Necessary conditions technique in structural optimization

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Abstract

The paper deals with the problem of optimal structures of multiphase conducting composites. This problem is formulated as an unstable multivariable extremal problem. It is known that the technique of the sufficient conditions (the translation method) fails to produce exact bounds. A variational technique is developed to investigate such problems. Special strong local variations are introduced and the necessary conditions of stability to these variations are derived. A minimal extension of the Lagrangian is introduced based on these necessary conditions. The extended problem preserves the cost of the original unstable problem and is stable against the class of the perturbations. Specifically, optimal structures of multiphase conducting composites in 2D are investigated. The following variation is considered: an infinitesimal elliptical inclusion of a material is inserted into a domain of a different material. The shape and the orientation of the inclusion is adjusted to the acting field. The necessary condition requires non-negativity of the increment of the goal functional. The minimal extension of the Lagrangian is found that is equivalent to enlarging the set of original materials by adding composites with extremal properties to that set. A more sophisticated test is introduced as well: a trial inclusion is made of a fine scale mixture of two materials; this inclusion is inserted into the domain of the third material. This test examines special boundaries in optimal three materials' structures.

Two examples are analysed. It is shown that the suggested necessary conditions conditions are equivalent to the known sufficient conditions (the translation bounds) for the problem of optimal two-component mixture. For three component mixtures, new structures are found that show the best conductivity; their properties are discussed.

1 Introduction.

Two-phase and multi-phase optimal composites. We consider the problem of determination of a structure of multiphase composites with extremal effective properties. The case of two-component optimal mixtures is well investigated: the bounds of effective properties and the matching microstructures that realize these bounds are described for a number of cases. The list of solved problems includes numerous examples of conducting, elastic, electromagnetic and other materials and is continuing to grow (see for the references Cherkaev and Kohn [7]). Optimal multiphase mixtures are investigated much less. Several approaches to the problem and some examples of optimal structures have been suggested in the papers by Milton [21], Kohn & Milton [13], Lurie & Cherkaev [18], Cherkaev & Gibiansky [5], Golden [10], by using different methods. However the total picture is still obscure.

From a physical viewpoint, the problem of optimal mixture of two materials has an intuitively expected solution: to find the best periodic mixture assembled from 'good' and 'bad' materials with prescribed volume fractions one must place the 'bad' material into compact nuclei surrounded by continuum of the 'good' material which forms the connected construction. This topology increases the influence of the good material and reduces the disadvantage of the bad one. Of course, it is a long way from this intuitive idea to a mathematically rigorous description of optimal microstructures, but the topology is clear from a beginning. The variety of known optimal solutions: high-rank laminates, 'coated spheres' or ellipsoids, 'truly' periodic structures (see Hashin and Shtrikman [12], Lurie & Cherkaev [17, 19], Milton [22], Vigdergauz [32]) demonstrate different realizations of this intuitive idea.

On the contrary, multi-material optimal mixtures can hardly be treated using similar intuitive ideas. The question where to place the material with intermediate properties is more delicate, and intuitive answer is not so clear. As we demonstrate below, the optimal topology depends not only on the ordering of the material's properties, but also on the concentrations of the materials.

The optimal multiphase mixtures is important for many theoretical and practical applications. Indeed, the majority of natural and artificial composites involve more then two materials, especially those which show unusual properties. In structural optimization, knowledge of the geometry of two-component optimal mixtures has allowed to the solutions of the problems of 'topology optimization' (see the book by Bensdsoe [1] for detailed discussion). This problem corresponds to one of the 'materials' being void, then the problem of optimal structures reduces to the problem of optimal topology of the solid phase. Therefore, a meaningful consideration of optimal design of even two-component mixtures requires knowledge of the geometry of optimal multicomponent mixtures, since void is the always available additional 'material' that can be added to the list of available materials. For a problem of the phase equilibrium in solids, differently oriented anisotropic materials participate in optimal mixtures; therefore optimal multiphase mixtures are natural objects

to address (see, for example, the paper by Bhattacharya and Kohn [2]).

Sufficient conditions and optimal structures for two-phase composites. The problem of optimal composite is formulated as a variational problem of minimization of the sum of stored energies in the composite. Assume that an unknown periodic structure is exposed to several uniform orthogonal external fields. To find the optimal composite, we compute the sum I of stored in the composite energies caused by these fields; then we minimize I upon all structures which possess the same volume fractions of the materials but possess different geometries (see the next section for mathematical formulation). The obtained variational problems are irregular (ill-posed, non-classical), since the minimizers include the index functions of materials which show what material is used in a point of the structure. These functions that take only two values: one, if the material is presented in a point, and zero, if it is not. The standard variations methods cannot deal with discontinuous functions and they must be adjusted to these specific functions.

The technique that is widely used for solving the problems of optimal mixtures includes two mutual supplementary elements. On one hand, one establishes *sufficient optimality conditions* in the form of classical Wiener bounds, Hashin-Shtrikman bounds [12], Translation bounds (Lurie & Cherkaev [17], Tartar and Murat [31], Milton [24], Zhikov [33]), Gibiansky & Torquato [9], bounds for multicomponent mixtures recently obtained by Nesi [28], etc. Bounds for composites of non-linear materials have been found by Talbot and Willis [30], Ponte-Castaniedo [29], et al. These bounds constrain the stored in a fixture energy by inequalities between it and a function of the mean fields, properties, and the volume fractions of mixed materials. The inequalities are valid for all mixtures regardless of their geometry.

On the other hand, one finds (essentially by guessing) specific structures of *optimal geometry*. For two-material case, a number of structures has been suggested that realize the bound so that the corresponding inequalities in the sufficient conditions become equalities. This match allows one to prove simultaneously the optimality of the structures (which represent minimizing sequences of the variational problem) and the sharpness of the bounds. Besides a guessing for the geometry of optimal structures, it is possible to obtain information about the fields in them using the above mentioned sufficient conditions (see Milton [25], Grabovsky [11]) if they are attainable.

Notice, that this technique (at least in its present form) is mainly restricted to two-component mixtures and does not provide a guide for non-trivial generalization to multimaterial case. Although the translation method leads to sufficient conditions also for multicomponent mixtures, there are no guarantee that the bounds are exact. Moreover, the sufficient conditions based on the translation method fail to be exact for a whole range of volume fractions of phases. (see the discussion in the papers by Kohn and Milton [13], Cherkaev and Gibiansky [5], Nesi [28]). These conditions prescribe the algebraic form of bounds and are adjusted to deal with a special form of the Lagrangians. It is hard to spec-

ulate on an optimal structure if the sufficient conditions do not lead to exact inequalities. Besides, the problem of an optimal multicomponent mixture depends on large number of prescribed parameters: properties of materials and their fractions. This makes the straightforward optimization in a class of arbitrary chosen structures ineffective. One needs either an *a priori* hint of optimal topology and a corresponding factorization of the class of 'suspicious' structures.

Some optimality conditions can be established by the consideration of the so-called 'stability under lamination' approach by Francfort and Milton [8, 26]. In principle, his approach allows to derive effective properties of a large class of arbitrarily high-rank laminate mixtures, which could include the optimal structures. In this method, a set of laminates with varying volume fractions is viewed as a curve in the space of characteristics of effective tensors; this curve joins the points that represent the mixing materials. The set of effective properties of all laminates corresponds to a body in that space. To find this body, one uses the following iterative procedure. Starting with a given set of materials, one first connects all of them by laminate curves, enlarging the initial set of materials' properties by all two-component laminates. Then one connects any pair of points in these curves, further enlarging the initial set, and so on. Finally, one ends up with a body in the space of effective properties of all mixtures that correspond to 'layers of some rank'. Although this scheme is geometrically clear, its implementation for specific problems is far from trivial. The difficulties correspond to the necessity to describe a rather complicated surface of the final extension.

Necessary conditions of Weierstrass type and an extension. The purpose of this paper is to enlarge the arsenal of available tools for structural optimization problems. We suggest to modify the classical variational technique which require the comparing of two proximate configurations, rather then the comparing all configurations as sufficient methods do. Namely, we develop a technique of *strong necessary conditions of optimality* for these variational problems with provide a regular method for their examination. These conditions detect instabilities of solutions. Based on the same necessary conditions, we also suggest *a minimal extension* of unstable variational problems that make it stable with respect to these variations and preserves its cost.

The type of considered optimality condition (strong variations of the material's properties) have been introduced by Lurie in a series of pioneering papers in the seventeens [14, 15, 16], where he considered the optimization problem for a conducting medium. The method is a multidimensional analog of the Weierstrass variations in classical calculus of variations and of Pontriagian's maximum principle in one variable control problems. This technique has been used by Lurie [14, 16], and by Lurie, Cherkaev and Fedorov [20] for analysis of various problems of optimal design of conducting and elastic media.

Structure of the paper. The structure of the paper is the following. In section 2 we formulate the problem and sketch a procedures of Weierstrass type necessary conditions and of the corresponding minimal extension. In section 3 we derive formulas for a Weierstrass type variation, in section 4 we apply this technique to the problem of two-component mixtures and show that the derived necessary conditions are also sufficient for this problem, in section 5 we derive the necessary conditions for three component mixtures, and in section 6 we analyse them and demonstrate new optimal structures.

2 Variational technique for unstable variational problems.

2.1 Formulation of the problem.

Consider the problem of optimizing the mixture of conducting materials in a two-dimensional domain, clearly the simplest problem of structural optimization. We develop the necessary conditions of optimality for this problem.

Consider a unit square $\mathcal{O} = \{\boldsymbol{x} = (x_1,\ x_2) :\ 0 \le x_i \le 1,\ i = 1,2$ divided into N subdomains $\mathcal{O}_i,\ i = 1,2,..N$: $\mathcal{O} = \cup \mathcal{O}_i$ and repeated periodically in a plane. Suppose that the subdomains $\mathcal{O}_i$ are fulfilled with isotropic materials with conductivities $\sigma_1, \sigma_2, ...\sigma_N$ so that a material σ_i occupies the domain $\mathcal{O}_i$. The conductivities are ordered as following:

$$0 \le \sigma_1 < \sigma_2 < ... < \sigma_N \le \infty$$

The variable conductivity $\sigma(\boldsymbol{x})$ in $\mathcal{O}$ is equal to :

$$\sigma(\boldsymbol{x}) = \sum_{i=1}^{N} \sigma_i \chi_i(\boldsymbol{x}) \tag{2.1}$$

where $\boldsymbol{x}$ the point in $\mathcal{O}$ and χ_i is the index function of $\mathcal{O}_i$:

$$\chi_i = \begin{cases} 1 & \text{if} \quad \boldsymbol{x} \in \mathcal{O}_i \\ 0 & \quad\ \ \text{elsewhere} \end{cases} \qquad \sum_{i}^{N} \chi_i(\boldsymbol{x}) = 0 \ \ \forall \boldsymbol{x} \in \mathcal{O}. \tag{2.2}$$

Suppose that the periodic structure is exposed to a uniform external electrical field $\boldsymbol{P}_1$ which causes the variable field $\tilde{\boldsymbol{E}}_1(\boldsymbol{x})$ in each point of $\mathcal{O}$. The field $\tilde{\boldsymbol{E}}_1(\boldsymbol{x})$ is a solution of the variational problem

$$J(\boldsymbol{P}_1, \chi_i) = \min_{\tilde{\boldsymbol{E}}_1 \in \tilde{\mathcal{E}}_1} \int_{\mathcal{O}} \tilde{\boldsymbol{E}}_1 \cdot \sigma(\boldsymbol{x}) \tilde{\boldsymbol{E}}_1, \quad \tilde{\mathcal{E}}_1 = \left\{ \tilde{\boldsymbol{E}}_1 :\ \nabla \times \tilde{\boldsymbol{E}}_1 = \boldsymbol{0},\ \int_{\mathcal{O}} \tilde{\boldsymbol{E}}_1 = \boldsymbol{P}_1 \right\}. \tag{2.3}$$

The Euler-Lagrange equations of this problem

$$\nabla \cdot \sigma \tilde{\boldsymbol{E}}_1 = 0, \quad \nabla \times \tilde{\boldsymbol{E}}_1 = 0, \quad \int_{\mathcal{O}} \tilde{\boldsymbol{E}}_1 = \boldsymbol{P}_1, \quad \tilde{\boldsymbol{E}}_1 \text{ is periodic in } \mathcal{O} \tag{2.4}$$

give the equations of the steady state conductivity.

Let us formulate an optimization problem. Suppose that composite cells are exposed sequentially to two different external fields $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ which produce two interior fields $\tilde{\boldsymbol{E}}_1$ and $\tilde{\boldsymbol{E}}_2$. It is required to minimize the sum of energies $J(\boldsymbol{P}_1, \chi_i) + J(\boldsymbol{P}_2, \chi_i)$ stored in the cell by choosing the best shape of domains $\mathcal{O}_i$ (or the functions $\chi_i(\boldsymbol{x})$) subject to constrains

$$\int_{\mathcal{O}} \chi_i - m_i = 0, \quad m_i \geq 0, \quad \sum_i^N m_i = 1, \quad i = 1, ..N \tag{2.5}$$

which fix the amounts (volume fractions) m_i of the materials.

We add the constrains (2.5) with the Lagrange multipliers γ_i to the minimizing functional and formulate the minimization problem:

$$I' = \min_{\chi_i \in (2.2)} J(\boldsymbol{P}_1, \chi_i) + J(\boldsymbol{P}_2, \chi_i) + \sum_i^N \gamma_i \left(\int_{\mathcal{O}} \chi_i - m_i \right) \tag{2.6}$$

Let us introduce convenient notations for the problem. First, we introduce a 2×2 matrix $\tilde{\boldsymbol{E}}$ of the fields:

$$\tilde{\boldsymbol{E}} = \left(\tilde{\boldsymbol{E}}_1, \tilde{\boldsymbol{E}}_2 \right) \tag{2.7}$$

and a symmetric positively defined matrix $\boldsymbol{E}$

$$\boldsymbol{E} = \left(\tilde{\boldsymbol{E}}^T \cdot \tilde{\boldsymbol{E}} \right)^{1/2} = \begin{pmatrix} E_a & 0 \\ 0 & E_b \end{pmatrix} \tag{2.8}$$

with nonnegative eigenvalues $E_a \geq 0$, $E_b \geq 0$. The matrix $\boldsymbol{E}$ allows a convenient representation of the sum $J(\boldsymbol{P}_1, \chi_i) + J(\boldsymbol{P}_2, \chi_i)$:

$$\sum_{i=1}^{2} \left(\tilde{\boldsymbol{E}}_i \cdot \sigma \tilde{\boldsymbol{E}}_i \right) = \mathrm{Tr} \, \left(\sigma \boldsymbol{E}^2 \right). \tag{2.9}$$

Notice that if the fields $\tilde{\boldsymbol{E}}_1$ and $\tilde{\boldsymbol{E}}_2$ are mutually orthogonal then $\boldsymbol{E} = \tilde{\boldsymbol{E}}$ and the eigenvalues E_a and E_b become the magnitudes of these fields.

Similarly, let us introduce the matrix $\boldsymbol{P} = (\boldsymbol{P}_1, \boldsymbol{P}_2)$ of the external loadings. We will consider orthogonal loadings which corresponds to the orthogonal matrix $\boldsymbol{P}$.

Finally, we rewrite problem (2.6) using the introduced matrix $\boldsymbol{E}$ (2.1) and omitting the constant term $\left(- \sum_i^N \gamma_i m_i \right)$:

$$I(\boldsymbol{P}; \gamma_1, ...\gamma_N : \sigma_1, ...\sigma_N) = \min_{\chi_i \in \ (2.2)} \min_{\boldsymbol{E}} \int_{\Omega} W, \tag{2.10}$$

$$W = \left(\mathrm{Tr} \, \left(\sum_i^N (\sigma_i \boldsymbol{E}^2(\boldsymbol{x}) + \gamma_i) \chi_i(\boldsymbol{x}) \right) \right). \tag{2.11}$$

This functional depends on external fields $\boldsymbol{P}$, the Lagrange multipliers $\gamma_1, ...\gamma_N$ (the materials' costs), and the conductivities $\sigma_1, ...\sigma_N$. The cost of the problem determines the effective tensor $\boldsymbol{\sigma}_*$ as following:

$$I = \min_{\boldsymbol{\sigma}_* \in G_m \ closure} \left(\boldsymbol{P}^T \cdot \boldsymbol{\sigma}_* \boldsymbol{P} + \sum_i^N \gamma_i \ m_i \right) \tag{2.12}$$

where G_m -closure is the set of all effective tensors of composites that are made from given materials with given volume fractions.

After the problem (2.11) is solved, we use (2.5) to determine γ_i. Also, we find the optimal volume fractions and the optimal degree of anisotropy of $\boldsymbol{\sigma}_*$ as a function of the mean field $\boldsymbol{P}$. After this, we are able to exclude $\boldsymbol{P}$ and γ_i from the problem and express the conductivity tensor as function of the volume fractions m_i and of its degree of anisotropy. In other words, we establish a relation

$$\Psi(\lambda_1, \lambda_2, m_1..m_N, \sigma_1, ...\sigma_N) = 0 \tag{2.13}$$

where λ_1, λ_2 are the eigenvalues of $\boldsymbol{\sigma}_*$ that describes the component of the G_m-closure of three materials mixture.

Remark 2.1 *Note, that the last problem is easily solved for two-component mixtures: the volume fraction of one of the components and the degree of anisotropy of the effective tensor is in one to one correspondence with the parameters of the external field* $\boldsymbol{P}$.

2.2 Variation of properties.

Here we introduce the 'structural variation' which is the suggested tool for checking the optimality of a structure. To perform this variation we implant an infinitesimal inclusion of an admissible material σ' in a point $\boldsymbol{x}$ in the domain $\mathcal{O}_i$ occupied by a different material, σ, and compute the difference in energies and cost of I due to this variation. The conductivities σ and σ' can take values $\sigma_1..\sigma_N$. If the examined structure is optimal, then the cost has to be positive.

The increment in energy depends on a shape of the region of variation; this shape must be adjusted to the field so that the increment reaches its minimal value (which remains non-negative!) We restrict ourselves to elliptical inclusions, which allows an analytical calculation of the increment. To compute the increment one solves the problem of an single elliptical inclusion in a plane.

Technically, we follow a slightly different equivalent procedure: we compute the increment caused by a replacing material σ, in a neighborhood of a point $\boldsymbol{x}$, by a quasiperiodic dilute composite of second-rank laminates in which the envelope is made of the host material σ and the inclusions are made of material σ'. In other words, we construct a second-rank

laminate from the isotropic host medium in the envelope. This matrix laminate composite of second rank is characterized by its effective tensor [4]:

$$\boldsymbol{\sigma}_* = \sigma \boldsymbol{I} + (1-m)\left((\sigma' \boldsymbol{I} - \sigma \boldsymbol{I})^{-1} + mQ(\sigma)\right)^{-1}, \tag{2.14}$$

where the matrix Q determines the degree of anisotropy:

$$Q = 1/\sigma \begin{pmatrix} \alpha_a & 0 \\ 0 & \alpha_b \end{pmatrix}, \quad \alpha_a + \alpha_b = 1, \quad \alpha_i \geq 0, \tag{2.15}$$

coordinates of Q coincide with directions of lamination, α_a and $\alpha_b = 1 - \alpha_a$ are inner parameters of the structures that determine the relative elongation (densities in orthogonal directions) of the inclusions, m is the volume fraction of material in the inclusions, $1-m$ is the volume fraction of enveloping material.

To compute the variation $\delta\boldsymbol{\sigma}$ of the tensor properties caused by the array of infinitely dilute nuclei with infinitesimal volume fraction $d\,m$, let us replace m by $\delta\, m << 1$, and $\boldsymbol{\sigma}_*$ by $\sigma + \delta\boldsymbol{\sigma}$. Then (2.14) becomes

$$\delta\boldsymbol{\sigma} = \delta\, m\left((\sigma' - \sigma)^{-1} + Q(\sigma)\right)^{-1} + o(\delta\, m) = \delta\, m \begin{pmatrix} \lambda_a & 0 \\ 0 & \lambda_b \end{pmatrix} + o(\delta\, m). \tag{2.16}$$

where

$$\lambda_i = \frac{\sigma(\sigma' - \sigma)}{\sigma + \alpha_i(\sigma' - \sigma)}, \quad i = a, b. \tag{2.17}$$

is the matrix of the variation. Note that the increment depends on α_a and on the orientation of the directions of laminates.

One can check by a direct calculation that the form of the increment (2.16) coincides with the increment caused by an single elliptical inclusion of the equal area. [1]

2.3 Increment of the functional.

Let us compute the variation δI of energy caused by the variation (2.17) of properties $\delta\boldsymbol{\sigma}$. For simplicity, we assume that the main axes of $\delta\boldsymbol{\sigma}$ are codirected with the principle axes of the matrix $\boldsymbol{E}$ (we will see that the obtained necessary conditions are the strongest ones, which justify this assumption). The cost consists of the increment in energy due to the variation of conductivity $\delta\boldsymbol{\sigma}$ and the 'direct cost' of the variation: the change in the total cost due to change of quantities of the used materials. The direct cost of the variation is determined by its type: we replace the material σ (the specific cost γ) with the material σ' (the specific cost γ'); therefore the change of the total cost is

$$(\gamma' - \gamma)\,\delta\, m. \tag{2.18}$$

[1] However, in the corresponding elasticity problem, the array of inclusions leads to more sensitive variation than a single inclusion of the same area, see the paper by Cherkaev, Grabovsky, Movchan, and Serkov [6]

The energy increment δW is equal to

$$\delta W \delta m = \mathrm{Tr}\ (\boldsymbol{E} \cdot \delta \boldsymbol{\sigma}) = \left(\lambda_a E_a^2 + \lambda_b E_b^2\right) \delta\, m \tag{2.19}$$

where E_a and E_b are defined by (2.41). Using (2.17) and (2.15) we transform the increment to the form

$$\delta W(\alpha) = \sigma(\sigma' - \sigma) \left(\frac{E_a^2}{\sigma + \alpha(\sigma' - \sigma)} + \frac{E_b^2}{\sigma + (1-\alpha)(\sigma' - \sigma)} \right). \tag{2.20}$$

where we use the notation $\alpha = \alpha_a$. The total cost of the variation is denoted by $v\delta m$::

$$v(\alpha) = (\gamma' - \gamma + \delta W(\alpha_1))\,; \tag{2.21}$$

it depends on the shape of the variation, specifically, on the parameter α of anisotropy (elongation) of the inclusions. If the structure is optimal, all variations, including the most sensitive one, must be nonnegative:

$$v \geq v_0, \quad v_0 = \gamma' - \gamma + \min_\alpha \delta W(\alpha) \geq 0. \tag{2.22}$$

because otherwise the cost can be reduced by this variation, and the structure fails the test.

Let us compute the most sensitive variations. Note, that the second derivative of v with respect to α has the sign of $(\sigma' - \sigma)$:

$$\frac{\partial^2}{\partial^2 \alpha_1} \Delta J(\alpha) = (\sigma' - \sigma)\psi^2, \tag{2.23}$$

where ψ is real, therefore α belongs to the boundary of the interval $[0,\ 1]$:

$$\alpha = 0 \quad or \quad \alpha = 1 \quad \text{if} \tag{2.24}$$

$$\sigma' - \sigma < 0. \tag{2.25}$$

In this case, v_0 is equal to

$$v_0 = \min_\alpha \Delta J(\alpha) = \gamma' - \gamma + \sigma(\sigma' - \sigma) F_2(\sigma, \sigma', E) \tag{2.26}$$

where

$$F_2(\sigma', \sigma, E) = \begin{cases} \frac{E_a^2}{\sigma'} + \frac{E_b^2}{\sigma} & \text{if} \quad \frac{E_a}{E_b} \geq 1, \\ \frac{E_a^2}{\sigma} + \frac{E_b^2}{\sigma'} & \text{if} \quad \frac{E_a}{E_b} \leq 1 \end{cases}. \tag{2.27}$$

In the plane of E_a, E_b the permitted region is the interior of the intersection of two mutually orthogonal ellipses. The elongation of these ellipses is determined by the ratio σ'/σ and their scale is defined by the difference in costs $\gamma' - \gamma$. Note that the boundary of the permitted set possesses the corner point $E_a = E_b = (\gamma - \gamma')\frac{\sigma'}{\sigma'^2 - \sigma^2}$.

Physically, the result can be interpreted as follows: If an inclusion is less conducting than the host material, then the best shape of the inclusion is a strip, elongated along the direction of the maximal field. The less conducting material 'tries to expose' itself by creating an elongated obstacle across the direction of the maximal current to maximally reduce the conductivity of the structure.

In the opposite case

$$\sigma' - \sigma > 0, \tag{2.28}$$

the optimal value α_0 of α varies in $[0,\ 1]$ depending on the field $\boldsymbol{E}$. We find the stationary value α' of α from the equation $\frac{\partial v}{\partial \alpha} = 0$ as:

$$\alpha' = \frac{E_b\sigma' - E_a\sigma}{(E_a + E_b)(\sigma' - \sigma)} \tag{2.29}$$

Optimal α_0 is

$$\alpha_0 = \begin{cases} \alpha', & \text{if } \alpha' \in (0,1), \\ 1, & \text{if } \alpha' > 1 \\ 0, & \text{if } \alpha' < 0 \end{cases} . \tag{2.30}$$

We substitute this value of α in (2.20) and find that the minimal increment Δ_0 is

$$v_0 = \gamma' - \gamma + \sigma(\sigma' - \sigma)F_1(\sigma, \sigma', E) \tag{2.31}$$

where

$$F_1(\sigma', \sigma, E) = \begin{cases} \frac{E_a^2}{\sigma'} + \frac{E_b^2}{\sigma} & \text{if} \quad \frac{E_a}{E_b} \le \frac{\sigma}{\sigma'} \\ \frac{(E_a+E_b)^2}{\sigma+\sigma'} & \text{if} \quad \frac{E_a}{E_b} \in \left[\frac{\sigma}{\sigma'},\ \frac{\sigma'}{\sigma}\right] \\ \frac{E_a^2}{\sigma} + \frac{E_b^2}{\sigma'} & \text{if} \quad \frac{E_a}{E_b} \ge \frac{\sigma'}{\sigma} \end{cases} . \tag{2.32}$$

In the plane E_a, E_b, the permitted region is the convex envelope supported by two mutually orthogonal ellipses; note that the boundary of the set possesses a straight component. The elliptical parts of the boundary of the permitted region correspond to the strip-like inclusions (the case when $\alpha_0 = 0$ or $\alpha_0 = 1$), the straight part corresponds to the elliptical inclusions (the case when $\alpha_0 \in (0,1)$).

Physically, we interpret the result in the following way: If the inclusion has a higher conductivity σ' than the host medium, its best shape is either a circle (if the field $\boldsymbol{E}$ is isotropic) , an ellipse (if the eigenvalues of the field $\boldsymbol{E}$ are close to each other), or a strip elongated across the direction of minimal field (if the ratio of the eigenvalues of the field $\boldsymbol{E}$ is large enough). The highly conducting inclusions 'want to hide' in the domain to minimize the decrease of the total conductivity.

Necessary conditions The obtained inequalities allow to determine the range of the field $\boldsymbol{E}$ in each material in an optimal structure. Put an inclusion of any admissible material σ_j into the host material σ_i. The resulting increment $\delta_{\sigma_j\sigma_i}\boldsymbol{E}_i$is non-negative. This provides constrains on the admissible field $\boldsymbol{E}_i$ in the i-th material. The field $\boldsymbol{E}_i$ is optimal if it corresponds to non-negative increments for all trial inclusions:

$$\Omega_i = \{\boldsymbol{E}_i : \ \delta_{\sigma_j\sigma_i}\boldsymbol{E}_i \geq 0, \quad \forall \ j = 1,..N\} \tag{2.33}$$

The region of optimality is denoted by Ω_i

2.4 Minimal extension.

Forbidden region The union $\Omega = \cup\Omega_i$ of all region Ω_i does not coincides with the whole space $\mathcal{E}$ of $\boldsymbol{E}$. The remaining part is 'forbidden': the fields never take the values in the region $\Omega_f = \mathcal{E} - \Omega$.

If the external field $\boldsymbol{P}$ lies in the forbidden region, it is split into several parts in an optimal structure, so that each part belongs to a region Ω_i and the mean field is equal to $\boldsymbol{P}$. This process of phase separation corresponds to optimality of a composite structure.

The necessary conditions allow to extent the Lagrangian, that is to determine a new Lagrangian in the forbidden region. Physically, the extended Lagrangian describes the minimal energy stored in a composite medium, assembled from the given materials. The extended Lagrangian has the following properties:
i) it preserves the cost of the variational problem (2.11)
ii) it leads to a classical solution defined for all mean fields (including whose in the forbidden region), that cannot be improved by considered class of variations.

Minimal extension Suppose that periodic structure is loaded by an homogeneous field $\boldsymbol{P}$. If the field $\boldsymbol{P}$ belongs to Ω_i, then the solution of the variational problem (2.11) is homogeneous as well, and the relaxed Lagrangian $\mathcal{S}W(\boldsymbol{E})$ coincides with the original Lagrangian. However, if the loading $\boldsymbol{P}$ belongs to the forbidden region Ω_f, no homogeneous solutions exist and the problem needs a reformulation. The optimality require the "phase separation": $\boldsymbol{E}(\boldsymbol{x})$ belongs to allowed regions Ω_i in any point of the structure, but its mean value is $\boldsymbol{E}$:

$$\boldsymbol{E} = \sum_{i=1}^{3} m_i\boldsymbol{E}_i, \quad m_i \geq 0, \quad \sum_{i=1}^{3} m_i = 1. \tag{2.34}$$

Note that the filed $\boldsymbol{E}_i$ in i-th material can take more than one value. Each pointwise value of the filed belongs to $|GO_i$. $\boldsymbol{E}_i$ admits the representation

$$\boldsymbol{E}_i = \sum_{i=k} \mu_k \boldsymbol{E}_i^{(k)}, \quad \boldsymbol{E}_i^{(k)} \in \Omega_i \tag{2.35}$$

The energy $\mathcal{S}W(\boldsymbol{E})$ of any structure that satisfies necessary conditions has the form

$$\mathcal{S}W(\boldsymbol{E}) = \sum_{i=1}^{3} m_i \sigma_i \boldsymbol{E}_i^2, \tag{2.36}$$

Then, we define the minimal extension $\mathcal{S}W(\boldsymbol{E})$ as the *maximum* of the energies of the type (2.36) that does not surpass W. The maximization brings the fields to the boundaries of Ω_i. We have

$$\mathcal{S}W(\boldsymbol{E}) = \max_{m_i} \max_{\boldsymbol{E}_i \in \partial\Omega_i} \sum_{i=1}^{3} m_i \sigma_i \boldsymbol{E}_i^2, \quad \boldsymbol{E} = \sum_{i=1}^{3} m_i \boldsymbol{E}_i, \quad m_i \geq 0, \quad \sum_{i=1}^{3} m_i = 1. \tag{2.37}$$

and simultaneously,

$$\mathcal{S}W(\boldsymbol{E}) \leq W(\ BE) \tag{2.38}$$

Remark 2.2 *The obtained extension is the upper bound of the energy of any structure. Indeed, there may be a new necessary conditions with further constrict Ω_i and there may be compatibility conditions which restrict the positions of the fields $\boldsymbol{E}_i$ in those sets. These potential restrictions lead to lower the value of $\mathcal{S}W(\boldsymbol{E})$.*

Effective properties A more specific construction allows to describe the effective properties tensor $\boldsymbol{\sigma}_*$ of optimal composites. Define *the extended Lagrangian* $\mathcal{S}W(\boldsymbol{P})$ in the form:

$$\mathcal{S}W(\boldsymbol{P}) = \mathrm{Tr}\ (\boldsymbol{P} \cdot \boldsymbol{\sigma}_*(\boldsymbol{P})) + \gamma_* \tag{2.39}$$

Here $\boldsymbol{\sigma}_*$ is a tensor that generally depends on $\boldsymbol{P}$; therefore (2.39) is a general form for the Lagrangian. Physically, $\boldsymbol{\sigma}_*$ represents an anisotropic effective tensor of composite, made of initially given materials. The structure of optimal composite varies together with the external field $\boldsymbol{P}$.

Denote by m_i the volume fractions of the materials in the composite; the cost γ_* of the composite is proportional to the quantities m_i of the used materials:

$$\gamma_* = \sum_{i}^{N} m_i \gamma_i. \tag{2.40}$$

To determine the extended Lagrangian we perform the following variation: put one of the original materials σ_i in the point $\boldsymbol{E} = \boldsymbol{P} \in \Omega_f$ instead of the unknown anisotropic material $\boldsymbol{\sigma}_*$. We call the extension $\mathcal{S}W(\boldsymbol{\sigma}_*,\ \boldsymbol{P})$ neutral to the variation, if the equality holds

$$\Delta_* = \min_i \{\Delta J(\sigma_i, \boldsymbol{\sigma}_*, \boldsymbol{P}) - \gamma_i + \gamma_*\} = 0. \tag{2.41}$$

This equality determines the unknown tensor $\boldsymbol{\sigma}_*(\boldsymbol{P})$ and the extended Lagrangian (see (2.39)). This condition of neutrality states that the composite with the properties $\boldsymbol{\sigma}_*$

and the cost γ_* is neutral to the most "dangerous" variations. The extended Lagrangian preserves the cost of the variational problem. Indeed, if

$$\Delta_* < 0, \tag{2.42}$$

then a material $\boldsymbol{\sigma}_*$ is not optimal, which means that it does not take part in the minimizing sequences. Therefore the problem has the same cost and minimizers as the problem without addition of the new material $\boldsymbol{\sigma}_*$, and none of the initial or extended materials is optimal in the field $\boldsymbol{P}$. In the the opposite case

$$\Delta_* > 0, \tag{2.43}$$

the new material $\boldsymbol{\sigma}_*$ provides the homogeneous optimal solution $SW(\boldsymbol{\sigma}_*, \boldsymbol{P})$, but its use decreases the cost of extremal problem.

Notice, that if the mean field $\boldsymbol{P}$ belongs to one of the regions Ω_i where the material σ_i is optimal, the suggested scheme shows that $\boldsymbol{\sigma}_* = \sigma_i$.

The equation (2.41) determines the effective tensor $\boldsymbol{\sigma}_*$ that corresponds to a composite with extremal conductivity. The set of these tensors form an interior boundary of the G_m-closure (the set of effective properties of all possible composites). The boundary corresponds to those composites which are stable under chosen type of variation. The extension gives the upper boundary of the 'final' extension – the quasiconvex envelope of the functional. Indeed, one could think of a wider class of variations that would lead to another extension with smaller minimal value for the variation. However, we demonstrate below that for the considered problem the extension is final.

It could be convenient to use another (equivalent) way to determine the extended Lagrangian. We perform the inverse variation: replace the material σ_i in a point of the allowed region $\boldsymbol{E}_i \in \Omega_i$ with the composite material $\boldsymbol{\sigma}_*$, and chose the best geometry of that inclusion. The resulting variation has the form

$$\delta W(\boldsymbol{\sigma}_*, \sigma_i, \boldsymbol{E}_i) - \gamma_* + \gamma_i \tag{2.44}$$

The minimal extension corresponds to the equality

$$\min_i \min_{\boldsymbol{E}_i \in \Omega_i} \delta W(\boldsymbol{\sigma}_*, \sigma_i, \boldsymbol{E}_i) - \gamma_* + \gamma_i = 0 \tag{2.45}$$

which determines the extended Lagrangian or the tensor $\boldsymbol{\sigma}_*$. The find the most dangerous variation, we vary the shape of the inclusion and the field $\boldsymbol{E}_i \in \Omega_i$ in the phase σ_i, and we examine all regions Ω_i. Again, the extended in that way Lagrangian preserves the cost of the variational problem.

3 Necessary conditions for two-phase mixtures.

First, we apply the technique to find optimal two-phase mixtures (G_m closure of the set of two conducting materials). This problem has been solved independently in the papers

by Lurie and Cherkaev [17] and by Tartar [31] by means of sufficient conditions (which are the early version of the translation method). Here, this problem is used as a test for the suggested technique. We obtain the results using exclusively the technique of necessary conditions and compare two approaches.

3.1 Optimality of the fields. Forbidden region.

Consider an optimal mixture made of two materials σ_1 and $\sigma_2 > \sigma_1$ and find the range of fields permitted by the described variations. Applying formulas (2.27) and (2.32) where the material constants σ and σ' are properly chosen, we get the following inequalities.

The increment of the inclusion of material σ_2 into the domain $\mathcal{O}_1$ is :

$$\delta_{12} = (\sigma_2 - \sigma_1)\sigma_1 F_1(\sigma_1, \sigma_2, \boldsymbol{E}_1(x)) + \gamma_2 - \gamma_1 \geq 0; \tag{3.1}$$

where $\boldsymbol{E}_1$ is the field in the domain $\mathcal{O}_1$ of the first material. The increment of the inclusion of material σ_1 into the domain $\mathcal{O}_2$ is:

$$\delta_{21} = (\sigma_1 - \sigma_2)\sigma_2 F_2(\sigma_2, \sigma_1, \boldsymbol{E}_2(x)) - \gamma_2 + \gamma_1 \geq 0; \tag{3.2}$$

where $\boldsymbol{E}_2$ is the field in a point of the domain $\mathcal{O}_2$ of the second material.

Let the set of permitted values of the field in the first material be Ω_1 and the set of permitted values of the field in the second material be Ω_2:

$$\Omega_1 = \{\boldsymbol{E} : \ \delta_{12} \geq 0\}, \quad \Omega_2 = \{\boldsymbol{E} : \ \delta_{21} \geq 0\}. \tag{3.3}$$

Assume that the eigenvalues of $\boldsymbol{E}$ are ordered as

$$0 \leq E_a \leq E_b. \tag{3.4}$$

Using (2.27) and (2.32), we have:

$$\left.\begin{array}{l} \sigma_1(\sigma_2 - \sigma_1)\left(\frac{E_A^2}{\sigma_2} + \frac{E_B^2}{\sigma_1}\right) + \gamma_2 - \gamma_1 \geq 0 \ \text{ if } \frac{E_A}{E_B} \leq \frac{\sigma_1}{\sigma_2} \\ \frac{\sigma_1(\sigma_2-\sigma_1)}{\sigma_1+\sigma_2}\left(|E_A| + |E_B|\right)^2 + \gamma_2 - \gamma_1 \geq 0 \ \text{ if } \frac{E_A}{E_B} \geq \frac{\sigma_1}{\sigma_2} \end{array}\right\} \text{ in } \Omega_1 \tag{3.5}$$

and

$$\frac{E_A^2}{\sigma_1} + \frac{E_B^2}{\sigma_2} \leq (\gamma_1 - \gamma_2)\frac{1}{\sigma_2(\sigma_2 - \sigma_1)} \quad \text{in } \ \Omega_2 \tag{3.6}$$

The corresponding graph of the permitted fields is presented on figure 1. Note that between the regions Ω_1 and Ω_2 lies the forbidden region Ω_f in which none of given materials is optimal.

Let us analyse the fields in an optimal structure. Assume that an external field $\boldsymbol{P}$ is given. The field distribution depends on the Lagrange multipliers or, equivalently, on the

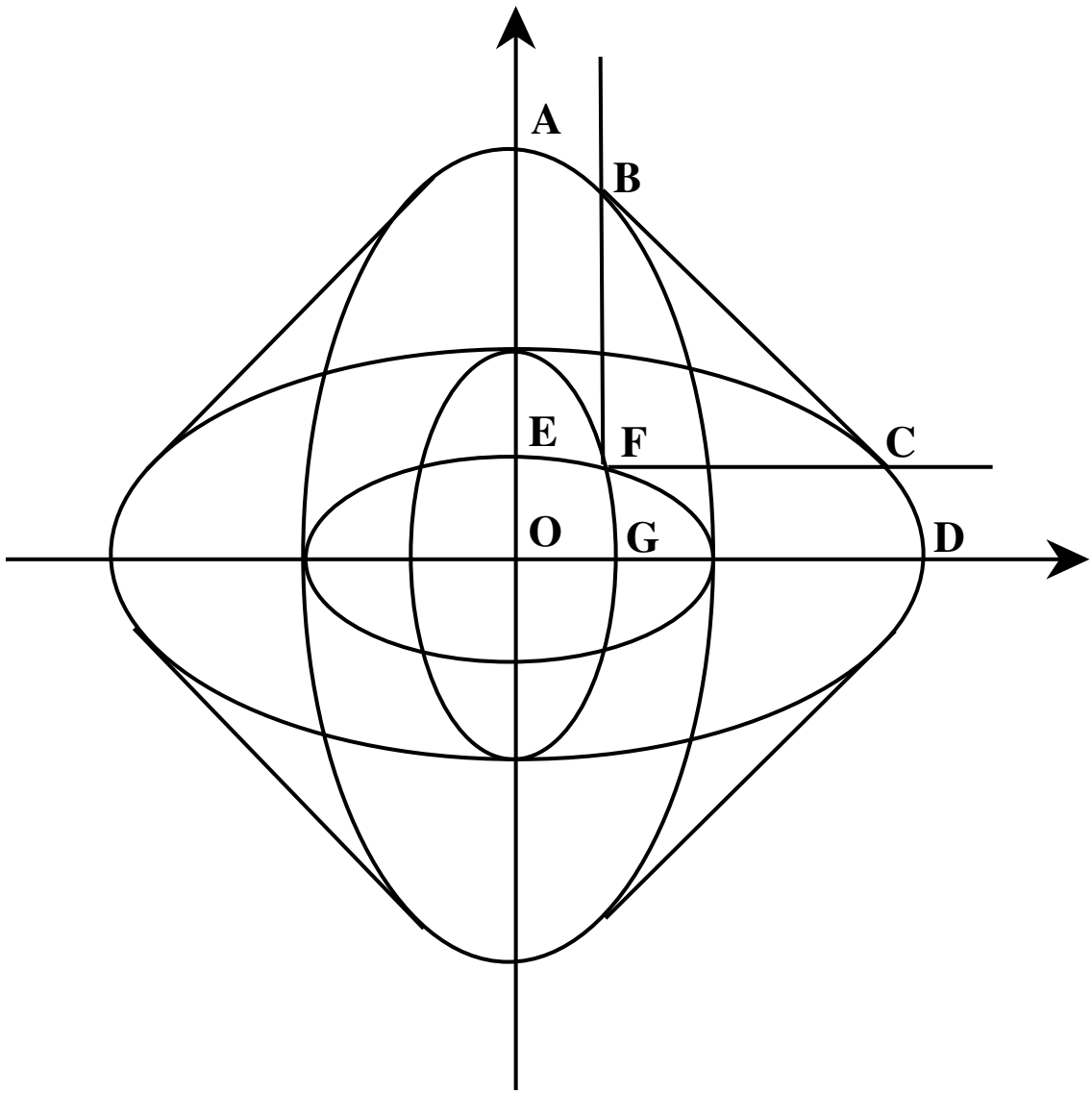

Figure 1: The permitted regions.
Region Ω_1 lies outside ABCD, region Ω_2 lies inside EFG , and in between lies the forbidden region Ω_f.

volume fraction of the materials in the mixture. If the Lagrange multipliers are chosen so that the external field belongs to Ω_1 or Ω_2, then the mixture consists of one material σ_1 or σ_2 only, the volume fractions are zero one, respectively, and the field is constant everywhere in the cell. The non-trivial case occurs when the mean field belongs to the forbidden region Ω_f. For any point in the cell $\mathcal{O}$, the field $\boldsymbol{E}$ cannot belong to this region, therefore $\boldsymbol{E}$ must belong to Ω_1 or to Ω_2. In this situation we are dealing with a true mixture and the solution of the variational problem is given by a non-smooth minimizer, i.e. the field $\boldsymbol{E}$ jumps on the boundary between these two regions.

3.2 Minimal extension.

Let us perform the minimal extension of the Lagrangian in the forbidden region. We show, that the minimal extension leads to determination of effective properties of optimal composites without the guessing optimal microstructures.

Compute the increment caused by replacing an isotropic material σ_1 by an anisotropic inclusion made of material $\boldsymbol{\sigma}_*$ with eigenvalues λ_1 and λ_2

$$\boldsymbol{\sigma}_* = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{3.7}$$

The procedure of calculating of the increment for the anisotropic inclusion is described above. Following that procedure, we find the increment caused by the inserting an unknown anisotropic material $\boldsymbol{\sigma}_*$ into the domain $\mathcal{O}_1$, where $\boldsymbol{E} \in \Omega_1$. The increment is

$$\delta_{*1}(\boldsymbol{E}, \beta) = \sigma_1 E_A^2 \frac{\lambda_1 - \sigma_1}{\beta\lambda_1 + (1-\beta)\sigma_1} + \sigma_1 E_B^2 \frac{\lambda_2 - \sigma_1}{\beta\sigma_1 + (1-\beta)\lambda_2} + \gamma_* - \gamma_1. \tag{3.8}$$

The difference $\gamma_* - \gamma_1$ of the cost of materials σ_* and σ_1 is proportional to the amount of the second material used in the formation of σ_*:

$$\gamma_* - \gamma_1 = (\gamma_2 - \gamma_1) m_2. \tag{3.9}$$

The increment depends on the parameter β that shows the elongation of the inserted inclusion. As before, we find the minimal increment (with respect to $\beta \in (0,\ 1)$) by calculating the optimal value of β and by substituting it into (3.8). The resulting increment is

$$\delta^0_{*1}(\boldsymbol{E}) = \sigma_1 \left(E_A + E_B\right)^2 \frac{(\lambda_1 - \sigma_1)(\lambda_2 - \sigma_1)}{\lambda_1\lambda_2 - \sigma_1^2} + (\gamma_2 - \gamma_1) m_2. \tag{3.10}$$

(It is assumed in this calculation that the optimal value of β lies in $(0, 1)$ and does not takes boundary values). The increment (3.10) is equal to zero, if the tensor $\boldsymbol{\sigma}_*$ (the minimal extension) is stable under the considered variations:

$$\delta^0_{*1}(\boldsymbol{E}) = 0, \quad \text{if } \boldsymbol{\sigma}_* \text{ is stable for the variation.} \tag{3.11}$$

Finally, we recall that the fields E_A and E_B on the boundary of the region Ω_1 (see (2.32)) satisfy the equality

$$(E_A + E_B)^2 = (\gamma_1 - \gamma_2)\frac{\sigma_1 + \sigma_2}{\sigma_1(\sigma_2 - \sigma_1)} \quad \text{if } \frac{E_A}{E_B} \geq \frac{\sigma_1}{\sigma_2}. \tag{3.12}$$

We substitute this value into (3.10) to get the equality for the eigenvalues of the effective tensor:

$$\frac{(\lambda_1 - \sigma_1)(\lambda_2 - \sigma_1)}{\lambda_1\lambda_2 - \sigma_1^2} = m_2\frac{\sigma_2 - \sigma_1}{\sigma_2 + \sigma_1}. \tag{3.13}$$

After obvious manipulations this equality is transferred to the familiar form

$$\frac{1}{\lambda_1 - \sigma_1} + \frac{1}{\lambda_2 - \sigma_1} = \frac{1}{m_2}\left(\frac{2}{\sigma_2 - \sigma_1} + \frac{m_1}{\sigma_1}\right), \tag{3.14}$$

that represent the boundary of G_m-closure (see Lurie and Cherkaev[17]) obtained by translation method.

In a similar way, one can obtain the minimal extension for the case when the external fields have a very different magnitudes so that the optimal value of β in (3.10) is either zero or one. One can check, that the extension corresponds to the effective tensor with eigenvalues $\lambda_1 = m_1/\sigma_1 + m_2/\sigma_2$ and $\lambda_2 = m_1\sigma_1 + m_2\sigma_2$, that also match the sufficient conditions (Wiener bounds).

We have demonstrated that the obtained necessary conditions of optimality lead to an extension that coincide with an extension provided by sufficient conditions (the translation method). This means that these conditions are exact for the considered problem.

3.3 Optimality conditions and the structures.

Compatibility. Let us analyse the system of necessary conditions of optimality aiming to obtain information about the microstructures of optimal composites. To begin, we find a form for these conditions that does not depend on γ_1 and γ_2. Consider the sum of the variations (2.27) and (2.32): we interchange two small inclusions of equal volume, putting the inclusion of σ_1 into the domain $\mathcal{O}_2$ occupied by σ_2 and vice versa. Note that inclusions may be of different shapes, only their volume are equal. Clearly, this variation does not change the total amounts of materials σ_1 and σ_2. Therefore the values of Lagrange multipliers γ_1 and γ_2 are irrelevant.

The total change in the functional due to the interchange of the materials is found from (3.1) and (3.2) :

$$\delta_{12} + \delta_{21} = (\sigma_2 - \sigma_1)\left(\sigma_2 F_1(\sigma_1, \sigma_2, E_1(x)) - \sigma_1 F_2(\sigma_2, \sigma_1, E_2(x))\right). \tag{3.15}$$

The variation cannot decrease the energy of the optimal structure (which possesses the minimal energy), therefore the inequality holds

$$\delta_{12} + \delta_{21} \geq 0, \quad \forall \boldsymbol{E}_1 \in \Omega_1, \ \forall \boldsymbol{E}_2 \in \Omega_2, \tag{3.16}$$

or

$$\sigma_2 F_1(\sigma_1, \sigma_2, E_1(x)) - \sigma_1 F_2(\sigma_2, \sigma_1, E_2(x)) \geq 0 \;\; \forall \boldsymbol{E}_1 \in \Omega_1, \;\; \forall \boldsymbol{E}_2 \in \Omega_2. \tag{3.17}$$

This inequality shows that the gap between the fields in different materials must be large enough. Consider the question: how these optimality conditions match the jump conditions on the phase boundaries?

The jump conditions. Let us compute the jump of the matrix $\boldsymbol{E}$ on the boundary line between zones $\mathcal{O}_1$ and $\mathcal{O}_2$. Denote the normal to the boundary as $\boldsymbol{n} = (\cos\theta, \; \sin\theta)$. The elements of the matrix (see (2.7)) $\tilde{\boldsymbol{E}} : \; \nabla \times \tilde{\boldsymbol{E}} = 0$ are discontinuous along the boundary. However, certain relations holds between the fields on the opposite sides of the boundary. The continuity conditions that follow from from the curl free nature of the fields $\tilde{\boldsymbol{E}}_i$ and the natural variational boundary conditions are

$$\left[\sigma \tilde{\boldsymbol{E}}_i \cdot \boldsymbol{n}\right]_-^+ = 0, \quad \text{and} \quad \left[\tilde{\boldsymbol{E}}_i \cdot \boldsymbol{t}\right]_-^+ = 0, \quad i = 1, 2. \tag{3.18}$$

where $[.]_-^+$ denotes the jump: $[z]_-^+ = z_+ - z_-$.

Remark 3.1 *The last condition means that rank$(\tilde{\boldsymbol{E}}_2 - \tilde{\boldsymbol{E}}_1) = 1$, it is called 'the rank-one connection' between the fields matrices $\tilde{\boldsymbol{E}}_2$ and $\tilde{\boldsymbol{E}}_1$).*

The conditions (3.18) can be rewritten as

$$\left[\begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \tilde{\boldsymbol{E}}\right]_-^+ = 0. \tag{3.19}$$

where ϕ in the angle of the decline of of the normal to the boundary.

We want to compare the jump conditions with the necessary optimality conditions. The technical difficulty is that the jump conditions are expressed in terms of the matrix $\tilde{\boldsymbol{E}}_i$ while the optimality condition as written in terms of the symmetric matrix $\boldsymbol{E} = \left(\tilde{\boldsymbol{E}}\tilde{\boldsymbol{E}}^T\right)^{-1}$. Therefore we need to express the jump conditions in terms of the matrix $\boldsymbol{E}$.

The conditions (3.19) imply in particular that the determinants of the matrices $\tilde{\boldsymbol{E}}$ and $\boldsymbol{E}$ on the opposite sides of the boundary are connected by an equality

$$[\sigma \det \tilde{\boldsymbol{E}}_i]_-^+ = [\sigma \det \boldsymbol{E}]_-^+ = 0. \tag{3.20}$$

applied to both matrices $\tilde{\boldsymbol{E}}$ and $\boldsymbol{E}$.

More complicated is the calculation of the jump of the trace of the matrix $\tilde{\boldsymbol{E}}$. It depends on the normal $\boldsymbol{n}$ to the boundary. Using (3.18), we compute the jump of the trace of $\tilde{\boldsymbol{E}}$ and compute the jump of the trace of the matrix $\boldsymbol{E}$ which depends, as on parameters, on the angle between the normal and an eigendirection of $\boldsymbol{E}$ and on the antisymmetric part of $\tilde{\boldsymbol{E}}$. Finally, we compute the bounds of the jump of Tr $\boldsymbol{E}$ by optimization with respect to those

parameters. We do not show here the details of this rather technical calculation performed by using Maple, but simply review the results.

The jump condition leads to inequality

$$\mathrm{Tr}\ (\boldsymbol{E}_1) \leq \frac{\sigma_1}{\sigma_2} \min\{(e_2)_a,\ (e_2)_b\} + \max\{(e_2)_a,\ (e_2)_b\} \tag{3.21}$$

The maximum value of the jump of the trace of $\boldsymbol{E}$ (3.21) corresponds to the following requirements on the fields close to dividing line:

1. the matrix $\tilde{\boldsymbol{E}}$ is symmetric, so that $\tilde{\boldsymbol{E}} = \boldsymbol{E}$ at both sides the boundary.

2. $\theta = 0$, which means that the normal to the boundary $\boldsymbol{n}$ is codirected with the eigendirection of $\tilde{\boldsymbol{E}}$ that corresponds to its minimal eigenvalue.

3. Unless the above conditions are satisfied, the magnitude of the jump is insufficient to pass from a point in Ω_1 to a point inΩ_2.

4. In the above conditions are satisfied, and if the field in phase σ_1 belongs to the boundary $\partial\Omega_1$ of the set of permitted fields Ω_1 in the neighborhood of the dividing line, then it is possible to jump over the forbidden region. In this case, the field on the boundary of the domain $\mathcal{O}_2$ of σ_2 belongs to the boundary $\partial\Omega_2$ of Ω_2. That is, only under these conditions the jump conditions and the necessary conditions are compatible with each other.

This result implies that the derived necessary conditions are the strongest possible for the considered problem, or that the considered variation is the most sensitive. If there were a variation which would lead to larger forbidden region, then it would be impossible to jump across this region. The system of such hypothetical necessary conditions would be inconsistent with the jump on the dividing line.

From the above consideration, the normal to the boundary between Ω_1 and Ω_2 is not uniquely determined only if field $\boldsymbol{E}$ in Ω_2 is 'isotropic': $\boldsymbol{E}_2 = \beta\boldsymbol{I}$. The field in the first material on the boundary satisfies equality

$$\frac{E_a}{E_b} = \frac{\sigma_1}{\sigma_2}.$$

3.4 Necessary conditions and the known optimal geometries.

To illustrate the obtained system of necessary conditions, two examples of the known optimal structures are examined.

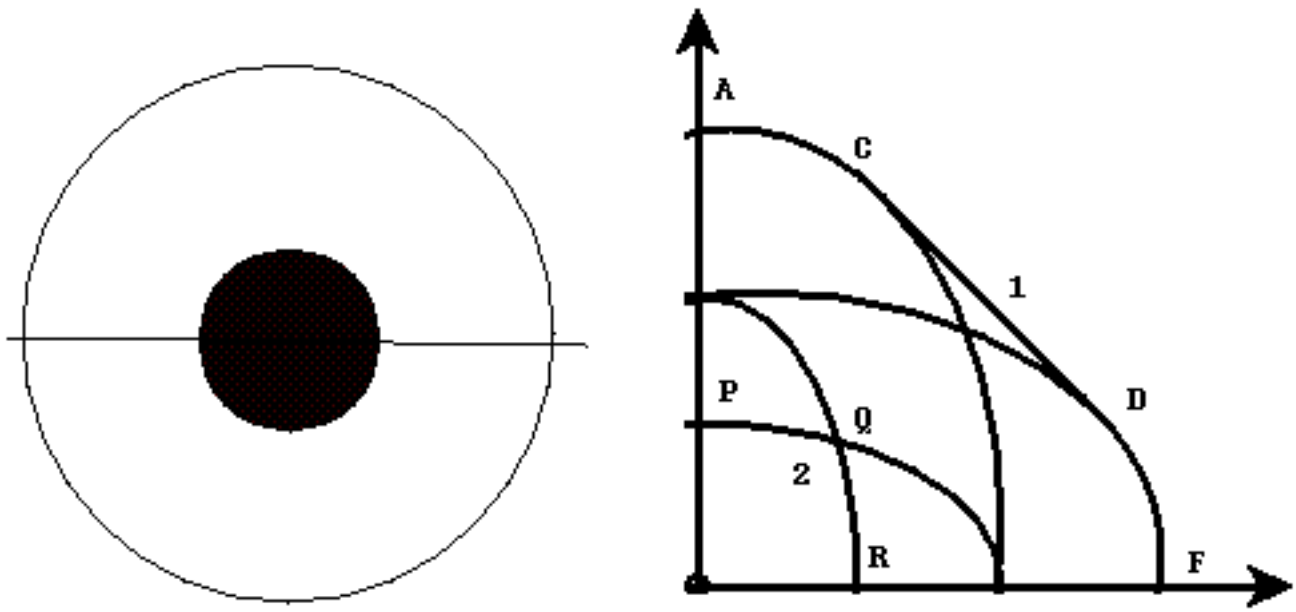

Figure 2: The structure of coated spheres.
The field inside the second material is isotropic (point Q), the field in the core varies along the interval CD, the boundary between regions corresponds to the point C.

Coated spheres. First, consider the construction of coated spheres (Hashin and Shtrikman, [12]), figure 2. This geometry provides the best isotropic effective modulus [12]. If placed in two equal orthogonal fields of the same magnitude ($\boldsymbol{P} = I$), it deliver the minimum of the sum of stored energy.

The effective properties of the structure are determined as following. The coated spheres structure, placed in the homogeneous medium with effective isotropic conductivity σ_*, leaves the outside field uniform and equal to the unit matrix. The effective conductivity of this structure found by Hashin and Shtrikman bound [12], is

$$\sigma_* = \sigma_{HS} = \sigma_1 + (\frac{m_1}{2\sigma_1} + \frac{m_2}{\sigma_1 + \sigma_2})^{-1}. \tag{3.22}$$

The potentials for this geometry are

$$u_1 = R(r)\cos\theta, \qquad u_2 = R(r)\sin\theta; \tag{3.23}$$

where r is the radius, and

$$R(r) = \begin{cases} a_0 r & \text{if} \quad 0 \le r \le r_0, \\ a_1 r + b_1/r & \text{if} \quad r_0 \le r \le 1, .. \\ 1r & \text{if} \quad 1 \le r \end{cases} \tag{3.24}$$

Here

$$\begin{aligned} a_0 &= \tfrac{2\,\sigma_2}{m_2\,\sigma_1 + (1+m_1)\,\sigma_2}, \\ a_1 &= \tfrac{\sigma_1+\sigma_2}{m_2\,\sigma_1 + (1+m_1)\,\sigma_2}, \\ b_1 &= \tfrac{m_1\,(-\sigma_1+\sigma_2)}{m_2\,\sigma_1 + (1+m_1)\,\sigma_2}. \end{aligned} \tag{3.25}$$

The eigenvalues of $\boldsymbol{E}$ are:

$$E_r = R', \quad E_\theta = R/r, \tag{3.26}$$

and the eigendirections coincide with the coordinate lines in polar coordinates.

From the solution (3.23), (3.24), one sees that the field in the nucleus is isotropic:

$$\boldsymbol{E}_r^1 = \boldsymbol{E}_\theta^1 = a_0 = \frac{2\,\sigma_2}{m_2\,\sigma_1 + (1+m_1)\,\sigma_2}. \tag{3.27}$$

The jump conditions on the boundary between the nucleus and the envelope are satisfied:

$$\begin{aligned} \boldsymbol{E}_r &= a_1 - b_1/r^2 = (\sigma_2/\sigma_1)\,a_0 \\ \boldsymbol{E}_\theta &= a_1 + b_1/r^2 = a_0 \end{aligned}, \tag{3.28}$$

the requirements for the fields on the dividing line are satisfied as well: The field on one side of the boundary is isotropic, on the other side the matrix $\tilde{\boldsymbol{E}}$ is symmetric, the eigendirections are codirected in each point with the radial direction, that is with the normal.

The field in the envelope has the constant trace:

$$\boldsymbol{E}_r^1 + \boldsymbol{E}_\theta^1 = 2a_1; \tag{3.29}$$

meaning that it belongs to the straight component of the boundary of Ω_1 everywhere.

Let us check the optimality conditions at the boundary point $r = 1$ where the materials σ_1 and σ_{HS} meet. We have

$$\boldsymbol{E}_r = a_1 - b_1 = \frac{(1+m_1)\,\sigma_1 + m_2\,\sigma_2}{m_2\,\sigma_1 + (1+m_1)\,\sigma_2} = \frac{\sigma_{HS}}{\sigma_1}$$
$$\boldsymbol{E}_\theta = a_1 + b_1 = 1 \tag{3.30}$$

so that the necessary condition are satisfied again on the boundary between the material σ_1 and the effective medium $\sigma_* = \sigma_{HS}$.

Therefore the structure satisfies the necessary conditions as equalities everywhere. The optimality of this structure has been established by the sufficient Hashin and Shtrikman bound [12]; this calculation shows the limiting character of the necessary conditions.

Matrix laminates. The conditions can be checked on the other known optimal two-phase geometry: the second rank matrix laminates (see 3). These structures are known (Lurie and Cherkaev [17]) to be optimal in anisotropic external fields. They are built as the laminate composite of the layers of the first material and the layer made of laminates of the first and second materials, mixed in smaller scale and oriented orthogonal to the larger layers. The calculation of the fields can be done using the technique described in Appendix. One can check that optimization of the sum of the energies leads to following results: ***i***) The field in the nucleus is isotropic and belongs to the corner point of Ω_2 ***ii***) the contact conditions between the fields $\boldsymbol{E}_2$ and $\boldsymbol{E}_{1,1}$ in neighboring layers are satisfied. ***iii***) Both fields $\boldsymbol{E}_{1,1}$ and $\boldsymbol{E}_{1,2}$ belong to the boundary of Ω_1. The mean field in $\mathcal{O}_1$ belongs to that boundary as well.

When the anisotropy of the exterior field varies, the field in the nucleus remains isotropic, and the field in the outside layer varies along the straight component of the boundary of Ω_1. When the field reaches the end of that component, the structure then degenerate into a simple laminate structure. In that case, both fields belong on elliptical parts of the boundaries of $\mathcal{O}_1$ and $\mathcal{O}_2$. Again, that all necessary conditions are satisfied as equality in every point.

The linearity of the components of $\partial\Omega_i$ and non-uniqueness of solution. The two above examples demonstrate that the fields are different in the structures of coated spheres and of matrix laminates. However, if both structures are isotropic, they possess the same extremal effective properties that also can be obtained by the formal procedure minimal extension. Why does this happen?

The minimal extension procedure treats the fields in each phase as single tensors not as functions of the point of the domain; these fields are in fact the mean fields within the

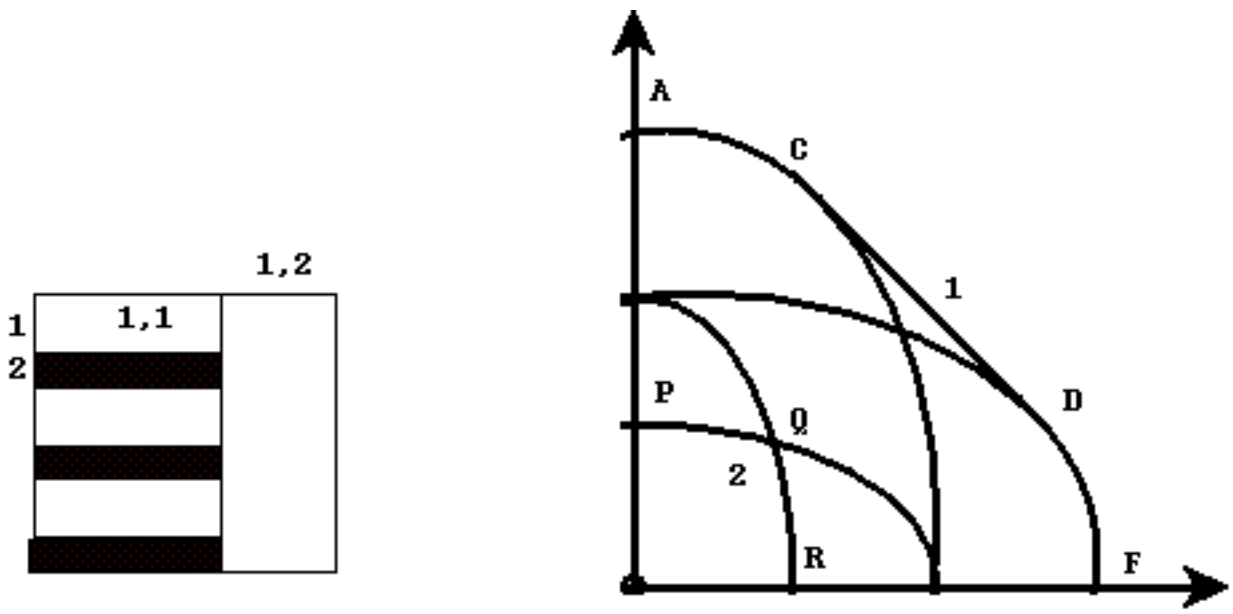

Figure 3: Matrix laminate structures.
The field on the second material is isotropic (point Q in Ω_2). The field on the region (1.1) is in rank-one connection with the field in Ω_2 (point C). The field on the region (1.2) varies with the external field, and it belongs to interval CD.

correspondent subdomains $\mathcal{O}_i$. The minimization among the fields $\boldsymbol{E}_i \in \Omega_i$ in the minimal extension procedure requires that the mean fields belong to the boundary $\partial\Omega_i$.

$$\boldsymbol{E}_i = \int_{\mathcal{O}_1} \boldsymbol{E}(\boldsymbol{x})\chi_i(\boldsymbol{x}) \quad \in \ \partial\Omega_i \tag{3.31}$$

On the other hand, the field $\boldsymbol{E}(\boldsymbol{x})$ belongs to Ω_i in each point $\boldsymbol{x}$. This shows that $\boldsymbol{E}(\boldsymbol{x})$ may vary from point to point in $\mathcal{O}_i$, if only it varies among a component $\partial_1\Omega_i \subset \partial\Omega_i$.

$$\boldsymbol{E}(\boldsymbol{x}) \in \partial_1\Omega_i, \quad \forall\, \boldsymbol{x} \in \mathcal{O}_i \tag{3.32}$$

and if his component is linear

$$\boldsymbol{E} = \sum c_k \boldsymbol{E}_k \in \partial_1\Omega_i, \quad \text{if } \ \boldsymbol{E}_k \in \partial_1\Omega_i, \ c_k \geq 0, \ \sum c_k = 1 \tag{3.33}$$

For example, the trace of the field in the first phase is constant everywhere in both checked structures. The mean field $\int_{\mathcal{O}} \boldsymbol{E}_1(\boldsymbol{x})$ obviously has the same trace. Therefore the necessary conditions (the constancy of the trace) are satisfied for $\boldsymbol{E}_1(\boldsymbol{x})$ in every point $\boldsymbol{x}$ and also for the mean field.

One could expect non-unique microstructures of optimal mixture if a permitted region Ω_i has a linear component of the boundary; also in this case one could expect that the field in the i-th material varies along this component. In the contrary, the strongly convex region Ω_i (like the region Ω_2 in the checked examples) suggests that the field in the i-th material is constant in every point of $\mathcal{O}_i$ and it may correspond to a compact inclusion like the nucleus in the checked geometries.

4 Variations for three materials mixtures.

4.1 Single variations.

Setting. Next, we apply the developed technique to an open problem of optimal three-material mixtures. Consider again a mixture with minimal conductivity assembled of three materials with conductivities

$$\sigma_1 < \sigma_2 < \sigma_3. \tag{4.1}$$

To simplify calculations, assume that

$$\sigma_3 = \infty, \tag{4.2}$$

i.e.the third material is an ideal conductor.

We normalize cost of the materials and we assign the cost γ_2 of the intermediate material to be between the costs of the extremal materials

$$\gamma_1 = 1, \quad \gamma_3 = 0, \quad 0 < \gamma_2 < 1. \tag{4.3}$$

The variations. To begin, we compute the necessary conditions for the fields in each of the materials, using the variational method described above. Namely, we place each of three materials into regions occupied by the other two materials, and calculate the corresponding inequalities for the admissible fields.

The trial inclusion of material σ_3 with infinite conductivity in the regions Ω_1 and Ω_2 occupied with materials σ_1 and σ_2 respectively, corresponds to the inequalities:

$$F_1(\infty, \sigma_1, \boldsymbol{E}_1) \geq 0 \quad \Longrightarrow \quad E_a + E_b \geq \sqrt{\frac{1}{\sigma_1}} \quad \text{in} \quad \Omega_1 \tag{4.4}$$

$$F_1(\infty, \sigma_2, \boldsymbol{E}_2) \geq 0 \quad \Longrightarrow \quad E_a + E_b \geq \sqrt{\frac{\gamma}{\sigma_2}} \quad \text{in} \quad \Omega_2. \tag{4.5}$$

These inequalities follow from (2.27) (2.32) if the materials are properly specified.

The trial inclusion of the material σ_2 in Ω_1 produces the inequalities (remind that $E_a \leq E_b$, see (3.4)):

$$F_1(\sigma_2, \sigma_1, \boldsymbol{E}_1) \geq 0 \quad \Longrightarrow \quad \left.\begin{array}{ll} \sigma_1(\sigma_2 - \sigma_1)\left(\frac{E_a^2}{\sigma_2} + \frac{E_b^2}{\sigma_1}\right) + \gamma - 1 \geq 0 & \text{if } \frac{E_a}{E_b} \leq \frac{\sigma_1}{\sigma_2} \\ \frac{\sigma_1(\sigma_2-\sigma_1)}{\sigma_1+\sigma_2}\left(E_a + E_b\right)^2 + \gamma - 1 \geq 0 & \text{if } \frac{E_a}{E_b} \geq \frac{\sigma_1}{\sigma_2} \end{array}\right\} \text{ in } \Omega_1 \tag{4.6}$$

The trial inclusion of materials σ_1 in Ω_2 produces the inequality:

$$F_2(\sigma_1, \sigma_2, \boldsymbol{E}_2) \geq 0 \quad \Longrightarrow \quad \sigma_2(\sigma_1 - \sigma_2)\left(\frac{E_a^2}{\sigma_2} + \frac{E_b^2}{\sigma_1}\right) + 1 - \gamma \geq 0 \quad \text{in} \quad \Omega_2 \tag{4.7}$$

Finally, placing the inclusions of the material σ_1 into Ω_3 leads to

$$F_2(\sigma_1, \sigma_3, \boldsymbol{E}_3) \geq 0 \quad \Longrightarrow \quad E_a = E_b = 0 \quad \text{in } \Omega_3 \tag{4.8}$$

The fields in the third superconducting phase is always zero, as they should be.

The topology of the obtained permitted regions Ω_i is described as follows. The region Ω_1 is permitted for fields of great magnitudes, region Ω_3 is permitted for zero fields only. The forbidden region Ω_f lies between these two regions, which makes the picture similar to that of the problem for two materials. Within the forbidden region is located the region Ω_2 where the second (intermediate) material is optimal. If this region is not empty, it either divides Ω_f into two disconnected parts (forms a 'belt' in Ω_f), or it leaves Ω_f connected (forms an 'island' in Ω_f). (see figure 4).

4.2 Composite variations.

Before the analysing the system of necessary conditions, let us discuss if these conditions are the strongest ones. The analysis of the section 3.3 is applicable to a common part of the

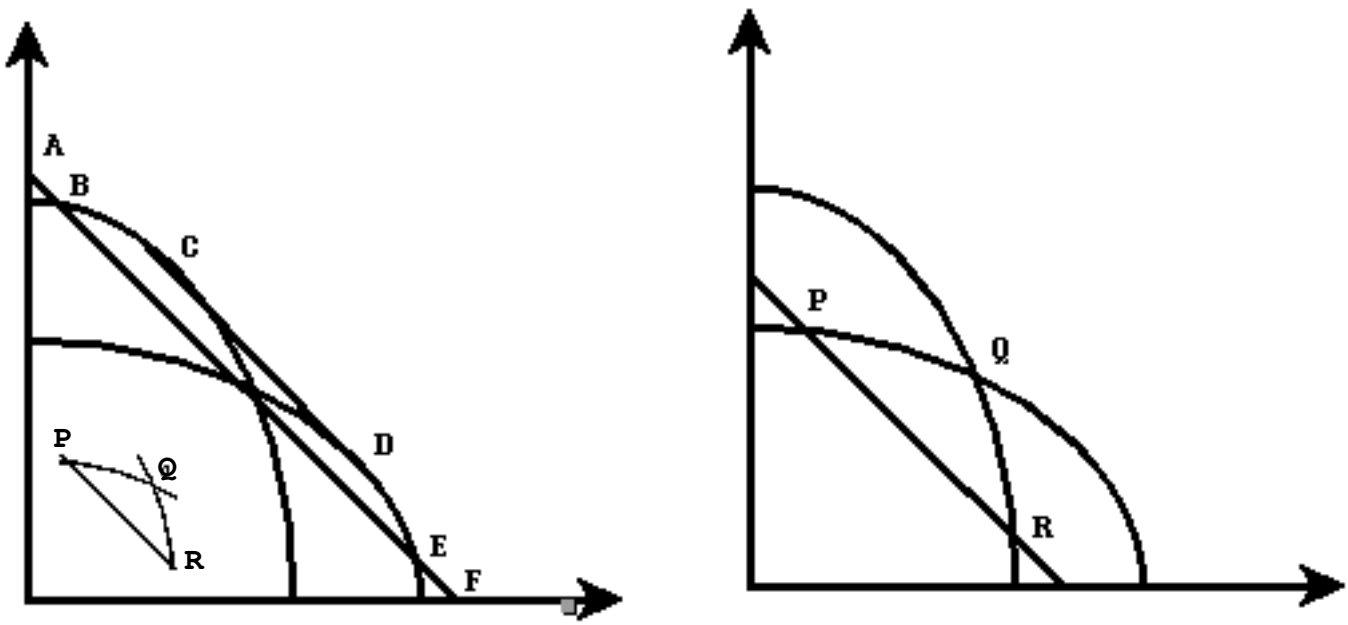

Figure 4: The permitted regions, based on single variations.
Left field - The regions Ω_1 and Ω_2, right field: magnified picture of the region Ω_2.
The boundary curve $B\,C\,D\,E$ comes from the inclusions of σ_2 into Ω_1, the boundary straight line $A\,F$ comes from the inclusions of σ_3 into Ω_1, the boundary due to intersection of two ellipses comes from the inclusions of σ_1 into Ω_2, the boundary straight line $P\,R$ comes from the inclusions of σ_3 into Ω_2.
The mutual positions of the boundary curves depends on γ.

boundary between domains of any two materials. It shows, that the considered variation and the corresponding inequalities lead to the maximal forbidden region that is consistent with the assumption that the pair of the available materials have a common boundary. One could therefore wrongly conclude that the no more variations is needed.

However, it is possible that the geometry of multi-materials composites includes zones where all three materials are densely mixed together. The dividing curves contains a dense set of points where $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$ meet. Our previous analysis is not applicable to the proximity of these points. The properties of such mixtures may be significantly different from a two-material mixtures.

To exam the optimality of such boundaries, one can also look for more complicated types of local variations. The variation is performed as following: A composite of two available materials is placed into a domain of the third one. A mixture inside the inclusion is described by its tensor of effective properties $\boldsymbol{\sigma}_{inc}$. We use knowledge of the bounds of effective properties of any two-component mixture (the G-closure problem) to solve this problem for three materials

4.2.1 Improving of Ω_2

Scheme. The scheme is the following. We form an inclusion of materials σ_1 and σ_3 and place it in the domain $\mathcal{O}_2$ of second material. The material in the inclusion is a composite from materials σ_1 and σ_3. Let c be the volume fraction of material σ_1 (the fraction of σ_3 obviously is $1-c$).

The change of materials' cost due to the variation is computed as the difference between the cost of the taken material σ_2 and the cost of the inserted materials:

$$\Gamma = -\gamma_2 + c\gamma_1 + (1-c)\gamma_3. \tag{4.9}$$

In our setting (see (4.3)) the cost is

$$\Gamma = \gamma + c \tag{4.10}$$

The increment has the form

$$\delta_{c,2}(\boldsymbol{E}) = \delta W + \Gamma \tag{4.11}$$

where Δ is the increment of the energy caused by a composite inclusion inserted into the domain of σ_2. The inclusion is anisotropic.

Compute the increment $\delta_{c,2}$. Denote the eigenvalues of the composite in the inclusion as $(l_1(c), l_2(c)$. Using (3.8), we compute the increment of the energy:

$$\delta W = \sigma_2 E_a^2 \frac{l_1(c) - \sigma_2}{\alpha l_1(c) + (1-\alpha)\sigma_2} + \sigma_2 E_b^2 \frac{l_2(c) - \sigma_2}{\alpha\sigma_2 + (1-\alpha) l_2(c)}; \tag{4.12}$$

as before, the parameter $\alpha \in [0,1]$ defines the rate of anisotropy of the second-rank laminate structure or the elongation (eccentricity) of the equivalent elliptical inclusion.

In order to get the maximally sensitive variation the inclusion is chosen as a composite of extremal anisotropic conductivity which belongs to the boundary of G-m closure for two materials. The boundary is realized by second rank laminates that have the eigenvalues (see [17]

$$\begin{aligned} l_1(c,\beta) &= \sigma_1 + (1-c)\left(\tfrac{1}{\sigma_3-\sigma_1} + \tfrac{c\,\beta}{\sigma_1}\right)^{-1} \\ l_2(c,\beta) &= \sigma_1 + (1-c)\left(\tfrac{1}{\sigma_3-\sigma_1} + \tfrac{c\,(1-\beta)}{\sigma_1}\right)^{-1}, \end{aligned} \tag{4.13}$$

where $\beta \in [0,1]$ defines the degree of anisotropy of a composite (the rate of the elongation of the inclusions of $\sigma_2 3$) and c is the fraction of material σ_1. When β varies in the interval$[0,1]$, each eigenvalue l_i varies in the interval $[(1/(c/\sigma_1 + (1-c)/\sigma_3), (c\sigma_1 + (1-c)\sigma_3]$. If $\sigma_3 = \infty$, the formulas become

$$\begin{aligned} l_1(c,\beta) &= \sigma_1\left(1 + \tfrac{1-c}{c\beta}\right) \\ l_2(c,\beta) &= \sigma_1\left(1 + \tfrac{1-c}{c(1-\beta)}\right) \end{aligned}; \tag{4.14}$$

each eigenvalue l_i varies in the interval $[\sigma_1/c,\ \infty]$.

The increment δW depends on three parameters: $(\boldsymbol{i})$ the volume fraction c of material that form the composite in the inclusion, $(\boldsymbol{ii})$ the degree of anisotropy of that composite β, and $(\boldsymbol{iii})$ the relative elongation (eccentricity) α of the inclusions, Each parameter varies in the interval $[0,\ 1]$. To perform an extremal variation, we solve the problem

$$\delta_0 = \min_{c\in[0,\ 1]}\ \min_{\alpha\in[0,\ 1]}\ \min_{\beta\in[0,\ 1]}\ \delta W(\alpha,\beta,c) + \Gamma(c). \tag{4.15}$$

where $\delta_{c,2}$ is defined by (4.11)

Laminate variation. The minimization of (4.15) with respect to β and α is independent of the cost of the materials since the change only affects only geometrical structure of an inclusion but not the quantity of materials involved. Each parameter β and α, varies in the interval $[0,\ 1]$. Therefore the optimal point corresponds either to an inner point of the square of parameters values $\beta \in (0,\ 1),\ \alpha \in (0,\ 1)$ or to its side ($\beta = 0\ or\ ,\ \beta = 1,\ \ \alpha \in (0,\ 1)$) or ($\beta \in (0,\ 1),\ \alpha = 0\ or\ ,\ \alpha = 1$).

First, let us check the case ($\beta = 1$): the inclusion is a laminate composite with eigenvalues

$$l_1(c) = \sigma_1/c, \quad l_2 = \infty. \tag{4.16}$$

The increment (4.12) becomes

$$\delta = \sigma_2 E_a^2 \frac{\sigma_1 - c\sigma_2}{\alpha\sigma_1 + c(1-\alpha)\sigma_2} + \sigma_2 E_b^2 \frac{1}{(1-\alpha)} + \gamma + c, \quad \alpha \in [0,\ 1]. \tag{4.17}$$

The stationary with respect to α points make the increment linear with respect to c. Therefore the optimal values of c are either zero or one, which reduces the complex variations to the already investigated case of single variations.

Check the corners of the region of parameters ($\alpha = 0$ *or* $\alpha = 1$). This case corresponds to a laminate composite is placed in a laminate shaped inclusion. The value $\alpha = 1$ leads to infinite increment, and is obviously not optimal, and the value $\alpha = 0$ leads to condition:

$$\delta_{c,2}(\boldsymbol{E}) = \Delta - \gamma + c = (\sigma_1/c - \sigma_2)E_a^2 + \sigma_2 E_b^2 - \gamma + c \geq 0 \quad \forall\, c \in [0,1] \tag{4.18}$$

It depends only on volume fraction c.

The optimal value c_0 is found from the condition

$$\frac{\partial}{\partial c}\delta W_{c,2} = 1 - \frac{\sigma_1}{c^2}E_a^2 \tag{4.19}$$

and is equal to

$$c_0 = \begin{cases} \sqrt{\sigma_1}E_a & \text{if } E_a \leq \frac{1}{\sqrt{\sigma_1}} \\ 1 & \text{if } E_a \geq \frac{1}{\sqrt{\sigma_1}} \end{cases} \tag{4.20}$$

The case $c_0 = 1$ corresponds to two-materials variation. The inequality $E_a \leq \frac{1}{\sqrt{\sigma_1}}$ is surely satisfied for all fields $\boldsymbol{E} \in \Omega_2$, that are optimal from the viewpoint of single variations (see (4.7))

The case leads when $c_0 \in (0,1)$ leads to the new necessary condition

$$\Delta_0 = 2\sqrt{\sigma_1}E_a - \sigma_2 E_a^2 + \sigma_2 E_b^2 - \gamma \geq 0, \tag{4.21}$$

or

$$\left(\sqrt{\sigma_2}E_a - \sqrt{\frac{\sigma_1}{\sigma_2}}\right)^2 - \sigma_2 E_b^2 \leq -\frac{\sigma_1}{\sigma_2} + \gamma. \tag{4.22}$$

This inequality provides an additional restriction on Ω_2. The condition checks the optimality of a boundary between the phase σ_2 and the pack of orthogonal to the boundary laminates of σ_1 and σ_3. This boundary is is not a dividing line between any two of available phases, and the jump conditions on it involves all three phases. Clearly, the variation that involves only two materials is not enough selective to judge about its optimality.

Note, The conditions (4.21) and (4.22) are supplemented by twin conditions in which the eigenvalues E_a and E_b are interchanged. Finally we get the inequalities (see figure 5)

$$F_3(\sigma_2, \sigma_1, \infty; E) = \max\left\{\left(\sqrt{\sigma_2}E_a - \sqrt{\frac{\sigma_1}{\sigma_2}}\right)^2 - \sigma_2 E_b^2,\ \left(\sqrt{\sigma_2}E_b - \sqrt{\frac{\sigma_1}{\sigma_2}}\right)^2 - \sigma_2 E_a^2\right\} \leq -\frac{\sigma_1}{\sigma_2} + \gamma \tag{4.23}$$

The obtained inequalities restrict the fields in the set Ω_2 by the requirement that these fields lie above the intersections of two hyperbolas with asymptotes:

$$(\pm E_a \pm E_b) = \frac{\sqrt{\sigma_1}}{\sigma_2} \tag{4.24}$$

These inequalities are added to the restrictions obtained by single variations (4.5), (4.7).

The graph of Ω_2 is represented on figure 5.

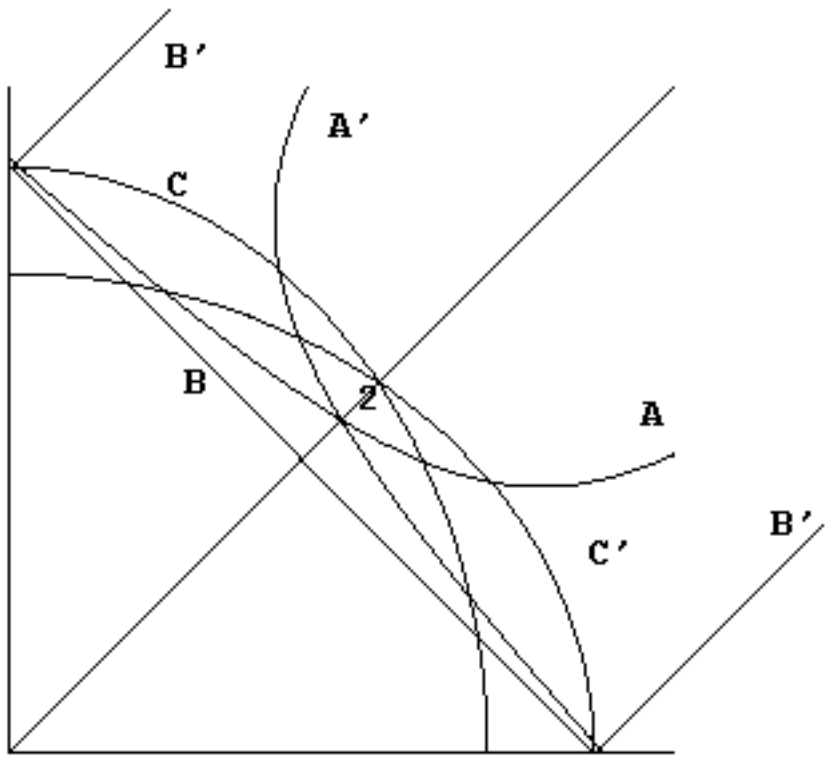

Figure 5: Permitted region Ω_2, based on complex variations.
A, A' - Hyperbolic bounds, obtained from composite variations. B, B' - The asymptotes of the hyperbolas. B - The (improved) bound, obtained from single variation. C, C' - The bound, obtained from single variation.

Other variations. It can be regularly shown that no other choice of parameters β, α, and c improves the bound given by the obtained inequality (4.21) and the single permutations. The formal investigation is routine but a long one, and Maple is a real help. Instead of presenting here the details of corresponding calculations, let us discuss physical reasons of why the other variations fail to improve the bounds.

Formally, one can check that the stationary points of the increment $\delta_{c,2}$ inside the square in the variables β, α corresponds to saddles and therefore are not optimal. This can be explained by the topology of these variations: if $\beta \in (0,\ 1)$ then the material σ_3 is placed inside the material σ_1 in the inclusions and the resulting structure is placed into the domain $\mathcal{O}_2$ occupied σ_2. This construction suggests the existence of the bounds between pairs materials σ_2 and σ_1 and between σ_1 and σ_3 (inside the inclusions). But the optimality of these boundaries corresponds to the variations of the single permutations, therefore the more complicated variations do not produce new more restricted inequalities.

4.2.2 Improving of Ω_1

To complete the investigation we need to consider two other schemes: the mixture of materials σ_2 and σ_3 placed into domain of σ_1 and the mixture of materials σ_1 and σ_2 placed into domain of σ_3. In our setting, the last case is trivial because the field in the third (superconducting) phase is always zero.

The complex variations lead to the decreasing of the domain Ω_1. The scheme of the variations is the same as in the previous case. Note that we can a priori restrict ourself to the case $\beta = 0$ because we look for inclusions that produce a common boundary between all three materials. The new inequality can be algebraically obtained from (4.11), (4.12), (4.14) where one interchange σ_1 and σ_2 and put $\Gamma = 1 - c\gamma$. The optimal variation corresponds to a strip-like inclusion assembled with perpendicular layers of σ_2 and σ_3; the fraction c of σ_2 is optimally chosen.

The corresponding graph is shown on figure 6. The new boundary component ABC of the boundary of Ω_1 corresponds to a hyperbola that joins the corner point A (where $c = 0$ and strip-like inclusions of σ_3 are optimal) and the elliptical boundary component C (where $c = 1$ and strip-like inclusions of σ_2 are optimal).

5 Some optimal structures for three-materials mixtures.

5.1 Range of values of the Lagrange multiplier.

The initial optimization problem of the composite structure is characterized by the parameters: the volume fractions m_1 and m_2 of the first and second materials in the mixture, and the degree of anisotropy of resulting composite. The volume fractions are subject to the obvious constrains

$$m_1 \geq 0,\ m_2 \geq 0,\ m_1 + m_2 \leq 1. \tag{5.1}$$

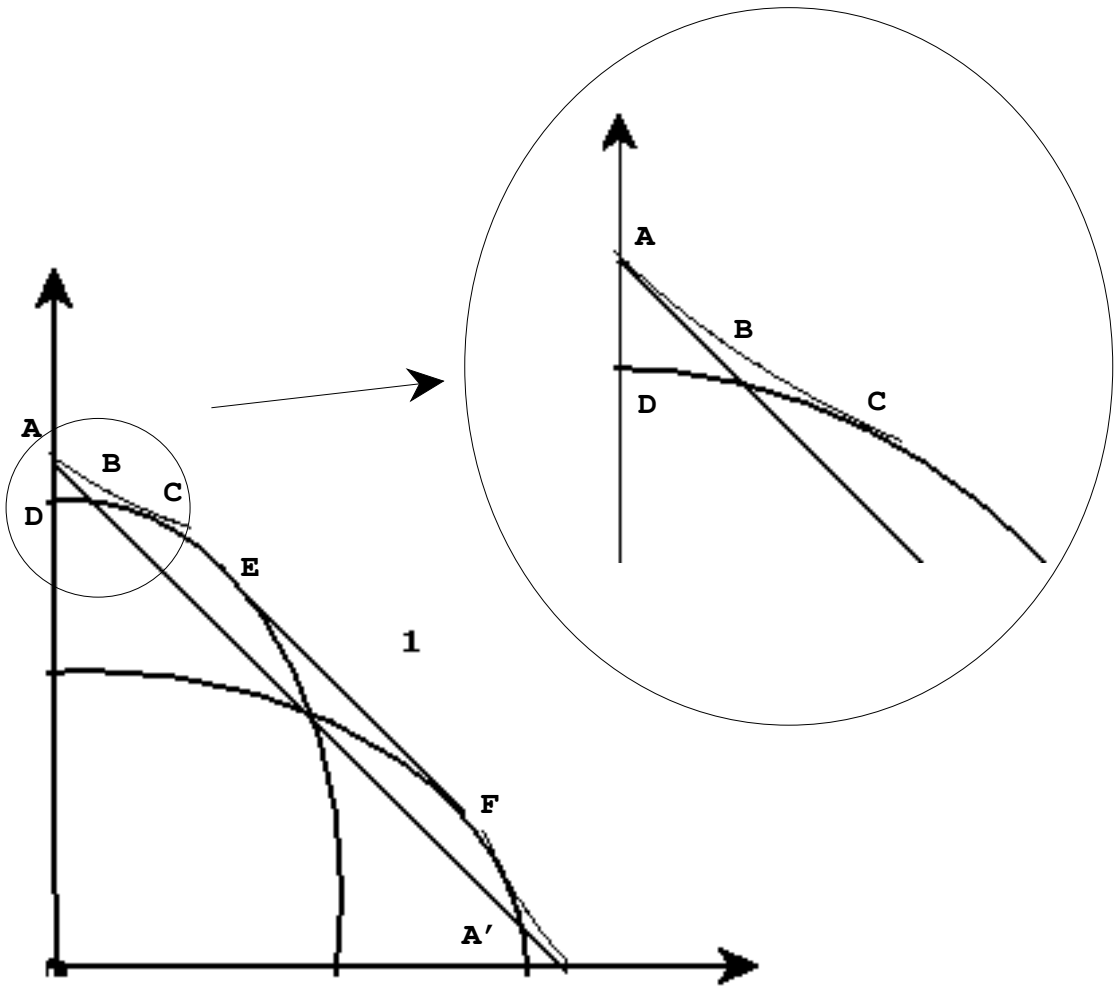

Figure 6: Permitted region Ω_1, based on complex variations.
A B C - Hyperbolic bounds, obtained from composite variations. A A' - The bound, obtained from variation with inclusions of σ_3. D C E F - The bound, obtained from variation with inclusions of σ_2.

In the process of solution, these parameters are replaced by three other parameters: magnitude of the external field $\boldsymbol{P}$, ratio of eigenvalues of $\boldsymbol{P}$, and the relative cost of the second material γ. However, not all values of these three parameters correspond to optimal volume fractions in their intervals $(0,1)$. For some parameters, an optimal solution corresponds either to two component mixture or to a solid material. We find the range of parameters $\boldsymbol{P}$ and γ that lead to three component mixture.

The mean field $\boldsymbol{P}$ must belong to the forbidden region

$$\boldsymbol{P} \in \Omega_f, \tag{5.2}$$

otherwise the solution is trivial: the field is constant everywhere and the structure is filled with one of initially given materials. Two volume fractions out of three are zero.

We find the range of γ that corresponds to three (not two) materials in optimal mixture. Let us show that the range

$$\gamma_2 \leq \gamma \leq \gamma_1, \quad \gamma_1 = \frac{2\sigma_1}{\sigma_1 + \sigma_2}, \quad \gamma_2 = \frac{\sigma_1}{\sigma_2} \tag{5.3}$$

corresponds to all three-materials optimal mixtures.

Remark 5.1 *The question of the materials costs which require a three-component mixture as the optimal solution is non-trivial. In the paper [3] by Burns and Cherkaev, it was considered the problem of three materials mixture, similar to the present one. The material was optimal optimal with respect to an energy $\sigma_*(\nabla u)^2$ in a field ∇u where u is a scalar potential. It was shown, that all three-component mixtures correspond to the unique value of the cost of the intermediate material. Indeed, the problem is reduced to convex envelope of three energies $\sigma_i(\nabla u)^2 + \gamma_i$; all three materials can coexist in an optimal mixture, if the convex envelope possesses a straight component supported by all these three energies simultaneously. Geometrically, the requirement is that three parabolas touch the same tangent. It uniquely determines the parameters γ_2, if γ_1 and γ_3. Interestingly, that in this case the optimal volume fractions are not defined uniquely.*

In the present problem, however, non-trivial three-component mixtures corresponds to a range of γ.

First, we review the inequalities that determine the fields in regions Ω_1 and Ω_2: The region Ω_1 is given by the inequalities:

$$F_2(\sigma_1, \sigma_2, \boldsymbol{E}_1) \leq \frac{1-\gamma}{\sigma_2(\sigma_2 - \sigma_1)} \tag{5.4}$$

$$F_2(\sigma_1, \infty, \boldsymbol{E}_1) = E_a + E_b \geq \sqrt{\frac{1}{\sigma_1}} \tag{5.5}$$

They are obtained from analysis of the single variations. (recall, that composite variations does not change Ω_1). The inequalities (5.4) are obtained by variation of an inclusion of

material σ_2 into Ω_1 and the inequality (5.5) is obtained by variation of an inclusion of material σ_3 into Ω_1.

The field in the second region Ω_2 satisfies the inequalities:

$$F_1(\sigma_2, \sigma_1, \boldsymbol{E}_2) \leq \frac{1-\gamma}{\sigma_2(\sigma_2 - \sigma_1)} \tag{5.6}$$

$$F_2(\sigma_2, \infty; \boldsymbol{E}_2) = E_a + E_b \geq \sqrt{\frac{\gamma}{\sigma_2}} \tag{5.7}$$

$$F_3(\sigma_2, \sigma_1, \infty; \boldsymbol{E}) \geq (\gamma\sigma_2 - \sigma_1)/\sigma_2^2 \tag{5.8}$$

Inequality (5.6) shows that the set Ω_2 belongs to an intersection of two ellipses, obtained by the consideration of inclusions of the first material. Inequality (5.7) shows that the set Ω_2 lies above the straight line, obtained from the consideration of inclusions of the third material. The inequality (5.8) shows that the set Ω_2 lies above the intersection of two hyperbolas, obtained by the consideration of inclusions of composite type.

• If $\gamma > \gamma_1$, then the set Ω_2 is empty and the second material is not optimal. We interpret it this as material σ_2 being too 'expensive' to use in an optimal composition; it is 'cheaper' to mimic material σ_2 as a mixture of the first and the third materials.

• If $\gamma = \gamma_1$, then the set Ω_2 degenerates to a point

$$\boldsymbol{E}_2 = \sqrt{\frac{\sigma_1}{2\sigma_2(\sigma_1 + \sigma_2)}}\boldsymbol{I}; \tag{5.9}$$

and the optimal structure keeps the field in the second phase constant and isotropic (see figure 7).

The set Ω_1 is determined by the only inequality (5.5) (see figure 7)

$$E_a + E_b \geq \frac{1}{\sqrt{\sigma_1}} \quad \text{in } \Omega_1. \tag{5.10}$$

The other inequalities are satisfied automatically. Notice that the inequalities (5.4), (5.5) correspond to regions that share the straight component of the boundary. This shows that the field in the first phase may vary within this component.

• If $\gamma \in [\gamma_1,\ \gamma_2]$, then the set Ω_2 is strongly convex:

$$F_1(\sigma_2, \sigma_1, \boldsymbol{E}_2) \leq \frac{1-\gamma}{\sigma_2(\sigma_2 - \sigma_1)} \tag{5.11}$$

$$F_3(\sigma_2, \sigma_1, \infty; \boldsymbol{E}) \geq (\gamma\sigma_2 - \sigma_1)/\sigma_2^2 \tag{5.12}$$

σ_2 is restricted by ellipses and hyperbolas, the straight component corresponds to a strong (inactive) inequality. This means, that the field in the second phase stays constant. In this case, the domain Ω_2 forms an 'island' in the forbidden region which leaves the possibility of optimal three-component mixtures. (see figure 8)

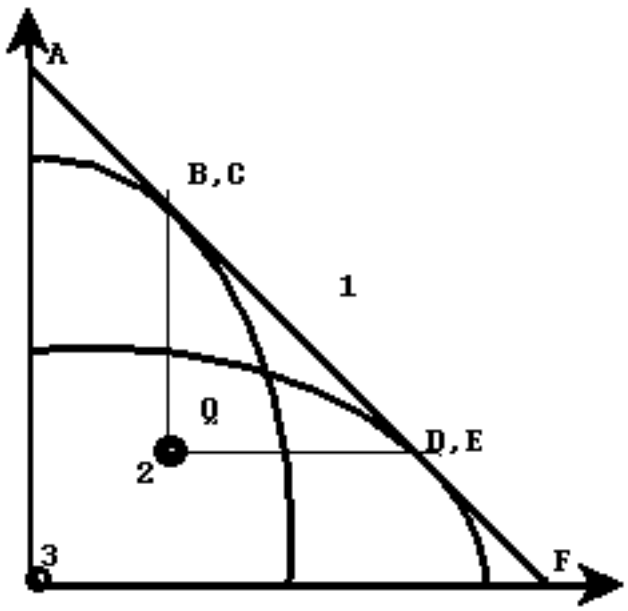

Figure 7:
The permitted regions, $\gamma = \gamma_1$.
Ω_2 and Ω_3 degenerates into single points, $\partial\Omega_1$ becomes straight line.

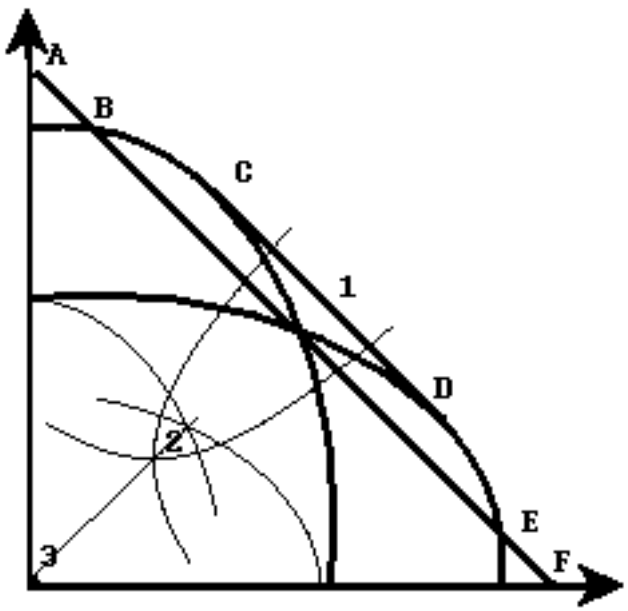

Figure 8: Permitted regions. Intermediate values of γ.
$\partial\Omega_1$ is represented by the curve $ABCDEF$. The set Ω_2 is strictly convex.

The boundary of Ω_1 is the intersection of restrictions provided by inclusions of material σ_2, of material σ_3 into the domain material Ω_1. (see (5.4), (5.5)) and of composite inclusions.

• If $\gamma = \gamma_2$, then Ω_1 is determined by the inequality (5.4)

$$F_1(\sigma_2, \sigma_1; \boldsymbol{E}) \geq \sqrt{\frac{\sigma_1}{\sigma_2}}. \tag{5.13}$$

The boundary $\partial\Omega_1$ is provided by consideration of an inclusion of σ_3 into Ω_1 (the inclusions of σ_2 leads to a strong inequality everywhere except from the corner point $E_a = 0, E_b = \sigma_1/\sigma_2$).

The set Ω_2 has a straight component:

$$\begin{array}{c} F_2(\sigma_2, \sigma_1; \boldsymbol{E}_2) \leq 1/\sigma_2^2 \\ F_1(\sigma_2, \infty; \boldsymbol{E}_2) = E_a + E_b \geq \frac{\sqrt{\sigma_1}}{\sigma_2} \end{array}. \tag{5.14}$$

Both hyperbolas degenerate into their straight asymptotes. The degenerated hyperbolic component of $\partial\Omega_2$ coincides with the straight line described by the inequality (5.7) that becomes active. The straight component of the $\partial\Omega_2$ shows that the field in the second phase could vary (see figure 9).

• If $\gamma < \sigma_2$, then the region Ω_2 forms a 'belt' which divides the forbidden region into the two disconnected parts Ω_f^{12} and Ω_f^{23}. If the mean field belongs to the inner part Ω_f^{23} of the forbidden region, then only connections between materials σ_3 and σ_2 are optimal. If the mean field belongs to the exterior part Ω_f^{12} of the forbidden region, then the only connections between materials σ_1 and σ_2 are optimal.

This range of γ corresponds to two-component mixtures. The type of optimal mixtures is determined by the mean field $\boldsymbol{P}$. When the mean field $\boldsymbol{P}$ belongs to the region Ω_f^{23} lying in the proximity of the origin, then optimal composites consist of materials σ_3 and σ_2. When the magnitude of the mean field increases, the fraction of σ_3 decreases and the fraction of σ_2 increases. When $\boldsymbol{P}$ reaches the belt-like region Ω_2, the optimal mixture degenerates into pure material σ_2. Further increase of the magnitude of $\boldsymbol{P}$ ($\boldsymbol{P} \in \Omega_f^{12}$) brings the field into the region Ω_f^{12} and leads to optimal mixtures of σ_2 and σ_1. The fraction of σ_2 in the mixture decreases with the increase of the magnitude of $\boldsymbol{P}$. When $\boldsymbol{P}$ reaches the exterior region Ω_1, it becomes zero.

Note, that three-composite mixtures never appear in that process. We interpret this as the second material being too 'cheap'. It is always better to use this material then a mixture of materials σ_1 and σ_3. (see figure 10).

Remark 5.2 *The described range of parameters $\boldsymbol{P} \in \Omega_f$, $\gamma \in \gamma_1$, γ_2 is sufficient to produce optimal solutions that involves all three materials. More detailed consideration could shrink this set by excluding certain combinations of these parameters.*

Remark 5.3 *The answer to the question: what is better to use, σ_2 or the mixture of σ_1 and σ_3 depends on the degree of anisotropy of $\boldsymbol{E}$. This feature is illustrated on figures 7 -*

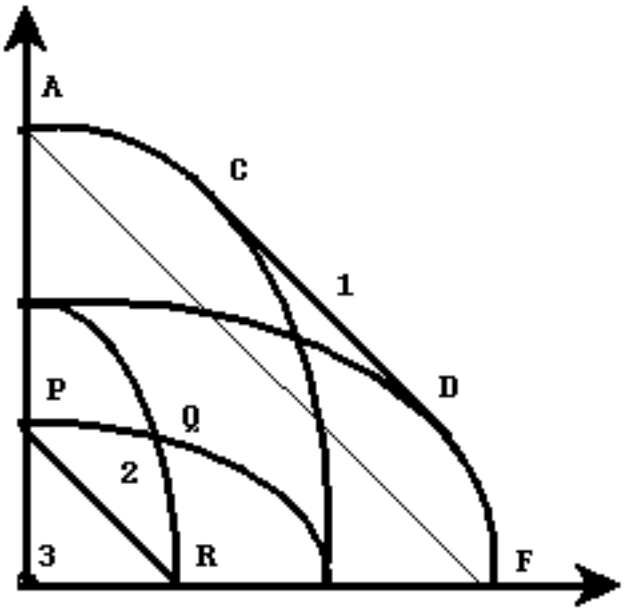

Figure 9: Permitted regions, $\gamma = \gamma_2$.
The boundary of Ω_2 has a straight component. Both hyperbolas degenerate and coincide with their asymptotic.

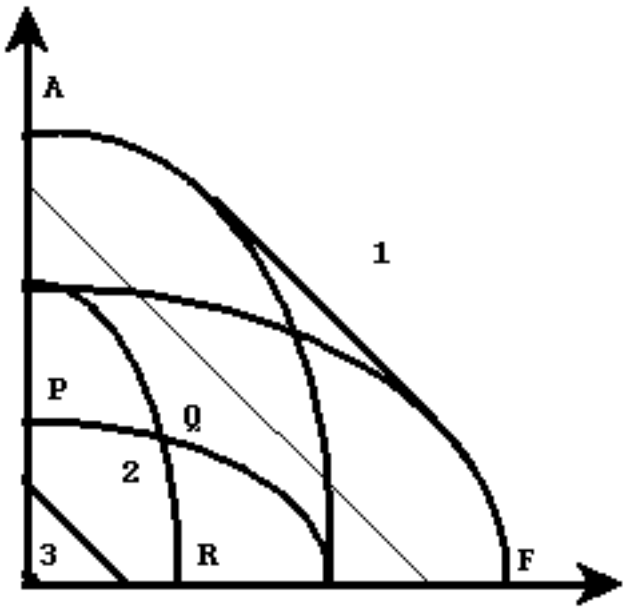

Figure 10: The permitted regions. γ is outside of $[\gamma_1,\ \gamma_2]$.
Three component mixtures are not optimal.

10. The closer to the isotropy is $\boldsymbol{E}$, the more 'useful' is σ_2 comparing with the mixture of other two materials. The dependence on the degree of anisotropy leads to the optimality of three component mixtures in a range of γ, in the contrary to the problem of optimization of one energy [3] which corresponds to the trajectory along the axis $E_B = 0$

5.2 Examples of optimal microstructures.

Let us now determine some optimal structures that satisfy the necessary conditions as equality everywhere. Our guess is guided by the derived conditions. We demonstrate how they is satisfied on the already known structures suggested by Milton [21] , Kohn and Milton [13], and we demonstrate several new types of optimal structures. In particular, structures are demonstrated that are optimal for a range of small volume fraction of σ_1. This is the case where structures are unknown, and the translation bounds give only rough estimates of the effective properties.

5.2.1 Large volume fractions of σ_1.

Consider the case $\gamma = \gamma_1$ (figure 7). Suppose that the mean field $\boldsymbol{P}$ belongs to the triangle CDQ . The optimal structures possess isotropic fields $\boldsymbol{E}_2$ and $\boldsymbol{E}_3$ in the second and in the third materials, and varying field $\boldsymbol{E}_1$ in the first material; and the trace of $\boldsymbol{E}_1$ is constant. Also, one can easily see from the conditions, that the following rank-one contacts are allowed: σ_2 with σ_1, σ_3 with σ_1, but the contact between σ_2 and σ_3 is not allowed.

The structures that satisfy all these requirements are known: they have been suggested by Kohn and Milton [13], following the earlier work by Milton [21]. Note, that these structures realize the sufficient (translation) conditions and therefore are surely optimal.

The topology of the structures is the following: σ_2 and σ_3 are placed in separated inclusions in the domain $\mathcal{O}_1$. (see figure 11)

More exactly, the construction is the following. The first material is divided into two parts with the fractions cm_1 and $(1-c)m_1$, where $c \in (0,\ 1)$ is a parameter. The first part of σ_1 together with the all amount of σ_2 forms a matrix layered composite with effective tensor $\boldsymbol{\sigma}_{12}$ with eigenvalues $\lambda_1(12)$ and $\lambda_2(12)$. The second part of σ_1 together with all amount of σ_3 forms a matrix layered composite with effective tensor $\boldsymbol{\sigma}_{13}$ with eigenvalues $\lambda_1(13)$ and $\lambda_2(13)$. Next, the fraction c is chosen so that the composites $\boldsymbol{\sigma}_{12}$ and $\boldsymbol{\sigma}_{13}$ have the same eigenvalues

$$\lambda_1(12) = \lambda_1(13) \quad and \quad \lambda_2(12) = \lambda_2(13) \tag{5.15}$$

and are equivalent to each other. Clearly, that a mixture of $\boldsymbol{\sigma}_{12}$ and $\boldsymbol{\sigma}_{13}$ has the same effective properties, independently of the geometry of that mixture. The effective properties of these structures satisfy the translation bounds; (see [13]); for isotropic composites, conductivity satisfies Hashin -Shtrikman bounds which in the discussing case $\sigma_3 = \infty$ becomes

$$\frac{1}{\sigma_* + \sigma_1} = \frac{m_1}{2\sigma_1} + \frac{m_2}{\sigma_1 + \sigma_2}. \tag{5.16}$$

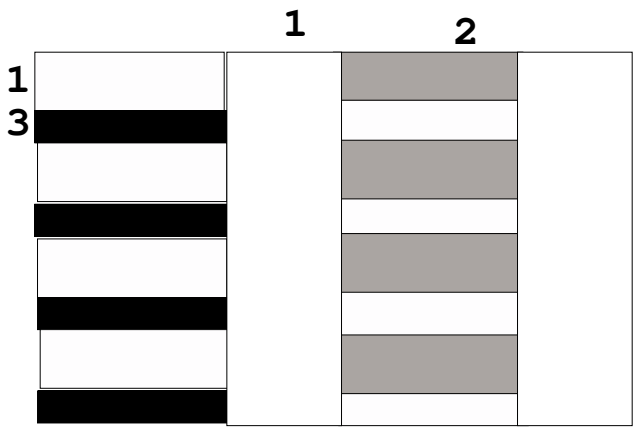

Figure 11: Optimal structures (Kohn, Milton); large volume fractions of σ_1.

The limitations of this construction come from the requirements (5.15). Clearly, that $\boldsymbol{\sigma}_{12}$ has the eigenvalues smaller that σ_2. One need to put a significant amount of σ_1 into $\boldsymbol{\sigma}_{13}$ to obtain the mixtures with such conductivities. (see the discussion in [13]) In particular, isotropic mixtures have the effective conductivity σ_* in the interval $[\sigma_1, \sigma_2]$. The requirement $\sigma_* \in [\sigma_1, \sigma_2]$, combined with (5.16) leads to the condition

$$m_1 \geq 2(1 - m_2)\frac{\sigma_1}{\sigma_1 + \sigma_2} \tag{5.17}$$

of attainability of the described isotropic mixtures (for more detailed discussion see [13])

The necessary conditions (the figure 11) show the optimality of the fields in that structures. Note that $\boldsymbol{E}_2$ and $\boldsymbol{E}_3$ are in rank-one contact with $\boldsymbol{E}_1$, but with each other. This contacts correspond to laminates of the pair σ_2 and σ_1, and the pair σ_2 and σ_1. These laminates are in rank-one contact if the volume fraction of σ_1 is properly chosen. The outside layer of $\boldsymbol{E}_1$ is in rank-one contact with the laminates.

The fields in three parts of the structure in the figure 11 belong to the following points of the boundary (see figure 7): F, D, and a point on the line $[C,\ D]$, their relative fractions are chosen to preserve the trace of the field $\boldsymbol{E}_1$ in these parts.

Remark 5.4 *Interesting, that there are several geometrically different realization of these structures. The first is the mentioned combination of matrix laminates. Other known realizations correspond to isotropic composites. Fist of such constructions has been suggested by Milton in [21]; the structure uses the constructions of 'coated spheres" of materials σ_1 and σ_2 and of materials σ_1 and σ_3, these two composites are mixed together. A topologically*

different realization was suggested by Lurie and Cherkaev [18]. The second material forms a ring; the coated sphere construction (σ_3 forms the nucleus, σ_1 forms the core) inside the ring has the effective conductivity equal to σ_2 due to the choice of the amount of σ_1 in this construction. The remain part of σ_1 forms the outside ring.

In the last construction, the fields $\boldsymbol{E}_2$ and $\boldsymbol{E}_3$ are constant, and the field $\boldsymbol{E}_1$ continuously varies along $\partial\Omega_1$ (the consideration is similar to the presented above example of two-phase coated spheres).

5.2.2 Small volume fraction of σ_1.

Next, we demonstrate structures which satisfy necessary conditions when $\gamma = \gamma_2$ and $\boldsymbol{P}$ lies between Ω_2 and the the origin. These structures correspond to small volume fractions of σ_1.

This time, the field $\boldsymbol{E}_2$ can vary, and $\boldsymbol{E}_1$ stays constant. σ_1 can be in a rank-one contact with σ_2 and with σ_3 only if it is in the corner point (one component is zero). Structures that realize the necessary conditions are easily guessed: they are matrices of σ_2 with inclusions of laminates of σ_1 and σ_3. (note that the domains σ_1 and σ_3 are not separated from each other).

Let us show how the necessary conditions are satisfied. The field in $\boldsymbol{E}_1$ is in the contact with $\boldsymbol{E}_3$ (the point F on the figure 9), the field in the $\mathcal{O}_2$ is piece-wise constant and its trace is constant, because it forms the matrix laminate across the anisotropic inclusion (point R and a point on the line $[P, R]$ on the figure 9). $\boldsymbol{E}_2$ is either in rank-one contact with the laminates from σ_1 and σ_3 (point R on the figure 9), or (the field in an orthogonal layer) in rank-one contact with the laminate of three materials. (see figure 12).

The effective parameters are found as following: the eigenvalues λ_1 and λ_2 of a matrix composite (σ_2 is in the core) satisfy the equation

$$g(\lambda_1, \lambda_2) = \frac{1}{\lambda_1 - \sigma_2} + \frac{1}{\lambda_2 - \sigma_2} = C. \tag{5.18}$$

where C is a constant. To determine C we observe that the structure degenerates into the 'T-structure' (see figure 13) that possesses the following eigenvalues (see Appendix)

$$\lambda_1^T = \frac{\sigma_2}{m_2}, \quad \lambda_2^T = m_2\sigma_2 + \frac{(1-m_2)^2}{m_1}\sigma_1 \tag{5.19}$$

This gives the final relations in the form $g(\lambda_1, \lambda_2) = g(\lambda_1^T, \lambda_2^T)$ or

$$\frac{1}{\lambda_1 - \sigma_2} + \frac{1}{\lambda_2 - \sigma_2} = \frac{1}{\lambda_1^T - \sigma_2} + \frac{1}{\lambda_2^T - \sigma_2}. \tag{5.20}$$

In particular, the isotropic conductivity σ_* satisfies the relation

$$\frac{1}{\sigma_* + \sigma_2} = \frac{m_1}{2\sigma_1} + \frac{m_2}{2\sigma_2} \tag{5.21}$$

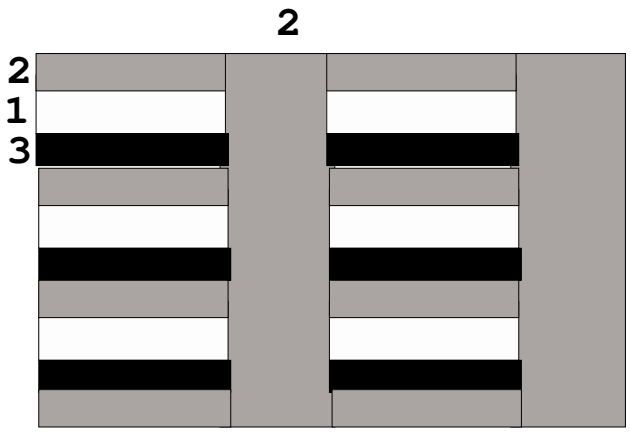

Figure 12:Structures of optimal composites, small volume fractions of σ_1.

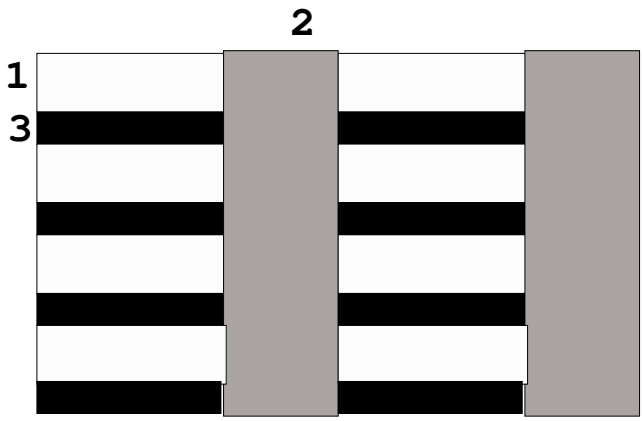

Figure 13: T - structure.

(compare with the Hashin-Shtrikman formula (5.16).

The described structures are optimal in a range of parameters. For isotropic composites, the limiting case correspond to isotropy of 'T-structures' : $\lambda_1^T \leq \lambda_2^T$ or (use (5.19))

$$0 \leq m_1 \leq \frac{m_2(1-m_2)}{1+m_2}\left(\frac{\sigma_1}{\sigma_2}\right) \tag{5.22}$$

Note that these structures have different topology than the previously discussed ones: (***i***) the second material forms an matrix and the laminates of the first and the third materials form inclusions. (***ii***) The first and the third materials are always glued together. (***iii***) The inclusions are highly anisotropic; for the overall isotropic structures the anisotropy of inclusion is compensated buy an eccentricity of the their shape.

In this case bounds give the rough estimates. Better bounds were suggested recently by Nesi [28]. The described structures exactly satisfy these bounds. The coincidence can be explained by the following: the sufficient condition by Nesi [28] are obtained by estimating the determinant of the field $\boldsymbol{E}_1$ by zero. (It was proven that it does not changes its sign everywhere in $\mathcal{O}$). This is exactly the case in the considered geometry: the field in Ω_3 is zero due to the infinite conductivity σ_3, therefore the field $\boldsymbol{E}_1$ in the neighboring layer of σ_1 has one zero component and its determinant is zero.

Remark 5.5 *One can check by direct calculations, that the described structures have better conductivity than the structures suggested in [13] and in [18] for the considered range of parameters; the last structures do not satisfy the necessary conditions.*

Postlude. The discussed examples do not complete the description of optimal structures for three materials. However, we have accomplished our goal which is to demonstrate the applications of the necessary conditions to structural optimization and to show variety of optimal topologies of multicomponent mixtures.

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A Appendix. Calculation of the fields inside the laminates

The following is a procedure for calculating the fields inside a laminate structure of some rank. Such structures are combined as laminates from substructures that could be also a laminates of substructures of a deeper level, etc. To compute the fields in the pure materials that form the deepest level of the structure an iterative scheme is needed. We first calculate the effective properties of a laminate as a function of the known properties of its components. Then we calculate the fields inside the layers of the laminate using the known average field and the effe3ctive properties.

On each rank of laminate structure we perform the following calculation. Suppose that laminates are assembled from two anisotropic materials with the codirected conductivity matrices $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ laminated in proportions c_1 and c_2 ($c_1 + c_2 = 1$). The normal to laminate $\boldsymbol{n}$ and the matrix of material's properties are:

$$\boldsymbol{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{R}_i = \begin{pmatrix} \rho_A^{(i)} & 0 \\ 0 & \rho_B^{(i)} \end{pmatrix}, \quad i = 1,\, 2. \tag{A.1}$$

The effective properties tensor $\boldsymbol{R}_*$ of the laminate is

$$\boldsymbol{R}_* = \begin{pmatrix} \frac{\rho_A^{(1)}\rho_A^{(2)}}{c_1\rho_A^{(2)}+c_2\rho_A^{(1)}} & 0 \\ 0 & c_1\rho_B^{(1)} + c_2\rho_B^{(2)} \end{pmatrix}; \tag{A.2}$$

the normal component is given by the harmonic mean of materials' properties, and the tangential component is given by the arithmetic mean.

Suppose that the laminate is placed in two mutually orthogonal fields that are codirected with the eigenvectors of $\boldsymbol{R}_*$. The matrix of the external fields $\boldsymbol{E}$ is described by (2.8). The field $\boldsymbol{E}_1$ in the laminates are computed from the requirements $c_1\boldsymbol{E}_1 + c_2\boldsymbol{E}_2 = \boldsymbol{E}$ and from the jump condition (3.19), and is equal to

$$\boldsymbol{E}_1 = \boldsymbol{K}_1\,\boldsymbol{E}, \quad \boldsymbol{K}_1 = \begin{pmatrix} \frac{\rho_A^{(2)}}{c_1\rho_A^{(2)}+c_2\rho_A^{(1)}} & 0 \\ 0 & 1 \end{pmatrix} \tag{A.3}$$

and similarly for $\boldsymbol{E}_2$. To perform the calculation of the fields of a sequential laminate one must first compute the effective properties of the substructures that form the composite and find the matrices $\boldsymbol{K}_i$. Then one computes the fields using (A.3).

As an example let us compute the effective properties of the T-structure shown in figure **??**. Three materials with the conductivities $\sigma_1,\ \sigma_2,\ \sigma_3$ and the volume fractions $m_1,\ m_2,\ m_3$ are combined as following: the first and the third material form a laminate $R_{(13)}$, then the second material and $R_{(13)}$ form laminate of second rank. The normal to the last laminate is orthogonal to the normal of the first rank laminate.

First, compute the properties of laminate substructure $R_{(13)}$ of the first and the third material (the normal to those laminate with the rotated on the right angle comparing with (A.1)). Note that the relative fraction of the first and the third materials in the substructure are $\frac{m_1}{m_1+m_3}$ and $\frac{m_3}{m_1+m_3}$. We have

$$\boldsymbol{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{R}_{(13)} = \begin{pmatrix} \rho_A^{(13)} & 0 \\ 0 & \rho_B^{(13)} \end{pmatrix}, \tag{A.4}$$

$$\rho_A^{(13)} = \frac{1}{m_1+m_3}(m_1\sigma_1 + m_3\sigma_3), \quad \rho_B^{(13)} = \frac{(m_1+m_3)\sigma_1\sigma_3}{m_1\sigma_3 + m_3\sigma_1}. \tag{A.5}$$

Then, compute the fields in the domain $\mathcal{O}_2$ and the average field in the composed domain $\mathcal{O}_{13}$

$$\boldsymbol{E}_2 = \boldsymbol{K}_2\,\boldsymbol{E}, \quad \boldsymbol{K}_2 = \begin{pmatrix} \frac{\rho_A^{(13)}}{(m_2+m_3)\rho_A^{(13)}+m_2\sigma_2} & 0 \\ 0 & 1 \end{pmatrix}, \tag{A.6}$$

$$\boldsymbol{E}_{(13)} = \boldsymbol{K}_{(13)}\,\boldsymbol{E}, \quad \boldsymbol{K}_{(13)} = \begin{pmatrix} \frac{\sigma_2}{(m_2+m_3)\rho_A^{(13)}+m_2\sigma_2} & 0 \\ 0 & 1 \end{pmatrix}. \tag{A.7}$$

Finally, compute the fields in the domains $\mathcal{O}_1$ and $\mathcal{O}_3$ that compose the first rank laminate $\mathcal{O}_{13}$:

$$\boldsymbol{E}_1 = \boldsymbol{K}_{(1)}\boldsymbol{E}_{(13)}, \quad \boldsymbol{K}_{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\sigma_3(m_1+m_3)}{m_1\sigma_3+m_3\sigma_1} \end{pmatrix}, \tag{A.8}$$

$$\boldsymbol{E}_{(3)} = \boldsymbol{K}_{(3)}\boldsymbol{E}_{(13)}, \quad \boldsymbol{K}_{(3)} == \begin{pmatrix} 1 & 0 \\ 0 & \frac{\sigma_1(m_1+m_3)}{m_1\sigma_3+m_3\sigma_1} \end{pmatrix}. \tag{A.9}$$

Finally we obtain the dependence of the fields in the materials on the geometric parameters of the structure (the tensors $\boldsymbol{K}_i$)

$$\boldsymbol{E}_1 = \boldsymbol{K}_{(1)}\,\boldsymbol{K}_{(13)}\,\boldsymbol{E}, \quad \boldsymbol{E}_2 = \boldsymbol{K}_2\,\boldsymbol{E}, \quad \boldsymbol{E}_3 = \boldsymbol{K}_{(3)}\,\boldsymbol{K}_{(13)}\,\boldsymbol{E} \tag{A.10}$$

The change of these parameters places the field into permitted regions.