# Stable optimal design of two-dimensional elastic structures

by

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**Abstract:** We study optimal layout of piece-wise periodic structures of linearly elastic materials. The effective tensors of these structures are constant within pre-specified regions, the optimality is understood as the minimum of complementary energy. The suggested formulation leads to a construction that is stable under variation of the loading and which does not degenerates into checkerboard type structures. We derive necessary conditions of optimality of such layouts and analyze them. Numerically, we find optimal structures for a number of examples, which are analyzed.

**Keywords:** structural optimization, composites, optimality conditions, optimal design.

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## 1. Introduction

The present paper deals with optimization of planar structures assembled from two isotropic materials characterized by different values of their elastic moduli. Given the amount of the two materials, the applied loads, the possible supports and a planar design domain $\Omega$, we seek the distribution of material in $\Omega$ that determines a structure with maximum integral stiffness. Maximization of the integral stiffness is equivalent to minimization of the total complementary energy, and thus expressing the equilibrium problem for the structure using the principle of minimum total complementary energy, we arrive at the following formulation for the maximum stiffness design problem

$$\min_{\boldsymbol{C}(\boldsymbol{x})\in\boldsymbol{C}_{ad}} \quad \min_{\boldsymbol{\sigma}(\boldsymbol{x})\in\boldsymbol{S}(\boldsymbol{x})} \quad \frac{1}{2}\int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{x}) : \boldsymbol{C}(\boldsymbol{x}) : \boldsymbol{\sigma}(\boldsymbol{x}) d\Omega \tag{1}$$

where $\boldsymbol{C}(\boldsymbol{x})$ is a fourth-rank material compliance tensor, $\boldsymbol{C}_{ad}$ a set of admissible compliance tensors, and where the inner minimization with respect to the stresses $\boldsymbol{\sigma}$ is taken over the set $\boldsymbol{S}$ of statically admissible stress fields,

$$\boldsymbol{S}(\boldsymbol{x}) = \{\ \boldsymbol{\sigma}(\boldsymbol{x}) \,|\, div\boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{p}(\boldsymbol{x}) = 0 \ in\ \Omega, \ \ \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{t}(\boldsymbol{x}) \ on\ \Gamma_T\ \} \tag{2}$$

with $\boldsymbol{p}$ and $\boldsymbol{t}$ as body forces and surfaces tractions, $\boldsymbol{n}$ as the normal vector to the surface, and $\Gamma_T$ as the traction part of the surface.

In situations where the material properties are allowed to change from point to point in structure we generally have the discrete choice of either "*Material 1*" or "*Material 2*" at each point of the structure and thus, denoting the compliance tensors for the two isotropic materials by $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ ($\boldsymbol{C}_1 < \boldsymbol{C}_2$), $\boldsymbol{C}_{ad}$ becomes expressed by

$$\boldsymbol{C}(\boldsymbol{x}) = \chi(\boldsymbol{x})\boldsymbol{C}_1 + (1 - \chi(\boldsymbol{x}))\,\boldsymbol{C}_2 \quad \textit{with} \quad \chi(\boldsymbol{x}) = \begin{cases} 1 \text{ if } \boldsymbol{x} \in \Omega_1 \\ \\ 0 \text{ if } \boldsymbol{x} \in \Omega_2 \end{cases} \tag{3}$$

giving the following formulation for the maximum stiffness design problem

$$\begin{aligned} &\min_{\chi(\boldsymbol{x})} \quad \min_{\boldsymbol{\sigma}(\boldsymbol{x}) \in \boldsymbol{S}(\boldsymbol{x})} \quad \frac{1}{2}\int_\Omega \boldsymbol{\sigma}(\boldsymbol{x}) : (\chi(\boldsymbol{x})\boldsymbol{C}_1 + (1-\chi(\boldsymbol{x}))\boldsymbol{C}_2) : \boldsymbol{\sigma}(\boldsymbol{x}) d\Omega \\ &\text{s.t. :} \quad \int_\Omega \chi(\boldsymbol{x}) d\Omega = V \end{aligned} \tag{4}$$

Here the constraint $\int_\Omega \chi(\boldsymbol{x}) d\Omega = V$ has been added to the formulation in order to avoid the trivial solution in which the stiffer material $\boldsymbol{C}_1$ is used everywhere.

## Class of optimum microstructures

A numerical solution of an optimization problem generally requires a finite dimensional approximation of it. We face here the following problem. The optimum material distribution is known to be characterized by an infinitely often alternating sequence of domains occupied by each of the two materials. A *well-posed* formulation of the optimization problem may then be obtained e.g. by extending the set of admissible compliance tensors $\boldsymbol{C}_{ad}$. Materials with an optimum composite microstructure assembled from the base materials should be added to the class of available materials.

For the design of structures of maximal rigidity it is known that optimal microstructures are found within the class of finite-rank matrix-layered composites see Gibiansky, Cherkaev (1997a), Avellaneda (1987), Avellaneda, Milton (1989), Lipton (1993), Diaz, Lipton, Soto (1994). For maximum stiffness design problems in plane elasticity optimum microstructures are second-rank and third-rank matrix-layered composites. Here, second-rank microstructures should be used in situations where we want maximum rigidity against a single loading, see Gibiansky, Cherkaev (1997a), while third-rank microstructures should be used in situations where we want maximum rigidity against several independent stress fields or against an over domain varying macroscopic stress field (or both), see Avellaneda (1987), Avellaneda, Milton (1989).

Effective properties of matrix-layered composites are given by simple analytic functions of the structural parameters describing their composition. The effective compliance tensor for the class of planar matrix-layered composites of given rank N is defined by

$$\boldsymbol{C}^{lamN} = \boldsymbol{C}_1 - (1-\rho)\left( (\boldsymbol{C}_1 - \boldsymbol{C}_2)^{-1} - \rho E_1 \sum_{n=1}^{N} p_n (\boldsymbol{t}_n \otimes \boldsymbol{t}_n) \otimes (\boldsymbol{t}_n \otimes \boldsymbol{t}_n) \right)^{-1} \tag{5}$$

in which all $p_n \geq 0$, and $\sum_{n=1}^{N} p_n = 1$, see Gibiansky, Cherkaev (1997b). In this formula, $\boldsymbol{C}_1$, $\boldsymbol{C}_2$ represent the compliance tensors for our isotropic base materials, $E_1$ the Young's modulus for the stiffer material $\boldsymbol{C}_1$, $\rho$ the volume fraction of the stiffer material in the microstructure, and $p_n$ and $\boldsymbol{t}_n$ are the relative layer thicknesses and the tangent vectors to the layers, respectively. In (5), the dyadic product $\boldsymbol{G} = \boldsymbol{a} \otimes \boldsymbol{b}$ of two vectors $\boldsymbol{a} = [a_1, \ldots, a_n]$ and $\boldsymbol{b} = [b_1, \ldots, b_n]$ is defined as a second rank tensor (matrix) with elements $G_{ij} = a_i b_j$. Similarly, the dyadic product of two second-rank tensors is defined as a fourth-rank tensor, and so on.

In particular, the case of $\boldsymbol{C}_2 = \infty$ corresponds to a material weakened by a system of infinitesimal holes. In this case, the effective compliance tensor $\boldsymbol{C}^{lamN}$ (5) for matrix-layered composite may be written as

$$\boldsymbol{C}^{lamN} = \boldsymbol{C}_1 + \boldsymbol{\Delta}^{-1} \tag{6}$$

where

$$\boldsymbol{\Delta} = \frac{\rho E_1}{1-\rho} \sum_{n=1}^{N} p_n (\boldsymbol{t}_n \otimes \boldsymbol{t}_n) \otimes (\boldsymbol{t}_n \otimes \boldsymbol{t}_n) \tag{7}$$

To describe the involved fourth-rank tensors, it is convenient to introduce the following orthogonal basis of second-rank tensors

$$\boldsymbol{a}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \boldsymbol{a}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{a}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{8}$$

In this basis, any symmetric second-rank tensor is represented as a vector, and any symmetric fourth-rank tensor as a symmetric three by three matrix.

Particularly, the fourth-rank tensor $\boldsymbol{\Delta}$ in (7) may be rewritten as

$$\boldsymbol{\Delta} = \frac{\rho E_1}{2(1-\rho)} \sum_{k=1}^{3} p_k \begin{bmatrix} \dfrac{1+\cos(4\theta_k)}{2} & \dfrac{\sin(4\theta_k)}{2} & -\cos(2\theta_k) \\ & \dfrac{1-\cos(4\theta_k)}{2} & -\sin(2\theta_k) \\ & & 1 \end{bmatrix} \tag{9}$$

Finally we note that the matrix $\boldsymbol{\Delta}$ in (9) has two linear invariants called the spherical trace $\mathbf{Tr_s}$ and the deviatoric trace $\mathbf{Tr_d}$. These quantities are defined as

$$\mathbf{Tr_s}\ \boldsymbol{\Delta} = [\Delta_{33}] = \frac{\rho E_1}{2(1-\rho)} \tag{10}$$

and

$$\mathbf{Tr_d}\ \boldsymbol{\Delta} = [\Delta_{11}] + [\Delta_{22}] = \frac{\rho E_1}{2(1-\rho)} \tag{11}$$

Further on, we will use the notation

$$c = \frac{\rho E_1}{2(1-\rho)}. \tag{12}$$

Furthermore, the inverse statement is also true:

Any matrix $\boldsymbol{\Delta}$ that

(1) is strongly positive definite: $\det \boldsymbol{\Delta} > 0$

(2) has two linear invariants as in ( 10, 11),

admits the representation (9), (see Avellaneda, Milton, 1989).

### 1.1. Optimal design problem

Application of (5) allow us to obtain a parameterization of an extended maximum stiffness material in (4) which definitely has a solution. Such a formulation is obtained by considering a division of the design domain $\Omega$ into a finite number of sub-domains $\Omega_i$, $i = 1, \ldots, I$, and allow an optimal matrix-layered composite to be generated within each of the sub-domains, i.e. by considering the following finite parameter approximation to the problem in (4)

$$\begin{aligned} &\min_{\boldsymbol{C}_i \in \boldsymbol{C}^{lamN}} \ \min_{\boldsymbol{\sigma}(\boldsymbol{x}) \in \boldsymbol{S}(\boldsymbol{x})} \ \sum_{i=1}^{I} \frac{1}{2} \int_{\Omega_i} \boldsymbol{\sigma}(\boldsymbol{x}) : \boldsymbol{C}_i : \boldsymbol{\sigma}(\boldsymbol{x}) d\Omega \\ &\text{s.t.}: \ \sum_{i=1}^{I} \rho^i v^i = V \end{aligned} \tag{13}$$

Allowing the density variables $\rho^i$ in (5) to vary continuously between 0 and 1, we obtain a formulation where within each of the sub-domains $\Omega_i$ we can generate either: pure "material 1", pure "material 2" or an optimum composite mixture thereof. Taking the size of the sub-domains $\Omega_i$ to correspond to the grid size in a finite element mesh and taking the second material in the matrix-layered composite as a very compliant material (representing void), above formulation $ha$ is zero, then the compliance tensor of the structure is finite, even if the compliance of the enveloped material $\boldsymbol{C}_2$ is infinite, and therefore they are more stable with respect to the change in the loading and with respect to the error in their orientation. Second-rank laminates as well as simple laminates correspond to the degeneration of the high-rank composites. Roughly speaking, the layers of strong material form triangles instead of rectangles, which provide the uniform rigidity to the microstructures.

A natural generalization of (13) can be written as

$$\begin{aligned} &\min_{\boldsymbol{C}_i \in \boldsymbol{C}^{lam3}} \quad \min_{\boldsymbol{\sigma}_j(\boldsymbol{x}) \in \boldsymbol{S}_j(\boldsymbol{x})} \quad \sum_{j=1}^{J}\sum_{i=1}^{I}\frac{1}{2}\int_{\Omega_i}\boldsymbol{\sigma}_j(\boldsymbol{x}) : \boldsymbol{C}_i : \boldsymbol{\sigma}_j(\boldsymbol{x})d\Omega \\ &\text{s.t.}: \quad \sum_{i=1}^{I}\rho^i v^i = V \end{aligned} \tag{14}$$

for $i = 1, \ldots, I$. In this formulation, the structure is supposed to have several loadings $\rho_1, \rho_2, \cdots, \rho_J$.

## 2. The formulation

Considering the single load case formulation in (13) and the multiple load case formulation in (14) we see that both problems involve minimization of a sum of integrals of complementary energy density over the sub-domains of the structure.

We rewrite the energy density $\boldsymbol{\sigma} : \boldsymbol{C} : \boldsymbol{\sigma}$ as

$$\boldsymbol{\sigma} : \boldsymbol{C} : \boldsymbol{\sigma} = \mathbf{Tr}(\boldsymbol{C} \cdot \boldsymbol{Z}) \tag{15}$$

where $\boldsymbol{Z}$ is a three by three matrix corresponding to the fourth-rank tensor $\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}$, or as tensor notation $Z_{\alpha\beta\kappa\gamma} = \sigma_{\alpha\beta}\sigma_{\kappa\gamma}$ in the basis given in (8).

This formulation is especially convenient to deal with multiple load case and with the case of domain-wise constant material properties. Indeed, introducing the 3x3 matrices $\boldsymbol{Z}_i$ which collect the information about the stress fields in the sub-domains $\Omega_i$, the multiple load case problem in (14) may be restated as

$$\begin{aligned} &\min_{\boldsymbol{C}_i \in \boldsymbol{C}^{lam3}} \quad \min_{\boldsymbol{\sigma}(\boldsymbol{x}) \in \boldsymbol{S}(\boldsymbol{x})} \quad \mathbf{Tr}\left(\sum_{i=1}^{I}\boldsymbol{C}_i \cdot \boldsymbol{Z}_i\right) \quad , \quad i = 1, \ldots, I \\ &\text{s.t.}: \quad \sum_{i=1}^{I}\rho^i v^i = V \end{aligned} \tag{16}$$

where the matrices $\boldsymbol{Z}_i$ correspond to positive definite fourth-rank tensors $\boldsymbol{Z}_i = Z^i_{\alpha\beta\kappa\gamma}$ defined by

$$\boldsymbol{Z}_i = \frac{1}{2}\sum_{j=1}^{J}\int_{\Omega_i}\boldsymbol{\sigma}_j(\boldsymbol{x}) \otimes \boldsymbol{\sigma}_j(\boldsymbol{x})d\Omega \tag{17}$$

Any positive symmetric definite fourth-rank tensor $\boldsymbol{Z}_i$ admits a representation via its non-negative eigenvalues $\zeta^k$ and its corresponding second order eigentensors $\boldsymbol{\alpha}^k = \alpha^k_{\alpha\beta}$, i.e.,

$$\boldsymbol{Z} = \sum_{k=1}^{3}\zeta^k\boldsymbol{\alpha}^k \otimes \boldsymbol{\alpha}^k, \quad \zeta^k > 0, \tag{18}$$

where $\boldsymbol{\alpha}^i : \boldsymbol{\alpha}^j = \delta_{ij}$. The tensor $\boldsymbol{Z}$ is a second order homogeneous function of the stresses. It is convenient to introduce the square root of this tensor as a positive symmetric fourth-rank tensor $\boldsymbol{S}$ such that $\boldsymbol{Z} = S \cdot S^T$.

Let $\lambda_k$ be eigenvalue of $S$. Then the energy density becomes

$$\mathbf{Tr}(\boldsymbol{Z} \cdot \boldsymbol{C}) = \sum_{k=1}^{3} (\lambda_k \boldsymbol{\alpha}_k) : \boldsymbol{C} : (\lambda_k^* \boldsymbol{\alpha}_k^T) \tag{19}$$

The minimizing quantity in (16) can be seen as a sum of complementary energy densities associated with the three orthogonal stress tensors $\boldsymbol{s}_i = \lambda_i \boldsymbol{\alpha}_i$, $k = 1, \ldots, 3$.

### 2.1. Necessary conditions

We use the above representations to derive necessary conditions of optimality. Let us formulate a local problem of the best microstructure of the composite. The volume fractions supposed to be fixed (they will be determined later).

The functional

$$W = \mathbf{Tr}(\boldsymbol{C} \cdot \boldsymbol{Z}) \tag{20}$$

is minimized with respect to the effective tensor of matrix laminates $\boldsymbol{C}$. The extremal tensor $\boldsymbol{C}$ corresponds to the minimal energy function for fixed volume of material. It is treated here as an arbitrary matrix of the form (5) subject to two linear constraints (11). By adding the last constraints with Lagrange multipliers $\mu_1$ and $\mu_2$, we get the extended Lagrangian

$$I = \mathbf{Tr}\ ((\boldsymbol{C}_1 + \boldsymbol{\Delta}^{-1}) \cdot \boldsymbol{Z}) + \mu_1\ \mathbf{Tr_s}\ \boldsymbol{\Delta} + \mu_2\ \mathbf{Tr_d}\ \boldsymbol{\Delta} \tag{21}$$

The stationary conditions include differentiation of the equation (21) with respect to the matrix $\boldsymbol{\Delta}$.

**Remark:** The derivative of a scalar function $\phi(\overline{\overline{\boldsymbol{A}}})$ with respect to the matrix $\overline{\overline{\boldsymbol{A}}}$ with components $[a_{ij}]$ is defined as a matrix $\overline{\overline{\boldsymbol{B}}}$ with components $[b_{ij}] = \dfrac{\partial \phi(\overline{\overline{\boldsymbol{A}}})}{\partial [a_{ji}]}$

We observe the following result for a matrix $\boldsymbol{\Delta} = [d_{ij}]_{3\times 3}$:

$$\frac{\partial}{\partial \boldsymbol{\Delta}}\ \mathbf{Tr}\ (\boldsymbol{\Delta}^{-1} \cdot \boldsymbol{Z}) = \left[ \frac{\partial}{\partial d_{ij}}\ \mathbf{Tr}\ (\boldsymbol{\Delta}^{-1} \cdot \boldsymbol{Z}) \right]_{3\times 3} \tag{22}$$

where

$$\frac{\partial}{\partial d_{ij}}\ \mathbf{Tr}\ (\boldsymbol{\Delta}^{-1} \boldsymbol{Z}) = -\ \mathbf{Tr}\ (\boldsymbol{S} \cdot \boldsymbol{\Delta}^{-2} \cdot \boldsymbol{S}^T \frac{\partial}{\partial d_{ij}} \boldsymbol{\Delta}) \tag{23}$$

and $\dfrac{\partial}{\partial d_{ij}}\boldsymbol{\Delta}$ is a matrix whose all entries are zero except that the $ij$-th entry is one. By using the definition of *matrix inner product*, (23) becomes

$$\frac{\partial}{\partial d_{ij}}\ \mathbf{Tr}\ (\boldsymbol{\Delta}^{-1}\cdot \boldsymbol{Z}) = -(\boldsymbol{S}\cdot\boldsymbol{\Delta}^{-2}\cdot\boldsymbol{S}^T) : \frac{\partial}{\partial d_{ij}}\boldsymbol{\Delta}. \tag{24}$$

The derivative of the remaining terms in (21) are

$$\begin{aligned}
\frac{\partial}{\partial\boldsymbol{\Delta}}\ \mathbf{Tr_s}\ \boldsymbol{\Delta} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\frac{\partial}{\partial\boldsymbol{\Delta}}\ \mathbf{Tr_d}\ \boldsymbol{\Delta} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{25}$$

Substituting (24) and (25) into (21) gives

$$\frac{\partial I}{\partial\boldsymbol{\Delta}} = \left(-(\boldsymbol{S}\quad\boldsymbol{\Delta}^{-2}\cdot\boldsymbol{S}^T) + \overline{\overline{\boldsymbol{A}}}^2\right) : \frac{\partial}{\partial d_{ij}}\boldsymbol{\Delta} = 0 \tag{26}$$

where

$$\overline{\overline{\boldsymbol{A}}} = diag[\sqrt{\mu_2}, \sqrt{\mu_2}, \sqrt{\mu_1}]. \tag{27}$$

The latter result is correct if and only if

$$-(\boldsymbol{S}\cdot\boldsymbol{\Delta}^{-1})\cdot(\boldsymbol{\Delta}^{-1}\cdot\boldsymbol{S})^{\mathbf{T}} + \overline{\overline{\boldsymbol{A}}}\cdot\overline{\overline{\boldsymbol{A}}}^{\mathbf{T}} = 0 \tag{28}$$

Thus, from the equation (28), we have the following equation for the optimal $\boldsymbol{\Delta}$ matrix

$$\boldsymbol{\Delta} = \boldsymbol{S}\cdot\overline{\overline{\boldsymbol{A}}}^{-1}. \tag{29}$$

Note that the positiveness and the symmetry of the involved matrices ensure the uniqueness of the square root of a matrix and therefore the uniqueness of this equation.

If the equations (29) and (27) together with (10) are used, it follows that

$$\sqrt{\mu_1} = \frac{1}{c}\ \mathbf{Tr_s}\ (\boldsymbol{S}) \tag{30}$$

and, similarly,

$$\sqrt{\mu_2} = \frac{1}{c}\ \mathbf{Tr_d}\ (\boldsymbol{S}) \tag{31}$$

which are the Lagrange multipliers needed in (21).

Finally, substitution of the Lagrange multipliers computed in (30-31) into (21) gives the minimal energy (for fixed volume fraction) as

$$W = \frac{1}{c}\left[\ \mathbf{Tr_s}\ {}^2(\boldsymbol{S}) + \ \mathbf{Tr_d}\ {}^2(\boldsymbol{S})\right] + \ \mathbf{Tr}\ (\boldsymbol{C}_1\boldsymbol{Z}). \tag{32}$$

### 2.1.1. Optimal volume fractions

To observe the dependence of the volume fraction, we rewrite the functional $I$ using and adding the cost $\gamma$ of material $\boldsymbol{C}_1$ Lagrange multiplier with the constraint (14);

$$I = \int_\Omega \left( B \frac{1-\rho_1}{\rho_1} \frac{1}{c} + \textbf{Tr}\ (\boldsymbol{C}_1 \cdot \boldsymbol{Z}) + \rho_1 \gamma \right) d\Omega \tag{33}$$

where $B = \textbf{Tr}_\textbf{s}{}^2(S) + \textbf{Tr}_\textbf{d}{}^2(S)$. The optimal material $\rho_1$ is determined by the Euler-Lagrange equation as

$$\rho_1^{opt} = \sqrt{\frac{B}{\gamma c^2}}$$

and the optimal energy density is

$$I(S) = \begin{cases} \left( (1+\frac{1}{c})\sqrt{B\gamma} - \frac{B}{c} + \textbf{Tr}\ (C_1 \cdot \boldsymbol{Z}) \right), & if \ \ \rho_1^{opt} < 1 \\ (\ \textbf{Tr}\ (C_1 \cdot \boldsymbol{Z}) + \gamma), & if \ \ \rho_1^{opt} = 1 \end{cases} \tag{34}$$

The last equality describes the quasi-convex envelope of the energy, that is the minimal energy stored in any body under given load. The specific energy stored in the material is equal to $I(S)/\rho(S)$.

## 3. Numerics

In this section we consider examples of optimal design of planar structures with domain-wise constant material properties. The design variables characterize third-rank matrix-layered composite within each of a number of prespecified sub-domains of the structure. The structures are determined from a condition of minimum total complementary energy. Therefore, we solve the optimization problems by means of design sensitivity analysis and a method of mathematical programming.

At this point we note that parametrization of the effective compliance tensor in terms of layer densities $p_n$ and layer orientations $\theta_n$, see (35), is known to lead to local minima of the total complementary energy, see, e.g., Pedersen (1989). For the solution of the present problem we therefore choose a parameterization of the effective compliance tensor in terms of the so-called moment variables, see, e.g., Francfort & Murat (1986) and Avellaneda & Milton (1989). This parameterization of the effective compliance tensor is, due to the convexity of the complementary energy in terms of the moments, perfectly suited for an iterative hierarchical design approach where the global distribution of material (the $\rho$ parameters) is improved in an outer loop while the optimal configuration of the microstructures are found numerically as solutions to a set of inner optimization problems in the moment variables. For a discussion of the convexity properties of the moment formulation the reader is referred to Lipton (1993), while examples on application of moment formulations to solution of topology optimization problems can be found in Lipton & Soto (1995); Lipton & Diaz (1994); Krog & Olhoff (1997).

For the sake of completeness we shall now first give a description of the moment formulation, before considering examples of optimal material design for structures with domain-wise constant material properties.

### 3.1. The moment formulation

A parameterization of the effective compliance tensor in terms of moments is obtained by applying the following variable substitutions to (5)

$$\begin{aligned} m_1 &= \sum_{n=1}^{N} p_n cos(2\theta_n) \quad ; \qquad m_2 = \sum_{n=1}^{N} p_n sin(2\theta_n) \\ m_3 &= \sum_{n=1}^{N} p_n cos(4\theta_n) \quad ; \qquad m_4 = \sum_{n=1}^{N} p_n sin(4\theta_n) \end{aligned} \tag{35}$$

whereby the effective compliance matrix in (5) becomes

$$\boldsymbol{C} = \boldsymbol{C}_1 - (1-\rho)\left((\boldsymbol{C}_1 - \boldsymbol{C}_2)^{-1} - \rho E_1 \boldsymbol{M}\right)^{-1} \tag{36}$$

with

$$\boldsymbol{M} = \frac{1}{2}\begin{bmatrix} \frac{1+m_3}{2} & \frac{m_4}{2} & -m_1 \\ & \frac{1-m_3}{2} & -m_2 \\ & & 1 \end{bmatrix} \tag{37}$$

The moment variables $\boldsymbol{m} = \{m_1, \ldots, m_4\}$ fulfill certain conditions. This set of conditions is easily established considering the case where we range over all possible layer directions. In this case the moments become

$$\begin{aligned} m_1 &= \int_0^{2\pi} p(\theta) cos(2\theta) d\theta \quad ; \qquad m_2 = \int_0^{2\pi} p(\theta) sin(2\theta) d\theta \\ m_3 &= \int_0^{2\pi} p(\theta) cos(4\theta) d\theta \quad ; \qquad m_4 = \int_{0pi}^{2\pi} p(\theta) sin(4\theta) d\theta \end{aligned} \tag{38}$$

With this definition of the moments and the solution to the trigonometric moment problem, see Krein & Nudelmann (1977), we easily get the set of feasible

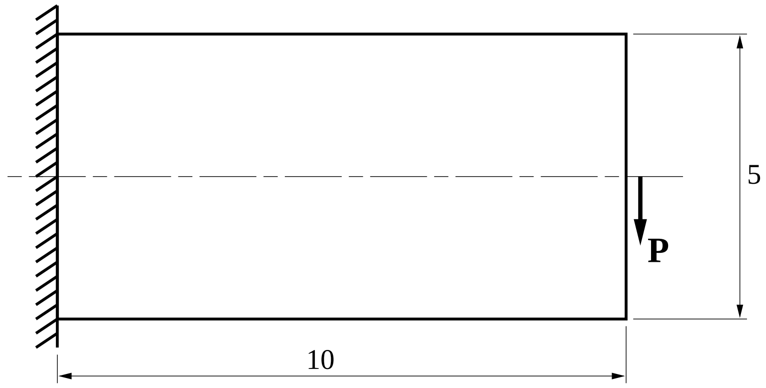



Figure 1. Design domain, load and boundary conditions

moments

$$H = \left\{ \boldsymbol{m} \in \Re^4 \;\middle|\; \begin{array}{c} m_1^2 + m_2^2 \leq 1 \quad , \quad -1 \leq m_3 \leq 1 \ , \\ \dfrac{2m_1^2}{1+m_3} + \dfrac{2m_2^2}{1-m_3} + \dfrac{m_4^2 - 4m_1 m_2 m_4}{1-m_3^2} \leq 1 \end{array} \right\} \tag{39}$$

The expressions in (36)-(39) give us the effective properties for the set of all finite-rank matrix-layered composites. A method for identifying the third-rank microstructure which corresponds to a given set of moments can be found in Lipton (1993).

Applying the above parameterization for the effective compliance matrix of the optimal material design problem in (4) we obtain the alternative form

$$\begin{array}{ll} \min\limits_{\rho_i} & \min\limits_{\boldsymbol{\sigma}_j \in \boldsymbol{S}_j} \min\limits_{\boldsymbol{m}_i \in H} \quad \dfrac{1}{2} \sum\limits_{i=1}^{I} \boldsymbol{C}_i \cdot \boldsymbol{Z}_i \\ \text{s.t.} \ \displaystyle\int_\Omega \rho = V & \end{array} \tag{40}$$

Here the outer optimization problem determines the macroscopic distribution of material in the structure, while the inner optimization problem determines the microstructure in each of the sub-domains $\Omega_i$. We note that the inner optimization problem actually may be solved as a set of smaller minimization problems, one for each of the sub-domains $\Omega_i$. Both the outer optimization problem and the inner optimization problem are suited for a solution by design sensitivity analysis and mathematical programming.

| | 1 | 2 | 4 | 8 | 16 |
|---|---|---|---|---|---|
| 1 | 1.000 | 0.881 | 0.827 | 0.794 | 0.774 |
| 2 | 0.765 | 0.658 | 0.598 | 0.565 | 0.548 |
| 4 | 0.679 | 0.588 | 0.537 | 0.508 | 0.493 |

Table 1. Normalized total complementary energy for optimal designs. (Rows correspond to alternative numbers of horizontal subdomains, while columns – of vertical ones).

### 3.2. Example

Applying the moment formulation described in the previous section, we now consider a series of examples of optimal design of planar structures with domain-wise constant material properties. A plane design domain which is subjected to a single concentrated load and supported as shown in Fig. 1 is considered, and by applying a symmetry condition we analyze only the upper half of the structure.

The upper half of the structure is initially discretized into 8x32 four-node anisotropic finite elements. Thereafter it is divided into a number of sub-domains which we will assume to have constant material properties. We use as design variables the ones which characterize the third matrix-layered composite in each of the sub-domains of the structure, and study the performance of the optimal designs as the number of these sub-domains is increased.

In all examples, the available amount of the stiffer material in the matrix-layered composite is set to be 40% of the design domain volume, and we specify the stiffness ratio between the stiff and the soft material in the microstructure to be 100, while both materials are taken to have the same Poisson's ratio of 0.3.

Table 1 shows the total complementary energy for a series of optimal designs, obtained using 1, 2 and 4 horizontal sub-domains with 1, 2, 4, 8 and 16 vertical sub-domains. As expected, it is seen that the total complementary energy decreases when the number of sub-domains with independent material properties are increased.

**Remark:** We notice the importance of actual division of the structure into sub-domains. Optimally this division of the structure should be obtained as a part of the solution to a more general optimization problem, rather than be performed manually.

The pictures of the optimal designs are shown in Figs. 2-6. It should be stressed that the pictures are only the illustrations of the optimum microstructures. The distance between the layers has been used to illustrate the different length scales in the third-rank microstructure, while layer thicknesses are used to illustrate the density of material in each level of the third-rank microstructure.

In reality both the distance between layers and the layer thicknesses should be infinitely small. It should also be mentioned that the microstructures shown represent just one of the possible realizations of a third-rank microstructure that will give exactly the same set of effective material properties. Indeed, the formula (4) does not specify which layer is "thicker", and therefore there are at least three equivalent solutions to it.

¿From the series of pictures of the optimal designs it becomes clear that the stiff material will be concentrated in the areas where the stress field has singularities, that is – where the concentrated load is applied and where the beam is attached to the rigid support. The smaller the sub-domains of the structure, the higher the concentration of the stiff material in these areas.

Next, when studying the microstructure of the material we see that the optimal material has everywhere a third-rank microstructure. We also see that the stronger layers of the stiff material become practically speaking co-aligned with what we should expect to be the principal stress directions in a domain. That means: at $\mp 45$degrees near the core of the beam where we have large shearing stresses and at zero degrees in areas where we have large bending stresses. Another effect which might be observed is that the stronger layers of stiff material bend when we get closer to the tip of the beam in order to catch the concentrated load at this point.

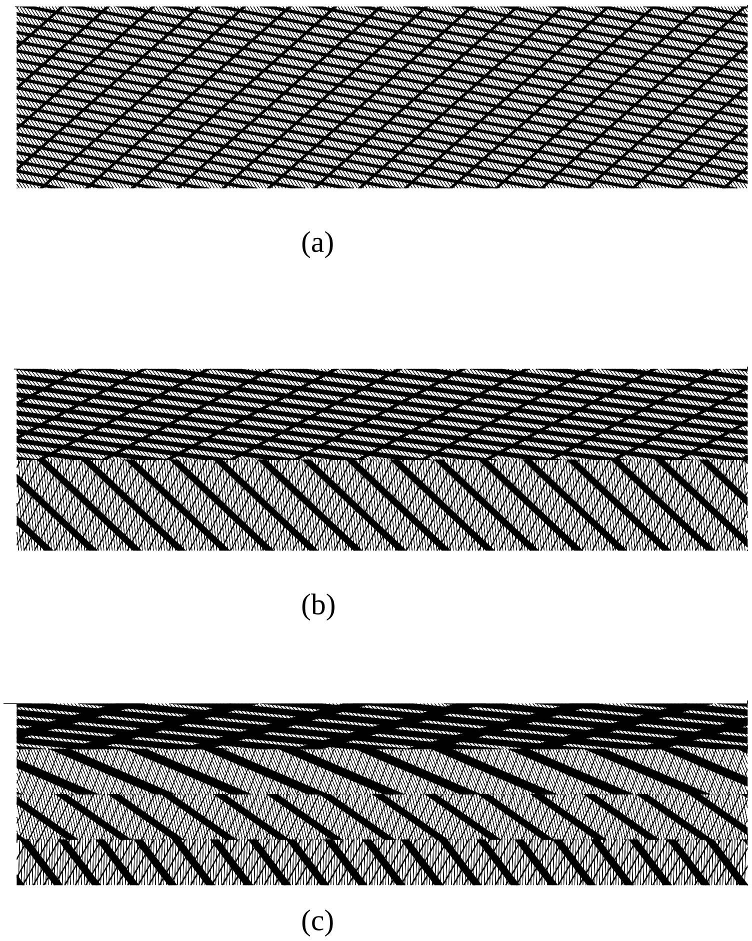
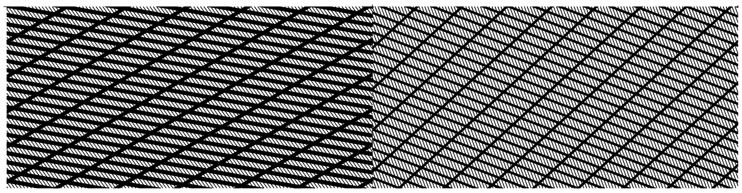
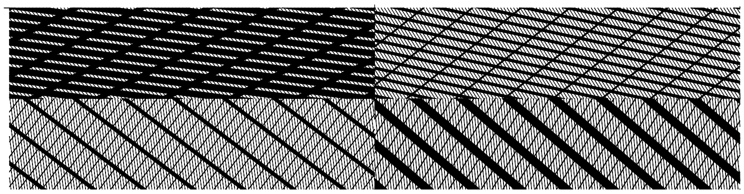


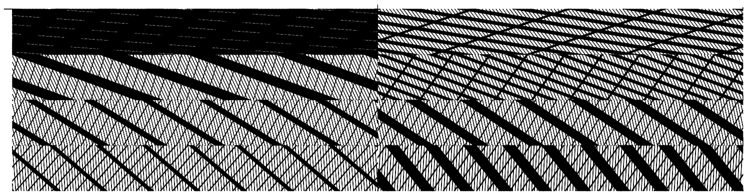

Figure 2. Optimal designs obtained using 1x1, 2x1, and 4x1 domains with constant material properties.



(a)

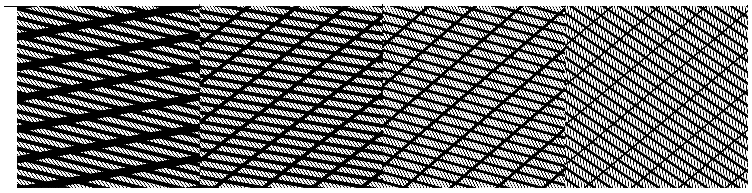


(b)

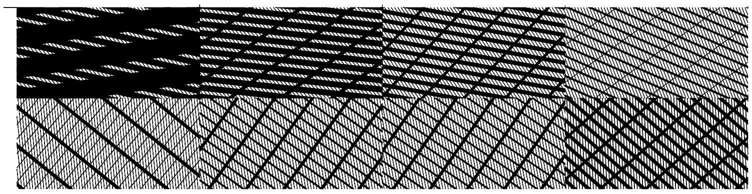


(c)

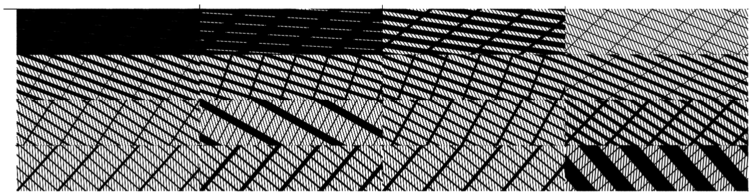
Figure 3. Optimal designs obtained using 1x2, 2x2, and 4x2 domains with constant material properties.



(a)

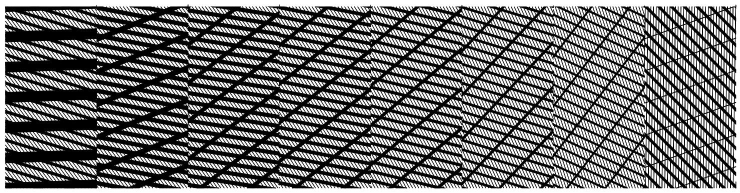


(b)

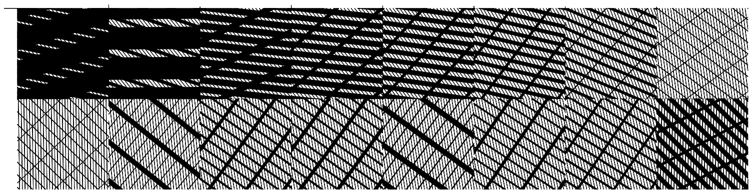


(c)

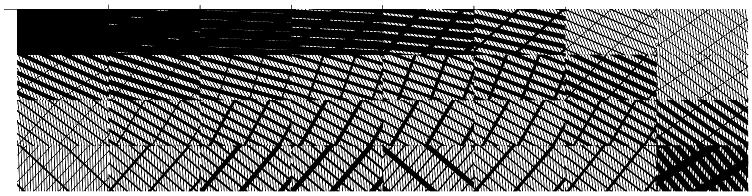
Figure 4. Optimal designs obtained using 2x4, 2x4, and 4x4 domains with constant material properties.



(a)

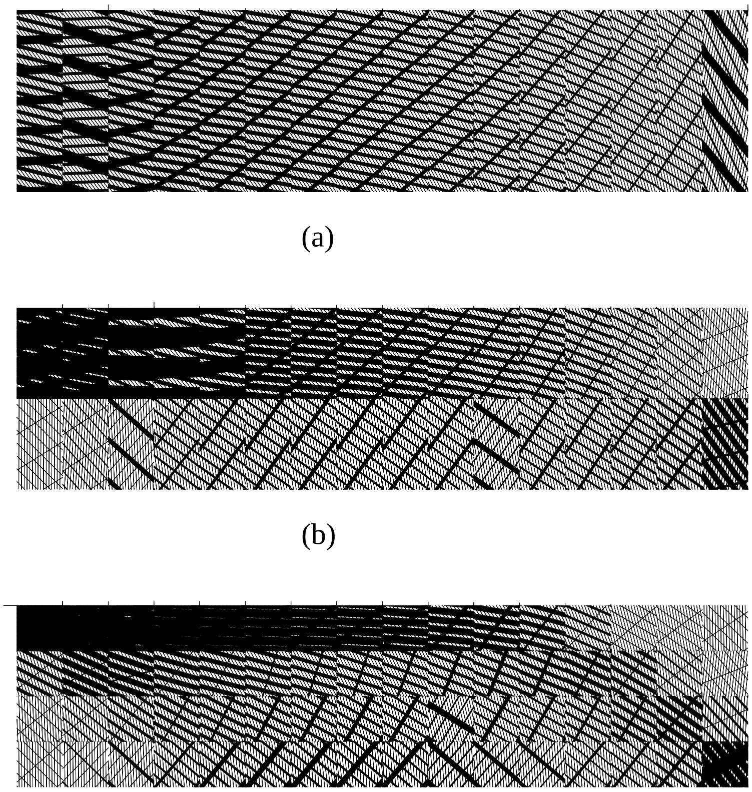


(b)



(c)

Figure 5. Optimal designs obtained using 1x8, 2x8, and 4x8 domains with constant material properties.



(a)

(b)

(c)

Figure 6. Optimal designs obtained using 1x16, 2x16, and 4x16 domains with constant material properties.

**Acknowledgements:** The authors gratefully acknowledge the financial support from the Danish Technical Research Council through the grant no. 9600827 and from National Science Foundation through the grant no. DMS- 9625129.

## References

AVELLANEDA, M. (1987) Optimal bounds and microgeometries for elastic two-phase composites. *SIAM J. Appl. Math.*, **47**, 6, 1216–1228.

AVELLANEDA, M., MILTON, G.W. (1989) Bounds on the effective elastic tensor of composites based on two-point correlations. In: Hui, D.; Kozik, T.J., eds., *Composite Material Technology*, 89–93. ASME.

DIAZ, A., LIPTON, R., SOTO, C.A. (1994) A new formulation of the problem of optimum reinforcement of Reissner-Mindlin plates. *Comp. Meth. Appl. Mechs. Engrg.*

FRANCFORT, G.A., MURAT, F. (1986) Homogenization and optimal bounds in linear elasticity. *Arch. Rat. Mech. Anal.*, **94**, 307–334.

GIBIANSKY, L.V., CHERKAEV, A.V. (1997A) Design of composite plates of extremal rigidity. In: Cherkaev, A.; Kohn, R., eds., *Topics in Mathematical Modelling of Composite Materials*, 95–137. Birkhäuser. Translation from Russian, Ioffe Physico-Technical Institute, Academy of Sciences of USSR, Publication 914.

GIBIANSKY, L.V., CHERKAEV, A.V. (1997B) Microstructures of composites of extremal rigidity and exact bounds on the associated energy density. In: Cherkaev, A.; Kohn, R., eds., *Topics in Mathematical Modelling of Composite Materials*, 273–317. Birkhäuser. Translation from Russian, Ioffe Physico-Technical Institute, Academy of Sciences of USSR, Publication 1115.

KREIN, M.G., NUDELMAN, A.A. (1977) *The Markov Moment Problem and Extremal Problems.* Translation of Mathematical Monographs, 50. American Mathematical Society.

KROG, L.A., OLHOFF, N. (1997) Topology and reinforcement layout optimization of disk, plate and shell structures. In: Rozvany, G.I.N., ed., *Topology Optimization in Structural Mechanics*, Springer-Verlag, Wien.

LIPTON, R. (1993) On implementation of partial relaxation for optimal structural compliance problems. In: Pedersen, P., ed., *Optimal Design with Advanced Materials*, Elsevier, Amsterdam, The Netherlands.

LIPTON, R., DIAZ, A. (1995) Moment formulations for optimum layout in 3D elasticity. In: Olhoff, N.; Rozvany, G.I.N., eds., *Structural and Multidisciplinary Optimization*, Pergamon, Oxford, UK.

PEDERSEN, P. (1989) On optimal orientation of orthotropic materials. *Struct. Optim.*, **1**, 101–106.