exact definition, the relations are used:

$$\frac{\partial}{\partial F}\frac{d\ln F}{dt} > 0, \quad \text{for instability}, \quad \text{and} \quad \frac{\partial}{\partial F}\frac{d\ln F}{dt} < 0, \quad \text{for stability} \quad (P = \text{const}) \tag{16}$$

The incompressibility of material is assumed, when one have

$$Fl = \text{const}, \quad \ln F = -\epsilon + \text{const}$$

These relations lead to:

$$\frac{\partial}{\partial \epsilon}\frac{d\epsilon}{dt} > 0, \quad \text{for instability}, \quad \text{and} \quad \frac{\partial}{\partial \epsilon}\frac{d\epsilon}{dt} < 0, \quad \text{for stability} \quad (P = \text{const}) \tag{17}$$

Now the stability condition can be written as

$$\frac{\partial f}{\partial \epsilon} + \sigma\frac{\partial f}{\partial \sigma} < 0 \tag{18}$$

The second term of the above expression is positive for a positive stress. Therefore the stability against a necking is provided by the first term which is to be negative and have a large enough absolute value. One can see that this requirement looks similar to that discussed above for a plastic material: the stability is provided by the strain hardening. But in the case of the visco-plastic material, the hardening depends on the rate of strain. Of course, for stability it is enough to have sufficient hardening in the total range of strain. The material is globally stable in a given range of strain $0 < \epsilon < \epsilon_*$ if the maximal stress $\tilde{\sigma}_{max}$ is at the limiting strain ϵ_* which can be carried by the construction. Even if for some stresses, smaller then the maximal one, the above inequality is violated and the 'local instability' occurs, this instability does not lead to the fracture of a specimen. Indeed, the stress increases due to that instability until it become big enough to fit the stability condition again. It is important to note, that the unstable stress cannot reach $\tilde{\sigma}_{max}$, and that the corresponding strain remains bounded.

Note also that the pure shear loading, in contrast to the extension loading, does not lead to a decrease of the cross-section area. As a result, the second term in the stability condition inequality vanishes. This means that the first term of this inequality has to be only negative, and its absolute value can be arbitrary. Therefore the shear is a more stable type of strain than the extension. This conclusion is confirmed by the test results shown in section 4.

Let us come back to the equality (15). From this equality, one can obtain

$$\frac{d^2\epsilon}{dt^2} = \lambda\frac{d\epsilon}{dt}, \quad \lambda = \frac{\partial f}{\partial \epsilon} + \sigma\frac{\partial f}{\partial \sigma} \tag{19}$$

Assuming that $\lambda \approx \text{const}$ on a short time interval, and that the force P is constant one obtains the rate of the strain as an exponentially decreasing function of time if $\lambda < 0$, and vice versa, an exponentially increasing function $\epsilon(t)$ corresponds to the instability ($\lambda > 0$). These considerations can be applied to the two-dimensional extension as well.

Let us consider a plate of the thickness h under the tension forces $P_{1,2}$ with the constitutive equations

$$\frac{d\epsilon_{1,2}}{dt} = f_{1,2}(\epsilon_1,\ \epsilon_2,\ \sigma_1,\ \sigma_2) \tag{20}$$

where $\sigma_{1,2}$ are the normal stresses:

$$\sigma_1 = \frac{P_1}{hL_2} = \frac{P_1 L_1}{V},\ \sigma_2 = \frac{P_2}{hL_1} = \frac{P_2 L_2}{V}$$

and $L_{1,2}$ be the sizes of the plate in its plane, and $V = \text{const}$ be its volume. For the case of $P_{1,2} = \text{const}$, under a possible instability, one has

$$\frac{d^2\epsilon_1}{dt^2} = \lambda_{11}\frac{d\epsilon_1}{dt} + \lambda_{12}\frac{d\epsilon_2}{dt},\quad \frac{d^2\epsilon_2}{dt^2} = \lambda_{21}\frac{d\epsilon_1}{dt} + \lambda_{22}\frac{d\epsilon_2}{dt} \tag{21}$$

$$\lambda_{11} = \frac{\partial f_1}{\partial \epsilon_1} + \sigma_1\frac{\partial f_1}{\partial \sigma_1},\quad \lambda_{12} = \frac{\partial f_1}{\partial \epsilon_2} + \sigma_2\frac{\partial f_1}{\partial \sigma_2}$$

$$\lambda_{21} = \frac{\partial f_2}{\partial \epsilon_1} + \sigma_1\frac{\partial f_2}{\partial \sigma_1},\quad \lambda_{22} = \frac{\partial f_2}{\partial \epsilon_2} + \sigma_2\frac{\partial f_2}{\partial \sigma_2}$$

The power λ of the exponentially grown solution determines the stability. This parameter can be found from the equation one has the equations (21) as

$$\lambda = \frac{\lambda_{11} + \lambda_{22}}{2} \pm \sqrt{\frac{(\lambda_{11} + \lambda_{22})^2}{4} + \lambda_{12}\lambda_{21} - \lambda_{11}\lambda_{22}} \tag{22}$$

Now the stability conditions can be written as

$$\lambda_{11} + \lambda_{22} < 0,\quad \lambda_{11}\lambda_{22} > \lambda_{12}\lambda_{21} \tag{23}$$

3 The waiting elements of a construction

3.1 The model. Statics

Let us discuss again the typical stress – strain diagram (8) of a bar with the unit cross-section. The material with this diagram is characterized with the interval of stable deformation where the stress increases with the strain, and the unstable interval, where the stress decreases when the strain grows.

Under monotonic loading the material resists until the process is stable, the maximum stress is equal to $E\epsilon_1$, and the energy A_1 consumed by the stable mode of the material is defined by (9).

It is shown in subsection 2.1 that the fracture energy can be increased by the improvement of the stress – strain diagram [see (10) - (14)]. Now let us show how to obtain a similar result using a proper structure of the material. Let us cut the bar lengthwise into two bars with equal length. The first one has cross-section area Λ, and the second one has the cross-section area $1-\Lambda$. Now let us join these bars parallel to each other. Suppose that we have made the second bar a bit longer so that it does not resist the loading until the first bar is deformed by an enlargement ϵ_1. Suppose also that we choose ϵ_1 to be the limiting value of strain in the stable branch of the stress-strain diagram.

When the strain is small $\epsilon \leq 2\epsilon_1$ only the first bar resists the pulling, and $\sigma = \Lambda E\epsilon$. Then, when the strain becomes greater $\epsilon \geq 2\epsilon_1$, the second bar begins to resist as well and the total resistance of the system is equal to:

$$\sigma = \Lambda E\epsilon - [(E+E_t)\Lambda - E(1-\Lambda)](\epsilon - \epsilon_1)H(\epsilon - \epsilon_1) \tag{24}$$

where H is the Heviside function. This curve also has an unstable zone, but this time it is supported by a second 'hump' of the curve. In the unstable zone, the stress id redistributed between the first and the second bars (which also leads to the dynamic energy-consuming process we describe below).

It is expected that the system of two bars in the whole must satisfy the stability condition. Here we do not touch upon the design question: how to connect these elements together to prevent the instability of the first element on the range $\epsilon_1 < \epsilon < 2\epsilon_1$.

This time the material resists the loading until the stress does not decrease with the increasing of the strain. We calculate the derivative $d\sigma/d\epsilon$ as

$$\frac{d\sigma}{d\epsilon} = \Lambda E \;(\epsilon < \epsilon_1) \quad \frac{d\sigma}{d\epsilon} = (1-\Lambda)E - \Lambda E_t \quad (\epsilon_1 < \epsilon < 2\epsilon_1) \tag{25}$$

Let us calculate the maximum of the consumed energy. Assume that, in the limiting case,

$$d\sigma/d\epsilon = 0 \;(\epsilon > \epsilon_1) \tag{26}$$

That gives us

$$\Lambda = \frac{E}{E+E_t}, \quad \sigma = \frac{E^2}{E+E_t}[\epsilon - (\epsilon-\epsilon_1)H(\epsilon-\epsilon_1)] \;\; (\epsilon \leq 2\epsilon_1) \tag{27}$$

Definitely the second curve is weaker that the first one (Fig. 5). However the consumed energy in the modernized construction is bigger then in the original project. Indeed, this energy is equal to

$$A_2 = \int_0^{2\epsilon_1} \sigma d\epsilon = \frac{3}{2}\frac{E^2}{E+E_t}\epsilon_1^2 \tag{28}$$

The ratio of two energies is:

$$\frac{A_2}{A_1} = \frac{3E}{E + E_t} \tag{29}$$

For example, as it is shown on Fig. 5, this ratio is equal 2.4 if $E_t/E = 1/4$. On the Fig. 5 the first curve corresponds to the initial diagram, and the second one corresponds to the waiting element system.

The essence of the suggested procedure is that we perform work on the first bar even in its unstable mode, and it leads to consumption of additional energy.

The other feature of the suggested scheme of joining elements is that the rate of resistance could increase with the increasing of the strain. The normal material without interior structure never possesses this feature. In the opposite, increasing the strain leads to decreasing of the cross-section of the sample, to the necking, micro-cracking, and other effects that decrease the rate of resistance and, finally, to instability of the extension. Therefore the suggested 'waiting elements' counteracts the instability.

3.2 Many waiting elements system

Surely one can continue the described process by dividing the initial bar into more and more pieces. All of the pieces except one will work in both their stable and unstable modes. The consumption of the energy in the system increases but the limiting strength decreases. The energetic efficiency of the procedure is higher the greater is the range of unstable deformation on the stress-strain curve, i.e. the greater is the ratio E/E_t.

Let us consider the same system of N parallel bars ($N-1$ waiting elements). Let the cross-section area of n-th element be Λ_n. We assume again that the total cross-section is fixed:

$$\sum_{n=1}^{N} \Lambda_n = 1 \tag{30}$$

The stress -strain relation is given by the formula

$$\tilde{\sigma} = \sum_{n=1}^{N} \Lambda_n \{[\tilde{\epsilon} - (n-1)\tilde{\epsilon}_1] E H[\tilde{\epsilon} - (n-1)\tilde{\epsilon}_1] -$$

$$(\tilde{\epsilon} - n\tilde{\epsilon}_1) E_t H(\tilde{\epsilon} - n\tilde{\epsilon}_1)\}, \quad 0 \le \tilde{\epsilon} \le N\tilde{\epsilon}_1 \tag{31}$$

To obtain the upper limit of the energy increase, the system is assumed to be on the verge of instability until $\tilde{\epsilon} = N\tilde{\epsilon}_1 \le \tilde{\epsilon}_*$ (see Fig. 5). This leads to the following condition

$$\frac{d\tilde{\sigma}}{d\tilde{\epsilon}} = 0 \quad (\tilde{\epsilon}_1 < \tilde{\epsilon} < N\tilde{\epsilon}) \tag{32}$$

We calculate now

$$\Lambda_n = \frac{E_t}{E}\sum_{k=1}^{n-1}\Lambda_k, \quad n = 2, ..., N \tag{33}$$

The last expression and the condition (30) give us:

$$\Lambda_1 = \left(1+\frac{E_t}{E}\right)^{1-N}, \quad \Lambda_n = \frac{E_t}{E}\left(1+\frac{E_t}{E}\right)^{n-1-N}, \quad n \geq 2 \tag{34}$$

The limit of the stable extension energy (see curve 3 on the Fig. 5 which corresponds to $N = 5$) is:

$$A_N = \Lambda_1 E\frac{\tilde{\epsilon}_1^2}{2}[1 + 2(N-1)] \tag{35}$$

At last, the ratio of the consumed energy is

$$\frac{A_N}{A_1} = \Lambda_1(2N-1) = (2N-1)\left(1+\frac{E_t}{E}\right)^{1-N} \tag{36}$$

One can choose the parameter N which satisfies the condition $N\tilde{\epsilon}_1 \leq \tilde{\epsilon}_*$. Clearly, the equality (see Fig. 5) holds: $(\tilde{\epsilon}_* - \tilde{\epsilon}_1)E_t = E\tilde{\epsilon}_1$. Combining the last two relations we came to the inequality

$$\frac{E_t}{E} = \frac{\tilde{\epsilon}_1}{\tilde{\epsilon}_* - \tilde{\epsilon}_1} \leq \frac{1}{N-1} \tag{37}$$

which express the improvement by waiting elements.

Let it be the equality in (37). Then the consumed energy can be increased unlimited

$$\frac{A_N}{A_1} = (2N-1)\left(1+\frac{1}{N-1}\right)^{1-N} \sim \frac{2}{e}N = \frac{2}{e}\frac{\tilde{\epsilon}_*}{\tilde{\epsilon}_1} \quad (\frac{\tilde{\epsilon}_*}{\tilde{\epsilon}_1} \to \infty) \tag{38}$$

However, the maximal strength goes to zero.

3.3 The superplastic rope model

The same idea of waiting elements can be realized in the following model with elements acting under static and dynamic loading.

Consider the rope which consists of m threads, and has N knots fixing them (see Fig. 6a). Suppose that the diameters of threads are a little different and their lengths between the neighboring knots is also different: the smallest thread is the shortest and so on.

Upon being stretched, the structure resists as the weakest thread acts alone until at some place it breaks (Fig. 6b). In this moment all the energy stored in the rope disappears because the rope became longer and therefore unloaded: it cannot produce mechanical work. Being tensioned again, it collects elastic energy again until some other piece of the same weakest thread breaks and so on. After each moment of a break all the stored energy is wasted.

Because of the differences in the thicknesses of threads the process of failure is stable in the sense that the first weakest thread breaks $N-1$ times, than the second one begin to breaks, and so on. The total number of breaks is $(N-1)(m-1)+1$, then the rope breaks down completely.

Let us consider the behavior of the described system under a given low rate of elongation. Namely, it is assumed that oscillations caused by the sudden unloading, by breaking one of the thread sections, are damped towards the next step of the loading of the thread. In this case, where all the threads are expected to be almost the same, the complete strain energy consumed by the system considered is equal to the energy stored in a one thread, A_0, multiplied by the number of the breaks:

$$A_2 = [(N-1)(m-1)+1]A_0 \tag{39}$$

Compare this expression with the energy consumed by a normal rope. The rope made in usual way of m threads with equal lengths is m times stronger than one thread and it collects m times more energy

$$A_1 = mA_0 \tag{40}$$

before breaking.

Clearly, the postulated modern rope stores more energy then the usual one. The ratio R between energies is

$$R = \frac{(N-1)(m-1)+1}{m} \sim N \quad (N,\ m \gg 1) \tag{41}$$

Note that the model uses the combination of elasticity and breaks to approximate the stable plastic tension and stable plastic strain of the rope. The rope behaves as plastic material although it consists of elastic-breaking elements. Note also that as a two-dimensional system the model appears like fabric (see Hearle, Grosberg and Backer, 1969).

3.4 Energy dissipation in dynamics

In the above considerations we have concentrated our attention on the strain energy of elements of a structure that carry static or quasi-static loadings. In this section we consider the dynamic loading. We show that in dynamic situation the strain energy may be neglected in comparison with the kinetic energy of nonlinear oscillations that plays the main role in fracture energy consumption.

Consider a material with a 'rigid-nonlinear' static stress-strain diagram (it means that the diagram is convex in the strain axis direction). Here, a shock wave propagates in the material under a dynamic extension. As the shock wave propagates a part of the strain energy occurs on a 'micro-level' that involves the excitation of high-frequency oscillations.

We could mention, as an example, the compessive heating that occurs in shock waves that propagate in a gas (see, for instance, Zel'dovich and Raizer, 1966) or the energy dissipation occuring in viscous fluids. Other example is given by a compression wave in a porous material. Here, a large portion of strain energy goes into a heat thanks to the great difference between the static and dynamic paths on the stress-strain diagram. The point is that in the case of 'rigid nonlinear' stress-strain diagram, the dynamic path corresponds to the straight line through the initial and final points on the so-called Hugoniot diagram, in contrast to the Poisson's diagram for the static curve. The first one corresponds to a higher uniaxial stress under the same strain, and the difference is especially high for porous materials (see Zel'dovich and Raizer, 1966, 1967).

It is important to mention, that the same phenomenon may take place under extension processes, as it is shown below. Namely, the structure of the waiting element type provides the energy flow into the 'micro-level' thanks to the loading - unloading oscillations. The greater the time of 'waiting' between breaks, the greater the difference between stable and unstable modes of stress transmission, and the more the energy is dissipated on the macro-level by high-frequence oscillations.

Consider now some examples to make these speculations clearer.

The simplest system is a wavy, inextensible fibre without any bending stiffness. In this case, the initial point on the static diagram (Fig. 7) is $\tilde{\sigma} = \tilde{\epsilon} = 0$. Then, we have $\tilde{\sigma} = 0$ for a range of the 'macro-strain' $0 \leq \tilde{\epsilon} < \tilde{\epsilon}_*$. Finally, we have $\tilde{\epsilon} = \tilde{\epsilon}_*$ for the range $0 \leq \tilde{\sigma} < \tilde{\sigma}_*$, where $\tilde{\sigma}_*$ be the stress limit. This diagram is very 'rigid', and, on the macro-level, the dynamic extension of this fibre causes the shock wave. This wave corresponds to the Hugoniot diagram which differs from the static one.

In the dynamic case, the static strain limit $\tilde{\epsilon_*}$ is replaced by $\tilde{\epsilon_{**}}$, where $\tilde{\epsilon}_{**} < \tilde{\epsilon}_*$. The macro-level dynamic path on the diagram is a straight interval supported by the points $\tilde{\sigma} = \tilde{\epsilon} = 0$ and $(\tilde{\sigma}_d,\ \tilde{\epsilon}_{**})$, where $\tilde{\sigma_d}$ is the stress in the shock wave. This problem is investigating now by Krilov, Parnes, and Slepyan; the theoretical considerations and numerical simulations have demonstrated that $\tilde{\epsilon}_{**} = 2/3\tilde{\epsilon}_*$.

In the considered case of inextensible fibre, all the work produced by the tension force goes to the kinetic energy on the 'micro-level'. It manifests itself by lateral oscillations of the fibre that form a solitary wave. Thus the system cannot accumulate any energy in static deformation and at the same time it is a good an energy absorber in dynamic loading. The limiting level of the energy depends on the stress limit but not on limiting elongation of the material. The absence of the dependence on the elongation is an important factor for the consideration of an almost inextensible fibre of a high strength.

The use of wavy fibres opens the way of increasing of the total fracture energy of such a type of material under dynamic extension. The point is that the wavy fibres can accumulate a lot of kinetic energy by lateral oscillations in addition to the energy of dynamic extension of the material. Composites with wavy fibres are considered in the book by Chou (1992).

The next example shows that, in some situations, the lower the limiting strain energy density in static conditions the greater is the total fracture energy under dynamics condsitions. This observation is based on the fact that the lower the resistance a material shows under the same stress limit, the greater amount of the material is involved into a large deformation, and the more energy goes to the high-frequence oscillations (to the micro-level). In the following example the linear analysis is used; under this assumption there is no difference between $\tilde{\sigma}$ and σ as well as between $\tilde{\epsilon}$ and ϵ.

Consider a wedge (Fig. 8)

$$r \geq r_0, \ 0 \leq \theta \leq \theta_0 \tag{42}$$

made of the material that is characterized by the stress-strain diagram shown on the Figure 9, namely:

$$\sigma = \sigma_0 \ (0 < \epsilon < \epsilon_*), \quad \epsilon = \epsilon_* \ (\sigma_0 \leq \sigma < \sigma_*), \ \sigma_0 \leq \sigma_* \tag{43}$$

Suppose that initially this wedge was unloaded and immobile. Suppose also that a dynamic loading is applied at the boundary $r = r_0$ so that a constant radial velocity of the displacement arises as $v_r = -v_0 < 0$ at that boundary. The problem can be described as a dynamic problem for a plastic plate placed under a concentrated, lateral, dynamic loading. We want to find what is value of the material parameter σ_0 that maximizes the total fracture energy.

Note, that maximal energy density in statics corresponds to the maximum possible value $\sigma_0 = \sigma_*$. This density is equal to

$$A^{(s)}_{max} = \sigma_* \epsilon_* \tag{44}$$

Consider now the dynamical case; a shock wave propagates along the wedge. Let us estimate parameters of this wave. We simplify the problem a bit. Firstly, we neglect a tangent velocity , i.e. we consider the axisymmetric problem. Secondly, the stress field is assumed to be one-dimensional:

$$\sigma_\theta = \sigma_{r\theta} = 0 \tag{45}$$

The equation of motion can be derived from the following consideration. Before the shock wave comes to a point, the material is motionless. and after the shock wave has past this point the material moves with the constant velocity $v_r = v = -v_0$. From these equilibrium conditions we derive the stresses in the body.

We calculate the stress first in the domain lying in front of the shock wave, where $r > l(t)$. Under the considered piece-constant diagram, this domain is under stress field

$$\sigma_r = \sigma = \sigma_0 \frac{l}{r} \tag{46}$$

immediately after the extension is applied ($t > 0$).

After the shock wave passes the intervale $r_0 < r < l$ we have

$$\sigma = \sigma_+ \frac{r_0}{r} \tag{47}$$

where σ_+ is an unknown stress at $r = r_0$ $(t > 0)$. The stress distributions (46) and (47) follow from the equilibrium equations which are valid for both domains $r < l$ as well as $r > l$ because the first one moves with the constant velocity.

Now the usual conditions are used for the jump at $r = l$ (see Zel'dovich and Raizer, 1966). The puls-equation is

$$\rho v_0 l \,\frac{dl}{dt} = (\sigma_+ \frac{r_0}{l} - \sigma_0) l \tag{48}$$

where ρ be the mass density. The continuity follows from the mass conservation equation that looks in our case as

$$\frac{v_0 t}{l - r_0} = \epsilon_* \tag{49}$$

From this we came to the representations:

$$l = r_0 + \frac{v_0 t}{\epsilon_*}, \ \frac{dl}{dt} = \frac{v_0}{\epsilon_*}, \quad \sigma_+ r_0 = l\sigma_0 + \rho v_0 l \frac{dl}{dt} \tag{50}$$

Then, one has the limiting relation ($\sigma_+ \leq \sigma_*$, $\sigma_+ = \sigma_*$ when $t = t_*$)

$$\sigma_* r_0 = l(t_*)\sigma_0 + \rho v_0 l(t_*) \frac{v_0}{\epsilon_*}, \ l(t_*) = \frac{\sigma_* r_0}{\sigma_0 + \rho v_0^2/\epsilon_*} \tag{51}$$

and the limiting time can be found from the equation ($\sigma_+ = \sigma_*$)

$$\sigma_* r_0 = \left(r_0 + \frac{v_0 t}{\epsilon_*} \right) \left(\sigma_0 + \frac{\rho v_0^2}{\epsilon_*} \right) \tag{52}$$

Now one has all data for calculation the total dynamic fracture energy

$$A_{total}^{(d)} = \int_0^{t_*} r_0 \sigma_+ v_0 \, dt =$$

$$\frac{\epsilon_*}{2} \left[r_0^2 \sigma_*^2 \left(\sigma_0 + \frac{\rho v_0^2}{\epsilon_*} \right)^{-1} - r_0^2 \left(\sigma_0 + \frac{\rho v_0^2}{\epsilon_*} \right) \right] \tag{53}$$

It is easy to see that the maximum of the total energy corresponds to the strain energy density at zero point, $\sigma_0 = 0$:

$$A_{total}^{(d)} = \frac{r_0^2}{2} \left(\frac{\epsilon_*^2 \sigma_*^2}{\rho v_0^2} - \rho v_0^2 \right) \tag{54}$$

In this case, the length of the part of the structure which is involved into the dynamic process, as it follow from (51), is

$$l(t_*) = \frac{\sigma_* \epsilon_* r_0}{\rho v_0^2} \tag{55}$$

One really can conclude from this example that 'the lower the energy density the greater the total energy'. It is true because the lower the energy density the greater the amount of material that is involved into the deformation before the failure of the construction. Finally, the integral value of the consumed energy is greater for the material with smaller energy density.

We note that several mechanisms have been neglected in our consideration. Namely, oscillations in the shock wave were not taken into account when the problem on the macro-level was considered. Therefore we neglected their influence on the stress-strain diagram and on the strength of the material. In this connection, it may be noted that the scale of the micro-oscillation does not influence the diagram but influences the strength, which is sensitive to the scale. It is the well known scale effect in strength. From this, one may conclude that the smaller the scale of vibrations is the lower is their influence. To improve the material properties, the best design is that which provides conditions for the direct transformation the macro energy into heat. Then the energy of melting and of heating to the melting temperature determines the limiting fracture energy.

4 Plastic helix extension. A theoretical consideration and experimental results.

Here we discuss a way to make a deformation under loading more stable. We want to transform the extension mode on the macro-level to a more stable mode of deformation on the 'micro-level'.

A good example of this method of stabilization is given in Figure 11. Here one can compare the stress–strain diagrams for a straight wire (curve – 1), and for a helix fabricated from the same piece of wire (curve – 2). In this test, as it is shown below, the total fracture energy of the helix may be many times greater.

Note that the elastic strain energy of the helix is lower than that for the straight wire due to the non-uniform distributionof the strains in the helix. Note also that a helix can be considered as the waiting elements type structure. Here, the highest failure resistance corresponded to the extension of the wire that 'waits' its time to carry the load during the main part of the helix elongation.

Consider a helix (Fig. 10) of radius R and length L which is made from a wire of radius r and length l. Let the spring be under tension; $\alpha,\ \kappa,\ \omega$ and ϕ be the angle between the wire axis and the cross-section of the spring, the curvature, the second curvature and the torsion angle of the spring, correspondingly. In this notations $R_0,\ L_0,\ \alpha_0,\ \kappa_0,\ \omega_0$ and ϕ_0 are corresponding initial values.

Assuming that the wire is inextensible, one can base on it geometrical relations

$$L = l \sin\alpha, \quad \kappa = \frac{\cos^2\alpha}{R}, \quad \omega = \frac{\sin 2\alpha}{2R}, \quad \phi = l\left(\frac{\cos\alpha}{R} - \frac{\cos\alpha_0}{R_0}\right) \tag{56}$$

We assume that the torsion of the helix (but not of the wire !) is excluded by the tension conditions, and hence

$$R = R_0 \frac{\cos\alpha}{\cos\alpha_0} \tag{57}$$

If the angle α_0 is small enough, the wire is mainly under torsion meanwhile the spring is under extension. Then, when this angle becomes large enough, the wire is mainly under bending. Really, the torsion and bending moments (let call them M_t and M_b respectively) can be approximately expressed as

$$M_t \approx PR\cos\alpha, \quad M_b \approx PR\sin\alpha \tag{58}$$

In this approximation, a possible torsion moment is neglected which acts in direction of the helix axis. This accuracy is enough for rough estimation.

To simplify the problem, assume the material of the wire is rigid-plastic and that the stresses in a cross-section of the wire are connected each other by the relation (see Kachanov, 1971)

$$\sigma^2 + 3\tau^2 = \sigma_Y^2 = \text{const} \tag{59}$$

where σ and τ are the normal and the shear stresses, σ_Y is the yielding limit. The last relation is valid under an assumption of the increasing strain. For the estimation of the energy, let us consider the strain energy due to the torsion and the bend separately, and let us neglect the energy of the wire elongation. The absolute values of the torsion and bending moments are equal to:

$$M_t = \frac{\sigma_Y}{\sqrt{3}}\frac{2\pi}{3}r^3, \quad M_b = \frac{4}{3}\sigma_Y r^3 \tag{60}$$

At the same time, the limiting change in ω is

$$\Delta\omega = \frac{\cos\alpha_0}{R_0}(\sin\alpha_{max} - \sin\alpha_{min}); \tag{61}$$

the variation in the curvature is

$$\Delta\kappa = \frac{\cos\alpha_0}{R_0}(\cos\alpha_{min} - \cos\alpha_{max}) \tag{62}$$

In this way, assuming that $\alpha_{min} = 0$, $\alpha_{max} = \pi/2$ one can obtain the upper estimations for the works per unit volume as following:

$$A_t = \frac{2}{3\sqrt{3}}\frac{r}{R}\sigma_Y, \quad A_b = \frac{4}{3\pi}\frac{r}{R}\sigma_Y \tag{63}$$

Finally, the estimation of the total work is:

$$A = \sqrt{A_t^2 + A_b^2} = k\frac{r}{R}\sigma_Y, \quad k \approx 0.6 \tag{64}$$

In fact, the limiting strain energy can exceed this value, mainly thanks to the strain hardening of the material. The tests worked out at the St.Petersburg Marine Technical University by V.F.Tscherbinin under the supervision by Slepyan show different results concerning the increase of the limiting strain energy density for different materials (Fig. 11). The results are also shown in Table 1, where A_0 is the strain energy of the straight wire.

Table 1 Experimental results

σ_Y	A_0	r	R	l	L	R/r	A	A/A_0	k
MPa	$MPa\ sm^3$	sm	sm	sm	sm	–	$MPa\ sm^3$	–	–
1050	17	0.08	0.27	46.5	6.7	3.4	382	23	1.3
			0.48		3.2	6.0	213	13	1.3
			0.59		2.7	7.4	170	10	1.3
890	130	0.1	0.3		7.2	3.0	390	3.0	0.9
			0.5		3.6	5.0	200	1.5	0.8
			0.6		3.0	6.0	150	1.2	0.7
730	26	0.1	0.3		7.2	3.0	200	7.7	0.6
			0.5		3.8	5.0	130	5.0	0.6
			0.6		3.0	6.0	98	3.8	0.6

One can see that the strongest effect of the strain energy increase corresponds to the wire of the greatest yielding limit. Also, the experimental dependence of the energy on the ratio R/r and, what is more, the least values of the coefficient k turn out rather close with the theoretical estimations.

This example shows us that the above discussed enhancements of fracture energy increases are not only theoretically possible. They can be achieved in an experiment as well.

REFERENCES

Chou, Tsu-Wei. 1992. Microstructural Design of Fiber Composites. Cambridge, Cambridge University Press.

Hearl, j. W. S., Grosberg, P. and Backer, S. Structural Mechanics of Fibers, Yarns, and Fabrics. 1969. New York, John Wiley & Sons.

Jones, N. and Wierzbicki, T. (Ed.). 1993. Structural Crashworthiness and Failure. London and New York, Elsevier Applied Science (Printed at The University Press (Belfast) Ltd.

Kachanov, L. M. Foundations of the Theory of Plasticity. 1971. Amsterdam, North-Holland Publishing Company. New York, American Elsevier Publishing Company.

Kachanov, L. M. Fundamentals of Fracture Mechanics. 1974. Moscow, Nauka (in Russian).

Kachanov, L. M. Introduction to Continuum Damage Mechanics. 1986. Dordrecht, Martinus Nijhoff Publishers.

Poluchin, P, Gorelik, S. and Vorontsov, V. Physical Principles of Plastic Deformation. 1983. Moscow, Mir Publishers (in English).

Zel'dovich, Ya. B. and Raizer, Yu. P. Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena, Vol. 1 (Ed. by Wallace D. Hayes and Ronald F. Probstain). 1966. New York, Academic Press.

Zel'dovich, Ya. B. and Raizer, Yu. P. Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena, Vol. 2 (Ed. by Wallace D. Hayes and Ronald F. Probstain). 1967. New York, Academic Press.