Reducing of optimal design problems to minimal variational problems

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Abstract

The problems of optimal design of conducting and elastic inhomogeneous bodies is considered; these problems are described as variational problem with differential restrictions which are the equations of the state of the body. A method is suggested for reducing the problem of restricted minimum to the simplest variational problem of the minimization of the sum of the energy and the complimentary energy correspondent to two orthogonal external fields; it is based on the Legandre transformation of slightly reformulated initial problem. The characteristic properties of optimally designed bodies are discussed. It is shown, in particular, that the optimal microstructures possess the maximal anisotropy in the sense that they minimize the difference between the energy densities caused by two orthogonal loadings.

1 Introduction

The goal of this paper is to discuss a new method of relaxation of optimal design problems. We consider the steady elastic or conducting state of the body made of several linear materials with different physical properties. These problem ask for the optimal layout of these materials throughout the designed body; the boundary conditions and the loading are supposed to be known and unchanged. The goal of the project is to minimize a lower weakly semicontinuous functional of the solution of the corresponding boundary value problem of conductivity or elasticity, that is, the functional of the thermal or deflection field.

For example, one may consider the minimization problem for the integral of a component of the displacement through some given subregion, or the maximization of the 'sensitivity' of a construction which is the norm of the displacement in some regions of the body. Similar problems arise in heat or electricity conducting bodies assembled from different materials. For example it may be required to minimize a norm of difference between the existing temperature field and some fixed field we would like to have.

Study of these problems has been in progress for more than decade, and several approaches have been developed. The most investigated class of optimal design problems of this kind is the minimization of the energy stored in an inhomogeneous material. Physically, the minimization of the energy means the maximization of the overall strength of the construction, the engineering justifications of it has been discussed in many times, see for example the book by Rozvany [44] or the paper by Kirsch [20]; mathematically, it is a simplest variational problem. Indeed, the problem can be formulated as a variational problem of minimizing the energy $\mathcal{E}$ with respect of both stresses σ (strains) and displacements of the materials (design variables), represented by the characteristic functions χ_i of the domain of i-th material:

$$\min_{\chi_i} \min_{\sigma} \int \mathcal{E}(\chi_i, \sigma) \ dx \tag{1.1}$$

This minimal variational problem is to be solved by the known and the recently developed methods of calculus of variations.

It has been found about two decades ago that this problem does not have a classical (smooth) solution and needs a regularization procedure. In the paper by Kohn and Strang [21] it has been pointed out, that the problems of this kind are ill-posed because of non-convexity of the integrand with respect to design variables; classical solutions of such problems do not exist. The nonexistence simply means that the optimal layout necessary includes infinitely small regions occupied with different initially given materials; these regions are to be mixed in a special way to form optimal microstructures. The most difficult and novel part of the whole problem is the problem of finding the microstructure of optimal energy (the so-called local problem). It has been shown (see the papers by Kohn and Strang

[21], by Tartar [46], by Lurie and Cherkaev [32], by Francfort and Murat [12] by Gibiansky and Cherkaev [13, 15], by Avellaneda [2]) that the local problem possesses in many cases an analytical solution which is given by so-called quasiconvex envelope (Kohn and Strang [21]) of the functional. The points of the envelope represent the energy of optimally designed microstructures and these microstructures picture minimizing sequences of layout of materials. Several methods have been developed for building the quasiconvex envelope. We mean the methods of Hashin-Shtrikman type (papers by Hashin and Shtrikman [18], by Milton [35, 36], by Allaire and Kohn [1]) and the Translation Method (see Lurie and Cherkaev [30], Murat and Tartar[41], Milton ([35, 36], Cherkaev and Gibiansky [7] which give the lower bound of the energy stored in microstructures of arbitrary geometry. Special minimizing sequences (special microstructures) have been constructed in the mentioned above papers and it has been proved that the energy reaches its minimum within these structures. The microstructures called 'laminates of some rank' which will be described later turn out to be of special importance; these structures have been investigated in many papers (see, for example, papers by Milton [34, 37], by Lurie and Cherkaev [30, 32], by Francfort and Murat [12], by Avellaneda [2], by Lipton [23], by Gibiansky and Cherkaev [14]).

The first solved problem of this kind is the design of an optimal inhomogeneous elastic bar with extremal rigidity (see the papers by Lavrov, Lurie, Cherkaev [22], by Kohn and Strang [21]). It was proved that only the properly oriented laminate materials must be used for both maximizing or minimizing the torsional rigidity of the bar. This conclusion also remains true for the bar made of non-linear elastic materials and of elastic-plastic ones (Gibiansky and Cherkaev [15]). The problem of optimal design of thin plates which store minimum elastic energy was solved in the same way (see papers by Gibiansky and Cherkaev [13], by Bendsoe and Kikuchi [3] , by Suzuki and Kikuchu [45] , and the recent papers by Allaire and Kohn [1] and by Jog, Haber and Bendsoe [19]) as well as the problem of optimal elastic three-dimensional composites (papers by Gibiansky and Cherkaev [14], Allaire and Kohn [1], and the related paper by Francfort and Murat [12]); it was found that the optimal structures here are the 'matrix lamination of second and third rank'.

The more general class of weakly lower semicontinuous goal functionals discussed here also requires the relaxation procedure because of the same reason: the absence of solutions in the initial problems. The evidence of the absence of solution of the problem and of the need of regularization procedure has been pointed in early papers by Lurie [24], by Murat [38], [39], by Tartar [46]. This time the problem is not more a simplest variational one; it turns to be a variational problem with differential restrictions which display the equation of the state. The technique of relaxation of these problem is less developed. However, several approaches have been already suggested for relaxation of this kind of structural design problem.

One approach (see the paper by Lurie et al. [16]) is based on the description of the so-called G-closure of the class of given material properties (see the papers by Lurie and Cherkaev [29], [30, 31, 32]), Francfort and Murat [11], Milton [36]). The G-closure is by definition the set of all possible tensors of effective properties of all microstructures, assembled from the given materials. Clearly, if the complete description of the G-closure set is available one can be sure that the appropriate element (the effective tensor of an optimal microstructure) belongs to it. Therefore the reformulation of the initial problem by replacing the initial set of materials by its G-closure makes the problem well-posed. (Dealing with optimal design problems with restrictions on the amounts of given materials we may also need the description of the so-called G_m-closure (Lurie and Cherkaev [30]) which is a set of of all possible tensors of effective properties of all microstructures, assembled from given materials taken in given volume fractions).

However, the G-closure problem itself is rather complicated and only few results are attained by now. Namely, the full description of G-closures and G_m-closures for a set of conducting media has been described in the papers by Lurie and Cherkaev [29, 30, 31, 32], by Murat and Tartar [41], by Francfort and Murat [11]). A more complicated example of G_m-closure has been obtained in the paper by Cherkaev and Gibiansky [6] where the problem has been completely solved for the electromagnetic dielectrics in two dimensions. In the mentioned papers there have been suggested an analytical procedure to determine the bounds of G-closures. It requires solving a simplest variational problem of minimization of the sum of densities of stored in a mixture energy and/or complementary energy. It is supposed that the periodic composite structure has been placed under action of several orthogonal external periodic fields. The minimizers in the variational problem are both the fields and the materials distribution. The solution of a corresponding variational problem has been found in an explicit form, thus the bounds of G-closure have been described explicitly. The description of this approach can be found in many papers, for example (by Lurie and Cherkaev [30],[32], by Milton [35], by Cherkaev and Gibiansky [7], [8]). Note, that we do not mention here numerous papers devoted to the partial description of G-closure, such that the description of the variety of isotropic composites. The reason for it is that the discussing approach to optimal design needs the knowledge of all G-closure set because it is not known a priori what composite is the best one.

The explicit representation of the G-closure of conducting composites has been used in the paper by Gibiansky, Lurie and Cherkaev [16] for solving the optimal design problem of the 'thermo-lens' i.e. of a body which condenses the thermal current due to its inhomogeneouty. In particular, it has been found that only pure materials and the most incomplicate laminate microstructures must be used to assemble an optimally designed lens.

For elastic composites, however, the problem of G-closure is far of complete, and there is little hope to obtain results for the general case in reasonable time.

(Some particular classes of the problem has been solved by Lurie and Cherkaev [31]). Besides this, such a description is to consist of a large number of inequalities which make difficult to work with them. On the other hand, the description of the G-closure is only sufficient but not necessary for determination of the optimal structures, and we need essentially much less information for this goal. The major point is that one could a priori restrict oneself to a special class of microstructures, which form an optimal solution of the correspondent variational problem. The problem of determination of this class is the main goal of the present paper; this problem turns out to be simpler than the G-closure problem.

We mention that there have been suggested more straightforward approaches to the relaxation of the optimal problems than the universal G-closure procedure. Raitum [42], [43] noted that the given values of the pair of current and gradient fields in the state low do not determine the tensor of properties completely. One can consider the equivalence class of anisotropic tensors which produce the same current under the given gradient field. Therefore one could restrict oneself to description of equivalence classes of effective tensors and to find a set of microstructures which represent each of them. This idea (the weak G-closure, as he called it) allows Raitum to prove that only laminates need to be used for optimal conducting media.

Lurie[25]), suggested a different approach based on direct bounds of the value of the min-max augmented functional that replaces the original optimization problem with differential constraints. He used lower and upper bounds for the augmented integrand, these bounds are generated, respectively, by a suitable laminar microstructure and by a specially constructed modification of the original integrand he called the 'polysaddle envelope'. If these bounds coincide, the problem becomes well posed. In the paper Lurie [27] there is given its extension of the technique from the conductivity problem to some specific cases in the problem of bending of a thin plate.

Here we develop a new method which allows to determine the optimal structures immediately. We simply reduce the problem to a regular minimal variational problem by demonstration that any optimally designed conducting (elastic) composite structure corresponds to the minimum value of the sum of specific energy caused by an exterior uniform gradient field (stress) and a specific complementary energy caused by an exterior uniform current (strain) field. This gives a qualitative information of the types of optimal microstructures and makes the problem of optimal composites a specific part of the general G-closure problem. Namely, we indicate components of the boundary of G-closure, to which optimal composites should belong.

2 The relaxed problem

The state of the body $\mathcal{O}$ is described as a solution of a conductivity (elasticity) problem with a self-adjoined elliptic differential operator $L(D)$:

$$L(D)w = q \tag{2.1}$$

where $D = D(x)$ is the tensor of physical properties of the material placed in the point x, and q is an exterior loading. Some boundary conditions should be fixed on the boundary $\partial\mathcal{O}$ of the body $\mathcal{O}$. They are of no importance for us, for simplicity, we will consider the case of uniform conditions:

$$w|_{\partial\mathcal{O}} = 0. \tag{2.2}$$

The operators $L(D)$ for the problems under consideration are equal either to the scalar operator $L'(D)$ (conductivity problem)

$$L'(D) = \nabla \cdot D(x) \cdot \nabla \tag{2.3}$$

where $D = D(x)$ is a symmetric positive second-rank tensor of specific conductivity of the material in the point x, the solution $w(x)$ means the temperature, or to the vector operator $L''(D)$ (elasticity problem)

$$L''(C) = \nabla : C(x) : (\nabla + \nabla^T) \tag{2.4}$$

where $C = \{c_{nmij}\}$ is a symmetric positive fourth order tensor of elastic constants of the material, the solution w means the displacement vector. The symbol (:) denotes a double contraction of indices of tensors.

The optimal design problem asks for the minimization of the functional $I(w)$ of the solution w of the corresponding boundary value problem of conductivity (elasticity):

$$I'(w) = \int_{\mathcal{O}} f(w, \nabla w)\ dx \tag{2.5}$$

where w is a scalar (in the conductivity problem) or vector (in the elasticity problem) potential. The function $f(\cdot,\cdot)$ is supposed to be convex with respect to the second argument (which does provide the lower semicontinuity of $I(w)$); see, for example, Dagorogna [10].

The augmented functional I_A is constructed by adding to the minimizing functional I the differential equation of the state (2.1) with a Lagrange multiplier $\lambda = \lambda(x)$.

$$I_A = \min_{D(x)} \min_{w(x)} \max_{\lambda(x)} \int_{\Omega} [r(x)w + \lambda(x)[L(D(x))w - q]]\ d\ x. \tag{2.6}$$

Applying the usual arguments we conclude that the cost of the augmented problem is equal to the value of the functional I.

The equation for the Lagrange multipliers may be found in the standard way by the variation the augmented functional (2.6) with respect to w:

$$\delta I_A = R\delta w = 0 \tag{2.7}$$

which implies

$$R = L^*(D)\lambda - \frac{\delta f}{\delta x} = 0, \quad \lambda|_{\partial\mathcal{O}} = 0. \tag{2.8}$$

Due to self-adjoint nature of L ($L = L^*$) the Lagrange multiplyer satisfies the equation similar to the equation (2.1) of the state with different boundary conditions and right hand sides. Physically it means that λ may be considered as a field of the same nature as the field w, as it corresponds to the same inhomogeneous media $D(x)$ but is caused by different external loading and boundary conditions (which generally may depend on the field w).

It is well-known that the evaluation of the control parameters cannot be achieved by straightforward use of the necessary condition technique because that the variables as well as the control function are characterized by fine-scale oscillations. The problem needs a relaxation: the algebraic procedure of solution of necessary conditions must be replaced by the solution of the so-called *local problem* which is the variational problem of optimization of the structure in an infinitesimally small neighborhood of a point of the designed body.

The weak continuity of the problem ensures that described program is correct: this property means that the value of the functional I_A (and the cost functional I) does not change if the fields are averaged throughout an infinitely small regular domain Ω; the average means

$$< (\cdot) >= \frac{1}{||\Omega||}\int_\delta (\cdot)\ d\ x \quad (||\Omega|| \to 0) \tag{2.9}$$

where Ω is a rectangle which shrinks to its center.

Doing the averaging, we replace the original problem (2.6) with the problem

$$I_A = \min_{D(x)} \min_{<w(x)>} \max_{<\lambda(x)>} \int_{\mathcal{O}} (< wr > + < f\lambda > + < \lambda L(D)w >)\ dx. \tag{2.10}$$

It is easy to calculate the first two terms of the integrand, supposing that functions f and q are sufficiently smooth almost everywhere:

$$< w \cdot r >=< w > \cdot r, \quad < q \cdot \lambda >= q\cdot < \lambda > \tag{2.11}$$

but the calculation of the last term is a difficult problem. This problem is in fact the main goal of the present paper. We want to calculate the average through Ω

$$B = \min_{D(x)} \min_{w} \max_{\lambda} \int_\Omega \lambda L(D)w\ dx \tag{2.12}$$

and to express the result in terms of the averaged fields ∇w and $\nabla \lambda$:

$$B = B(<\nabla\lambda>, <\nabla w>). \tag{2.13}$$

Doing this we divide the optimization problem into the local and global parts. The global problem has a form

$$I_R = \min_{<D(x)>} \min_{<w(x)>} \max_{<\lambda(x)>} \int_{\mathcal{O}} (<w> r + q <\lambda> + B(<\nabla\lambda>, <\nabla w>)\ dx \tag{2.14}$$

it is called the *relaxed* problem. This problem deals with the averaged fields, it must have a smooth solution basically because fast oscillations of a solution of the original problem are averaged by the local problem.

The local problem is the problem of finding the extremum of the functional B upon the rectangle Ω if the materials distribution $D(x)$ as well as the fields ∇w and $\nabla\lambda$ are supposed to be periodic. It asks for an optimal displacement of materials $D(x)$ in the element of periodicity Ω (considered as a neighborhood of a point of the body $\mathcal{O}$ in the large scale). The solution of the local problem represents the value of the integrand of B on the optimal microstructures; it depends explicitly on the average fields $<\nabla w>$, $<\nabla\lambda>$ which are treated as known parameters.

Consider the periodic structure with the cell of periodicity Ω displaced in the uniform external field. For simplicity we put the volume of the cell Ω equal to 1.

Firstly we deal with the local problem of an optimal conducting composite (the elasticity problem is considered below in the same way). Let us specify for definiteness the set of admissible materials, namely, suppose that the cell Ω is divided into two subdomains Ω_1 and Ω_2, and the conductivity $D(x)$ is equal

$$D(x) = \begin{matrix} D_1 = d_1 I, & if & x \in \Omega_1, \\ D_2 = d_2 I, & if & x \in \Omega_2 \end{matrix}, \quad 0 < d_1 \le d_2 < \infty \tag{2.15}$$

where d_1, d_2 are the scalar conductivity of components, I is the identity tensor. Also it is convenient to assume that the volume fractions

$$m_1 = \text{mes}\Omega_1,\ m_2 = 1 - m_1 = \text{mes}\Omega_2 \tag{2.16}$$

of two materials in a composite are given (they must be determined later in order to minimize the value of the local problem). In other words, we want to find first a function J:

$$J(<p>, <q>, m_1) = \min_{D(x)\in(2.15),(2.16)} <p\cdot D(x)\cdot q> \tag{2.17}$$

where

$$p(x) = \nabla w, \quad q(x) = \nabla\lambda. \tag{2.18}$$

J depend on the averaged fields and on the volume fraction of materials in a mixture. The cost of the local problem is:

$$B(<p>,<q>)=\min_{m\in[0,1]} J(<p>,<q>,m_1) \tag{2.19}$$

Note that the periodic in Ω fields w, λ satisfy (due to the self-adjoint nature of the operator L') the equality:

$$J(w,\lambda,D)=\int_\Omega \lambda L'(D(x))w=\int_\Omega W(p,q,D)dx \tag{2.20}$$

where

$$W(p,q,D)=p\cdot D(x)\cdot q. \tag{2.21}$$

Similarly, we get

$$J''(w,\lambda,C)=\int_\Omega \lambda L''(D(x))w=\int_\Omega W''(u,v,C) \tag{2.22}$$

where the symmetric tensors fields $u,\ v$ (strains) are associated with the elasticity operator L'') and are defined as

$$u(x)=\text{def }w, \quad v=\text{def }\lambda \quad \text{and} \quad W''(u,v,C)=u:C:v, \tag{2.23}$$

The differential operator def (short for deformation)

$$\text{def }w=(1/2)(\nabla w+\nabla w^T) \tag{2.24}$$

means the symmetric part of a gradient of a vector potential (displacement) w.

The functionals J and J'' can be easily defined by introducing the tensor of effective properties of a composite D_*. It is determined by the relation:

$$<j>=<D(x)\cdot\nabla w>=D_*\cdot<\nabla w>. \tag{2.25}$$

The functional (2.17) takes the form

$$J(<p>,<q>,m)=<p>\cdot D_*\cdot<q(x)>=W_*(<w>,<\lambda>,D_*). \tag{2.26}$$

The tensor D_* is determined only by the microstructures of a composite; it belongs to the G_m-closure set mentioned above

$$D_*\in G_mU. \tag{2.27}$$

The cost of the local problem depends now only upon the effective tensor of a mixture, because the mean fields are supposed to be known:

$$\begin{aligned}B(<p>,<q>)&=\min_{m\in[0,1]}\min_{D_*\in G_mU}<p>\cdot D_*\cdot<q>\\&=\min_{m\in[0,1]}\min_{D_*\in G_mU}W(<p>,<q>,D_*).\end{aligned} \tag{2.28}$$

Note, however, that we must calculate (2.28) directly without reference to the description of G_mU-set which is considered as unknown. Instead, we formulate the local problem as a variational problem of calculating the functional J.

$$J(p_0, q_0, D) = \min_{D(x)\in\mathcal{D}} \min_{p(x)\in\mathcal{P}} \max_{q(x)\in\mathcal{Q}} < W(p, q, D) > \tag{2.29}$$

where the sets $\mathcal{P}$, $\mathcal{Q}$,$\mathcal{D}$ are:

$$\mathcal{P} = \{p : \quad \nabla \times p(x) = 0, \quad < p(x) >= p_0, \quad p(x) \text{ is } \Omega - \text{periodic}\} \tag{2.30}$$

$$\mathcal{Q} = \{q : \quad \nabla \times q(x) = 0, \quad < q(x) >= q_0, \quad q(x) \text{ is } \Omega - \text{periodic}\} \tag{2.31}$$

$$\mathcal{D} = \{D(x) : \quad D(x) = \begin{array}{ll} d_1 I, & \text{if} \quad x \in \Omega_1 \\ d_2 I, & \text{if} \quad x \in \Omega_2 \end{array}, \quad \text{vol} \quad \Omega_1 = \mathrm{m}_1, \tag{2.32}$$

$$D(x) \text{ is } \Omega - \text{periodic.}\}$$

We have used here the differential restrictions $\nabla \times p(x) = \nabla \times q(x) = 0$ equivalent to the restriction of potentiality of p, q (2.18), because $\nabla \times p = 0$ if and only if p is a potential field: $p = \nabla w$.

The solution depends only on local characteristics $< p >$, $< q >$, m_1 which are treated here as known parameters.

The local problem for the elasticity operator is formulated in the same way, with differences only in notations.

The key question is to calculate analytically the cost J or to solve explicitly the local problem which is the problem of optimization of the structure in an infinitesimal small neighborhood of a point of the designed body. Note that parameters of an optimal medium are adaptive to the fields w and λ.

3 Reducing to a minimal variational problem. Conductivity

Our goal is to reduce the min-max problem (2.29) - (2.32) to the regular minimal variational problem which could be solved by the already developed methods (we mean the methods of relaxation of the non-convex minimal variational problems which have been well developed for a piece-wise quadratic integrands (see, for example, Kohn and Strang [21], Milton [35]).

3.1 New variables

To reduce the problem to the minimal one , we use a two-step procedure. We note first that the cost J of the local problem depends only on magnitudes of the vectors $|p_0|$ and $|q_0|$ and on the angle 2θ between them. Moreover, J is proportional to the amplitudes $|p_0|$ and $|q_0|$ of both external fields p and q, so it can be rewritten in the form:

$$J = J(< p_0 >, < q_0 >, D) = |p_0||q_0| J(< p_* >, < q_* >, D) \tag{3.1}$$

where p_* and q_* denote the normalize fields p, q respectively. The functional $J(< p_* >, < q_* >, D)$ depends only on one parameter θ.

Now we seek for more convenient form of the functional J. We use the similarity of the fields p_* and q_*; each of them is a normalized curl free field. Let us introduce new variables (Fig. 1):

$$a = p_* + q_*; \quad b = p_* - q_*. \tag{3.2}$$

which obviously are curl free fields as well; they can be treated as a new pair of external fields. The normalization of $< p_* >$, $< q_* >$ implies that the mean values of these fields are orthogonal

$$< a > \cdot < b >= 0. \tag{3.3}$$

The moduli of the mean values of the variables $< a >$ and $< b >$ depend on the angle 2θ:

$$| < a > | = 2\cos\theta \quad | < b > | = 2\sin\theta \tag{3.4}$$

The value $< W >$ is transformed into the form:

$$< W(p, q, D) >= | < p > || < q > | \quad < W_R(a, b, D) >; \tag{3.5}$$

$$W_R(a, b, D) =< a \cdot D(x) \cdot a > - < b \cdot D(x) \cdot b > . \tag{3.6}$$

Note that the self-adjoint cases ($p(x) = q(x)$) see (2.16) or ($p(x) = -q(x)$) correspond either to $a = 0$ or to $b = 0$.

The original local problem (3.1) is equivalent to the problem:

$$J_R(a, b, D) = \min_D \min_{a\in\mathcal{A}} \max_{b\in\mathcal{B}} W_R \tag{3.7}$$

where the sets $\mathcal{A}$ and $\mathcal{B}$ are:

$$\mathcal{A} = \{a : \quad \nabla \times a = 0, < a >= 2\cos\theta\}, \tag{3.8}$$

$$\mathcal{B} = \{b : \quad \nabla \times b = 0, < b >= 2\sin\theta, \quad < a > \cdot < b >= 0\}. \tag{3.9}$$

The local problems for the functionals J_R could also be rewritten in terms of the effective tensors of a mixture:

$$J_R = \min_{D_*\in G_m U} < a > \cdot D_* \cdot < a > - < b > \cdot D_* \cdot < b > \tag{3.10}$$

Now is is immediately clear that the optimal tensor D_* must be oriented so that the vectors $< a >$ and $< b >$ become its eigendirections; its minimal eigenvalue d_{min}

must correspond to the eigenvector directed along $< a >$ and its maximal eigenvalue d_{max} - to the eigenvector directed along $< b >$; such orientation minimizes simultaneously both terms of (3.10). J_R takes the form:

$$J_R = \min_{D_* \in G_m U} [d_{min} < a >^2 - d_{max} < b >^2] =$$
$$4 \min_{D_* \in G_m U} [d_{min} \cos^2 \theta - d_{max} \sin^2 \theta]. \tag{3.11}$$

The last formula displays the basic qualitatively property of an optimal mixture: *its effective tensor D_* possesses maximal difference between weighted maximal and minimal eigenvalues*; in other words these structures should be extremely anisotropic.

Coming back to the original fields $< p >$ and $< q >$ we see that the vectors $< p >$ and $< q >$ lie in the plane of maximal and minimal eigenvalues of D_* and that the direction of its minimal eigenvalue bisects the vectors $< p >$ and $< q >$.

3.2 The minimal problem

To reduce the problem (3.10) to the minimal one we could use the Legandre transformation on the variable b. We will replace the problem of minimization of conductivity of a composite in the direction b with the problem of minimization of its resistance in this direction. Simply speaking, we observe that *the maximization of the conductance is the same as the minimization of the resistance.*

So, we observe that the last term in the right hand side of the representation (3.10) can be replaced by:

$$< b > \cdot D_* \cdot < b > = \max_{<j>} E^*(< j >, < b >) \tag{3.12}$$

$$E^*(< j >, < b >) = 2 < j > \cdot < b > - < j > \cdot D_*^{-1} \cdot < j > \tag{3.13}$$

where $j(x)$ is a current (the variable, dual to b) form the set $\mathcal{J}(j_0)$ of divergence-free periodic vectors with the mean value j_0:

$$\mathcal{J}(j_0) = \{ j : \quad \nabla \cdot j(x) = 0, \quad < j(x) > = j_0, \quad j(x) \text{ is } \Omega - \text{periodic} \}. \tag{3.14}$$

The representation (3.12) implies that

$$- < b > \cdot D_* \cdot < b > = - \max_{<j>} E^*(< j >, < b >) = \min_{<j>} \{ -E^*(< j >, < b >) \}. \tag{3.15}$$

If substitute this expression into (3.10), we get

$$J_T = \min_{<j>} T(< j >, < b >) \tag{3.16}$$

where

$$T(< j >, < b >) = \min_{D_* \in G_m U} [-2 < j > \cdot < b > + < a > \cdot D_* \cdot < a > + < j > \cdot D_*^{-1} \cdot < j > . \quad (3.17)$$

Now we can show that

$$T_*(< j >, < b >) = \min_{D_* \in G_m U} T(< j >, < b >, D) \quad (3.18)$$

is the value of the following minimal variational problem:

$$T_*(< j >, < b >) = \min_{D(x)} \min_{a \in \mathcal{A}} \min_{j \in \mathcal{J}(j_0)} \int_\Omega [-2b \cdot j + a \cdot D(x) \cdot a + j \cdot D(x)^{-1} \cdot j] \ dx; \quad (3.19)$$

Indeed, the first term in the right hand side of (3.19) can be calculated by using the 'compensated compactness' theory (see Tartar[46], Murat and Tartar [40]):

$$\int_\Omega b(x) \cdot j(x) \ dx =< j > \cdot < b > \quad (3.20)$$

the last two terms of (3.19) are equal to the corresponding terms of (3.17) due to the definition of the effective tensor D_*.

Substituting the expression (3.19) into the functional of the local min-max problem (3.10) we replace it with the following minimal one:

$$J_T(< a >, < b >) = \min_{< j_0 >} [-2 < b > \cdot < j_0 > + \min_{D(x)} \min_{a \in \mathcal{A}} \min_{j \in \mathcal{J}(j_0)} \int_\Omega W_T(a, j, D) \ dx] \quad (3.21)$$

where

$$W_T(a, j, D) = a \cdot D(x) \cdot a + j \cdot D(x)^{-1} \cdot j. \quad (3.22)$$

We mentioned already that the last problem is well-known and that it has been well investigated and several methods has been suggested for for its solution. The problem is reduced to the problem of bounds of sum of the values of the energy of the media exposed in the external curl-free field a and in the external divergence-free field j.

The structures $D(x)$ which are optimal for the minimax functional J_R are also optimal for the minimal functional J_T and the variety of the optimal structures coincide. The exact value of the parameter j_0 may be determined only as a final step of the procedure.

Note, that the minimal form of functional is symmetric in the sense that it includes the field a and the current j in the same way. One may expect this property because the original conductivity problem could be formulated in two equivalent ways by using field potential or a current potential; surely the result must be invariant of this choice.

3.3 Bounds and optimal structures

For the problem under consideration, it is enough to use the simplest Reuss-Voigt bounds for effective tensors which estimate the tensors D_*^{-1}, D_* as if they were independent. We find the lower bound the functional $J_T'^*$ by substitution these bounds into (3.21 :

$$
\begin{aligned}
J_T^* \geq &< -a \cdot j > + \\
< a > \cdot < D^{-1}(x) > \cdot < a > + &< j > \cdot < D(x) >^{-1} \cdot < j >= \\
- < b > \cdot < j > + \frac{d_1 d_2}{m_1 d_2 + m_2 d_1} < |a| >^2 &+ \frac{1}{m_1 d_1 + m_2 d_2} < |j| >^2 .
\end{aligned}
\tag{3.23}
$$

The bounds are realized by the appropriate oriented laminates, which are the sought structures of extremal anisotropy: they possess simultaneously the maximal conductance in some direction(s) (along the layers) and the minimal conductance in the perpendicular direction (across the layers).

3.4 Discussion

We outline the basic steps of the described method of relaxation of optimal design problem. We assume that the problem is described by self-adjoint elliptic equations; the shape of the domain, boundary conditions, and external loadings (right hand side terms) are fixed, the minimizing functional is lower weakly semicontinuous. The problem asks for optimal distribution of several given materials in the body. To solve this problem one can follow the scheme:

(i) Formulate the local problem as a variational min-min-max problem for integral of bilinear form of the field and of the Lagrange multiplier, the coefficients of the form represent properties of materials.

(ii) Normalize the bilinear form and transform it to the diagonal form by introducing new potentials. End up with min-min-max problem for the integral of difference of two quadratic forms (or of difference of two energies).

(iii) Use Legandre transformation to reduce the problem to the minimal variational problem, end up with the problem of minimization of the sum of the energy and the complementary energy upon all structures .

(iv) Use the translation method or similar variational methods to find the value of the minimized functional (or its lower bound) and use lamination technique to build a minimizing sequence (i.e. the optimal microstructure).

(v) Return to the original notations, formulate and numerically solve a relaxed variational problem 'in the large' to find the actual distribution of optimal microstructures throughout the body and the optimal distribution of volume fractions of the materials in microstructures.

4 Optimal structures in planar elasticity

4.1 The minimal problem

Clearly, the outlined procedure can be applied for the problems of elasticity as well.

Again, we consider a local min-max problem

$$\min_{C} \min_{u} \max_{v} \int_{\Omega} u : C : v \; dx \tag{4.1}$$

where $u,\ v$ are the symmetric normalized ($u : u = v : v = 1$) two by two tensors of strains. As before, we transform (4.1) by introducing new variables.

$$\epsilon = u + v, \quad \epsilon' = u - v. \tag{4.2}$$

The new form of the problem is (compare with the conductivity problem)

$$\min_{C} \min_{\epsilon} \max_{\epsilon'} \int_{\Omega} \epsilon : C : \epsilon - \epsilon' : C : \epsilon'. \tag{4.3}$$

Again, the problem is to find such a structure of composite which minimizes the difference of weighted energy density caused by two orthogonal strain fields ϵ and ϵ'.

The transformation of this problem to the minimal one leads to the following problem: *Find the microstructure of a composite, assembled from two given materials, which minimizes the sum of stored energy, generated by some uniform strain ϵ and the complementary energy, generated by an orthogonal uniform stress $\sigma = C : \epsilon'$:*

$$J^{el} = \min_{\epsilon} \min_{\sigma} \min_{C \in \mathcal{C}} [-2 < \sigma >:< \epsilon' > + \int_{\Omega} W^{el}(\epsilon, \sigma, C) \; dx] \tag{4.4}$$

$$W^{el}(\epsilon, \sigma, C) = \epsilon : C(x)^{-1} : \epsilon + \sigma : C(x) : \sigma \tag{4.5}$$

In other words, the optimal microstructure must minimize the sum of its rigidity under one loading and its compliance under another orthogonal loading.

This time the Reiss-Voigt inequalities provide only bounds of the sum of the values of stored energy but not fine bounds of this sum. The more accurate methods could be applied here like the translation bounds or the Hashin-Shtrikman type bounds . These bounds have been used by Gibiansky and Cherkaev [13] and by Allaire and Kohn [1] to get the exact value for each term of our bound, i.e to get the exact separate bounds for the densities of the energy and the complementary energy. The bounds depend of the ratio of the bulk and shear parts of the external loadings ϵ and σ. As opposed to the Reiss-Voigt bounds, the translation method also provide the coupled bounds of the maximal rigidity and the maximal compliance in two orthogonal fields, as was demonstrated in the recent paper (Cherkaev and Gibiansky [8]).

4.2 Modified translation bounds

Here, we cannot apply directly the technique of the well-known translation bounds. The difficulty is connected with the problem of finding quasiconvex functions in the space of strains - a subspace of the gradients. In order to find such functions, we need to pass to the bigger space. In fact, we use the approach described in [8] in the contest of the G-closure problem.

First, we introduce a enlarged space strain and stress tensors. Namely, we consider the energy $\epsilon : C : \epsilon$ as a quadratic form in the four-dimensional space of gradients of the displacements:

$$\epsilon : C\epsilon = \nu : C^{\cap} : \nu, \quad \nu = \nabla u \quad (\epsilon = (\nu)^s \tag{4.6}$$

Let us introduce a basis in the space of two by two tensors:

$$\sigma = \sum_{i=1}^{3} \sigma_i a_i, \quad \nu = \sum_{i=1}^{4} \nu_i a_i \tag{4.7}$$

where

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; a_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; a_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; a_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.8}$$

The triplet $[\sigma, \nu] = [\sigma, \epsilon, h] = z$ is described as a seven-dimensional vector in $L_2(\mathcal{O})$, the divergence-free nature of σ and the curl-free nature of ν imply additional restrictions of the type (see,for example,[40, 10])

$$A_{ijk} \frac{\partial z_j}{\partial x_k} = 0 \tag{4.9}$$

Now we are going to apply the translation method to bound the quadratic form

$$z \begin{pmatrix} C^{-1} & 0 \\ 0 & C^{\cap} \end{pmatrix} z \tag{4.10}$$

associated with the above block-diagonal matrix. The four by four tensor $C^{\cap}$ for have the block representations:

$$C^{\cap} = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} \tag{4.11}$$

it has the fourth row and column equal to zero.

Then we can find the quasiaffine quadratic functions in the introduced space. Referring to the Cherkaev and Gibiansky [8] we note that to the following bilinear forms are quasiaffine:

$$\sigma T_1 \sigma, \quad \sigma T_2 \nu, \quad \nu T_3 \nu, \tag{4.12}$$

where $T_k,\ k=1,2,3$ are translation matrices, depended of several free parameters t_i. These matrices, found in [8], are equal to:

$$T_1=\begin{pmatrix} t_1 & 0 & 0\\ 0 & -t_1 & 0\\ 0 & 0 & -t_1\end{pmatrix},\ T_2=\begin{pmatrix} 0 & t_3 & t_4 & t_5\\ t_3 & 0 & -t_5 & -t_4\\ t_4 & t_5 & 0 & t_3\end{pmatrix},\ T_3=\begin{pmatrix} t_2 & 0 & 0 & 0\\ 0 & -t_2 & 0 & 0\\ 0 & 0 & -t_2 & 0\\ 0 & 0 & 0 & t_2\end{pmatrix}, \tag{4.13}$$

The set $\sqcup$ is defined by the requirements

$$\mathcal{U}=\{t_i:\begin{pmatrix} C_i+T_1 & T_3\\ T_3 & C_i^{-1}+T_2\end{pmatrix}\geq 0,\quad i=1,2.\} \tag{4.14}$$

The characteristic feature of T_i is the quasiafinness [10] of the corresponding quadratic form:

$$<z\cdot\begin{pmatrix} T_1 & T_3\\ T_3 & T_2\end{pmatrix}\cdot z>=<z>\cdot\begin{pmatrix} T_1 & T_3\\ T_3 & T_2\end{pmatrix}\cdot<z>\quad \forall t_i\in\sqcup, z\in(4.9) \tag{4.15}$$

The standard translation bound has the form:

$$W^{el}(\epsilon,\sigma,C)\geq\max_{t_i\in\sqcup}\ <z>\cdot P(t_i)\cdot<z> \tag{4.16}$$

where $[z=\sigma,\epsilon,h]$ ia the seven-dimensional vector of components of strains, stresses, and rotation:

$$z=[\sigma,\ \epsilon,\ h], \tag{4.17}$$

P is a seven by by matrix of the block form

$$P=\sum_{i=1}^{2}[m_i\begin{pmatrix} C_i^{-1}+T_1 & T_3\\ T_3 & C_i+T_2\end{pmatrix}^{-1}]^{-1}-\begin{pmatrix} T_1 & T_3\\ T_3 & T_2\end{pmatrix} \tag{4.18}$$

The bound depends on the mean values of stress and strain as well as on an undetermined parameter $h=(\nabla u)^A$ which means the rotation of the elastic element. To clarify this dependence, let us rewrite the expression zPz as

$$zPz=\sigma,\epsilon P'\sigma,\epsilon+2\sigma,\epsilon Kh+Hh^2 \tag{4.19}$$

where

$$P=\begin{pmatrix} P' & K\\ K^T & H\end{pmatrix}, \tag{4.20}$$

P' is the left upper 6 by 6 minor of P, K is the last column of P without the last element:$K=[P_{17},..P_{67}]$, $H=P_{77}$.

We do not know the value of $h = (\nabla u)^A$, and we want to eliminate the dependence of it in our bounds. To do it, let us enforce the bound supposing that h minimize the form zPz:

$$\min_h zPz = (\sigma, \epsilon) \cdot (P' - (1/H)KK^T) \cdot (\sigma, \epsilon) \tag{4.21}$$

The final bound takes the form

$$W \geq \max_{t \in \mathcal{T}} < (\sigma, \epsilon) > \cdot (P' - (1/H)KK^T) \cdot < (\sigma, \epsilon) > \tag{4.22}$$

Note that this bound has a more complicated structure that the standard translation bound because of the presence of the term $(1/H)KK^T$: unless $K \neq 0$ the bound looks different: the volume fractions of materials and the translation parameters come into the bound in the more complicated way through the elements $H(m, t, C_i)$, $K(m, t, C_i)$.

4.3 Some optimal structures

As we mentioned above, the pair of orthogonal tensors depend on three rotation-invariant parameters and by the volume fraction m_1. One can choose, for example, the traces of each tensor and the ratio between their magnitudes as such a parameters. We should expect that the parameters of microstructures will be in correspondence with these parameters, so they should form a three dimensional manifold.

Although the full picture of optimal structure is not clarified yet, we could describe some cross-section of the manifold, which correspond to several additional assumptions on the acting fields.

Firstly, we know already the structures which minimize the stored energy in the field ϵ (i) or in the field σ (ii), which means that we know the solution of (4.5) in two limiting cases.

$$u = v \quad \text{or} \quad \epsilon = 0 \tag{4.23}$$

(the first self-adjoint problem) and

$$u = -v \quad \text{or} \quad \epsilon' = 0 \tag{4.24}$$

(the second self-adjoint problem). To simplify the discussion, we assume here that materials can be ordered as following

$$C_1 \leq C_2. \tag{4.25}$$

The known structures of elastic composites which minimize the stored energy in the field σ are the so-called 'matrix laminates of second rank' [32]. The structures optimal for the problem (4.23) are obtained as laminates (with normal n_1)

assembled from the first material C_1 and homogenized laminates with normal n_2 made from first C_1 and second C_2 material (see fig. 2). We see that the material C_2 is situated in inclusions, and the material C_1 forms an envelope around these inclusions. The relative concentrations of the first material in laminates of the first and second rank ν_1, ν_2 can be changed assuming that the total volume fractions of materials should be kept constant.

The set of structures which gives the solution in the second case (4.24) is also known. This time the problem is reduced to the problem of minimization of the complementary energy which was solved in [13, 14]; the optimal microstructures are again the second rank matrix laminates, but now the material C_2 forms the envelope and the material C_1 is situated in inclusions.

The effective compliance tensor C^{2r} of second rank laminates (the material C_1 forms the envelope) has the form [12, 14]

$$C^{2r} = C_1 + m_2[(C_2 - C_1)^{-1} + c\sum_{i=1}^{2} \nu_i n_i \otimes n_i \otimes n_i \otimes n_i]^{-1} \qquad (4.26)$$

where C_1 and C_2 are compliances of two initially given isotropic materials. The constant c is equal to

$$c = \frac{m_1}{m_2(n_i \otimes n_i : C_1 : n_i \otimes n_i)} = \text{const} \quad \forall\ i; \qquad (4.27)$$

it is independent of the direction of the unit vector n_i because of isotropy of the tensor C. The parameters ν_i are nonnegative and normalized:

$$\nu_1 + \nu_2 = 1, \quad \nu_1 \geq 0, \quad \nu_2 \geq 0. \qquad (4.28)$$

It was proved in [13] (see also [14, 1]) that second-rank laminates correspond to the optimal rigidity if the three following conditions hold:
(i) the vectors n_1 and n_2 are orthogonal:

$$n_1 \cdot n_2 = 0 \qquad (4.29)$$

(ii) they are oriented along principle axis of the tensor σ
(iii) the relative concentrations ν_i are chosen in a proper way, depending on the ratio of eigenvalues of σ.

Note, the value of the functional calculated on these structures corresponds to the bounds (4.22), if the matrices T_i are chosen in such a way that

$$T_2 = T_3 = 0. \qquad (4.30)$$

Naturally this choice leads to the translation bounds of traditional structure.

Let us demonstrate now that the described classes of structures give a solution of a structural optimization problem for larger class of values of parameters then

(4.23), (4.24). We will show that translation matrices (4.30) gives the exact bound of the functional (?) for a two-dimensional manifold of parameters of the invariants of the stress and strain tensors. Indeed, they lead to the bounds

$$W \geq < \sigma >: C \;\; < \sigma > + < \epsilon >: (m_1 C_1 + m_2 C_2)^{-1} :< \epsilon > \tag{4.31}$$

where

$$C \;\; = ((m_1 C_1 + T_1)_{-1} + m_2 (C_2 + T_1)_{-1})^{-1} \tag{4.32}$$

Let us calculate the value of functional corresponded to the second-rank laminate structures. Note that the second-rank laminates (4.26) possess *maximum* compliance in the trace free tensor 'direction' $\mathbf{z}$:

$$\mathbf{z} = \frac{1}{\sqrt{2}} (n_1 \otimes n_2 + n_2 \otimes n_1), \quad ||\mathbf{z}|| = 1. \tag{4.33}$$

Because of the equalities

$$\mathbf{z} : n_1 \otimes n_1 = \mathbf{z} : n_2 \otimes n_2 = 0 \tag{4.34}$$

this compliance is equal to:

$$\mathbf{z} : C^{2r} : \mathbf{z} = \mathbf{z} : (m_1 C_1 + m_2 C_2) : \mathbf{z} \tag{4.35}$$

no matter what the values of parameters ν_i are. This means, that matrix laminates store minimum of complementary energy in a trace free strain field ϵ. In the same time, as it has been mentioned above, they store minimum of the elastic energy (equal to the $\sigma : C \;\; : \sigma$) in an arbitrary field σ if the parameters ν_i are chosen in an appropriate way and if σ is orthogonal to ϵ. Therefore they provide a solution of the general problem (4.5) if the field ϵ is trace free but the field σ is arbitrary.

An other type of optimal structures are simple laminates. Indeed, they show minimum compliance in the uniaxial stress field along the layers and minimum rigidity in all orthogonal tensor 'directions'. Therefore, laminates are optimal if the field σ is uniaxial. The energy stored in the laminates agrees with the bounds 4.21 in which it is set

$$T_1 = T_2 = 0. \tag{4.36}$$

For example, laminates assembled from materials with zero Poison ratio provide maximum shear strain under hydrostatic stress. Indeed, we suppose that the structure is placed into uniform strain field a' equal to

$$a' = \mathbf{i} \otimes \mathbf{i} + \mathbf{j} \otimes \mathbf{j} \tag{4.37}$$

and the minimizing strain is

$$b' = \mathbf{i} \otimes \mathbf{i} - \mathbf{j} \otimes \mathbf{j}, \tag{4.38}$$

(This means, that the strain component along $-b'$ is maximized). We have then

$$\epsilon = \mathbf{i} \otimes \mathbf{i}, \quad \epsilon' = \mathbf{j} \otimes \mathbf{j} \tag{4.39}$$

The best structure is laminates oriented along $\mathbf{j}$ axis. Indeed, the stress field $\sigma = C^{-1}\epsilon'$ is uniaxial and the compliance is minimal along this direction.

It easy to provide physical arguments to see why this structure is optimal here. Indeed, a square piece of laminated material, being extremely anisotropic, deforms under hydrostatic pressure into a rectangle with maximal ratio between its sides.

4.4 Asymptotic. Optimal structures.

Here we are going to discuss a class of composite structures which solve another special case of the problem. This time we pass to an asymptotic: we suppose that microstructures are assembled from two materials, one of them possesses infinitely small rigidity, and the other possesses infinitely big rigidity.

$$||C_1|| \to 0, \quad ||C_2^{-1}|| \to 0. \tag{4.40}$$

The bounds of the energy and the complementary energy became a trivial form:

$$I = \int_\Omega [\sigma : C_*(x) : \sigma + \epsilon : C_*^{-1}(x) : \epsilon] \; dx \geq 0. \tag{4.41}$$

We are going not to use the translation bounds directly; instead we refer to structures with the effective tensor C_{opt} which possess zero rigidity by applying any stress field σ and zero compliance (or infinite rigidity) by applying any other strain field ϵ orthogonal to σ; for these structures the functional (4.41) vanishes:

$$\sigma : C_{opt} : \sigma = 0, \quad \epsilon : C_{opt}(x) : \epsilon = 0 \tag{4.42}$$

The microstructures with the required properties have been found in the paper by Milton and Cherkaev [9] (following the approach suggested earlier by Milton [37]) where even more general problem was solved: the variety of asymptotic microstructures has been described which corresponds to any positively defined effective tensor of complience.

Note, that the volume fractions of materials in a mixture is of no importance in the asymptotic case and the problem of optimal structures becomes a purely geometrical one. Physically this follows from the fact that the infinitely thin element of structure made from absolutely rigid or absolutely soft material may make the structure's effective compliance infinitely big or infinitely small no matter what the volume fraction of this element is.

It is shown in [9] that the sought structures are either the matrix second rank composites with arbitrary orientation of the normal n_1 and n_2 (Fig. 2) to layers (the rigid material form the envelope, and the soft one form the kernel) or 'herring-bone' structures made of them.

Indeed, the matrix second rank composites possess the complience tensor with two infinite eigenvalues and with one zero eigenvalue. Infinite complience corresponds to the directions

$$a_1 = n_1 \otimes n_1, \qquad a_2 = n_2 \otimes n_2. \tag{4.43}$$

Zero complience corresponds to the eigentensor $\mathbf{z}$, orthogonal to both these tensors. This is the tensor which main axis bisects the direction of n_1 and n_2¿. It has the following representation.

$$\mathbf{z} = \begin{pmatrix} \cos\psi & 0 \\ 0 & \sin\psi \end{pmatrix} \tag{4.44}$$

in the main axes.

The parameter ψ depends on the angle ϕ between normals of layers of first and second ranks as following:

$$\tan\psi = -\tan^2\phi. \tag{4.45}$$

If the angle ϕ is varied in the interval $[0,\ \pi/2]$ then the variety of eigentensors include every tensor with the non-positive determinant:

$$\det \mathbf{z} = \sin\psi\cos\psi = -tan^2\phi \sin^2\psi \le 0. \tag{4.46}$$

This shows, that the matrix second rank laminates structure has infinite compliance (zero resistance) in a direction $\mathbf{z}$ and it has zero compliance in all orthogonal directions.

Clearly a tensor of this class corresponds to zero value of the functional (4.5) if it is oriented in such a way that the tensor $\mathbf{z}$ is proportional to the field ϵ, the orthogonal field σ obviously corresponds to infinite rigidity. Note that the described structures provide solutions of optimal problem only for the points where the tensor ϵ has non-positive determinant.

To find structures which complete the set of optimal composites is is enough to refer the construction suggested by Milton in [37]. Following his approach [37] let us consider a 'herring-bones structure': a laminate structure assembled from differently oriented second rank laminates (described above) which we treat as given materials.

These structures possess additional parameters which should be chosen in appropriate way. It is easy to shown following Milton [37] that the variety of eigentensors $\mathbf{z}$ formed by varying the parameters of a herring bones structure includes all possible normalized symmetric tensors, both with positive and negative determinants.

One can be sure that this class contains an element which solve the problem (4.5) because the sum of the stored energy and the complementary energy in a properly oriented structure of this kind is clearly zero.

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