What Structures are Optimal for a General Problem of Structural design?

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An optimal structural design problem for elastic inhomogeneous bodies is considered. The problem is to find the optimal layout of two materials with different material constants; the boundary and loading conditions are supposed to be known. The goal is to minimize a lower weakly semicontinuous functional of the solution of the corresponding boundary value problem. For example we could minimize the integral of weighted deflection, its norm, etc.

It has been found about two decades ago that this problem does not have a classical (smooth) solution (see papers by Olhoff, Lurie & Cherkaev, Cheng & Olhoff, by Kohn & Strang). The nonexistence of solution simply means that the optimal layout necessary includes infinitely small alternating regions occupied with different materials; these regions are mixed in a special way to form optimal microstructures.

Solutions of such problems require special relaxation methods. They have been well developed for the simplest problem of minimization of total energy stored in the designed construction. The problem of a body with minimal stored energy could be divided in two parts. The first part is called the local problem; it asks for a periodic structure of a composite which corresponds to minimum energy stored in it under uniform loading. It can often be solved analitically using methods of quasiconvexity like the translation method (see for example the papers by Lurie & Cherkaev, Tartar, Murat, Milton, Francfort, Cherkaev & Gibiansky, where this problem is addressed in different settings). The second part (the global problem) asks for a distribution of optimal microstructures throughout a body; usually it must be solved numerically. Doing this problem one replaces an inhomogeneous material of optimal microstructures with an equivalent homogenized material.

Here an approach is developed to solution of the local problem for an arbitrary lower semicontinous functional, not only for the energy. This approach allows one to determine optimal structures for this general problem.

The problem of optimal design is a variational problem with the equations of elasticity taken as differential constrain. The Lagrange multiplier for this constrain formally coinsides with the displacement of the original elasticity problem under a different loading. The local problem asks for minimax of the bilinear form of the gradient of optimal displacement and the gradient of Lagrange multiplier.

We reformulate this problem and we prove a basic qualitative requirement to optimal composites: they maximize the weighted difference of energy density caused by two orthogonal strain fields ϵ and ϵ'. The last two fields are build as linear combination of gradients of optimal displacement and of Lagrange multiplier. The new form of the problem shows that all optimal structures are *extremely anisotropic* as they possess maximal difference in rigidities in two orthogonal directions; this propertiy guides us to find corresponding composite structures.

The further changeover of the problem by means of Legandre transformation leads to a minimal variational problem, similar to the simplest problem of energy minimization. It asks for: *a microstructure of a composite, assembled from two given materials, which minimizes the sum of stored energy, generated by some uniform strain ϵ and the complementary energy, generated by an orthogonal uniform stress $\sigma = C : \epsilon'$:* In other words, an optimal microstructure must minimize the sum of a rigidity under one loading and a compliance under another orthogonal loading. This formulation permits the application of previously developed techniques of quasiconvexity to the problem.

We find examples of optimal microstructures. They include, in particular, second-rank laminates known as optimal structures for the special problem of energy minimization. The complete description of optimal microstructures is obtained in asymptotic case when elastic properties of one of components are infinitely small. The optimal structure adapts itself to both loading and functional by varying its inner parameters. The class of optimal composites includes second rank matrix laminates with arbitrary angle between normals to the laminates of the first and second ranks. This set is completed by 'herring-bones' structures suggested by Milton; the herring-bones structure is a laminate assembled from differently oriented second rank matrix composites. The volume fraction of 'zero' material tends to zero in an optimal mixture; nevertheless, the optimal mixture still shows zero resistance in some directions.