Invariant properties of the stress in plane elasticity and equivalence classes of composites

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Abstract

Attention is drawn to the invariance of the stress field in a two-dimensional body loaded at the boundary by fixed forces when the compliance tensor $\mathcal{S}(\mathbf{x})$ is shifted uniformly by $\mathcal{S}^I(\lambda, -\lambda)$ where λ is an arbitrary constant and $\mathcal{S}^I(\kappa, \mu)$ is the compliance tensor of a isotropic material with two-dimensional bulk and shear moduli κ and μ. This invariance is explained from two simple observations: first, that in two-dimensions the tensor $\mathcal{S}^I(\frac{1}{2}, -\frac{1}{2})$ acts to locally rotate the stress by 90^0 and the second that this rotated field is the symmetrized gradient of a vector field and therefore can be treated as a strain. For composite materials the invariance of the stress field implies that the effective compliance tensor $\mathcal{S}^*$ also gets shifted by $\mathcal{S}^I(\lambda, -\lambda)$ when the constituent moduli are each shifted by $\mathcal{S}^I(\lambda, -\lambda)$. This imposes constraints on the functional dependence of $\mathcal{S}^*$ on the material moduli of the components. In particular it explains why the effective Young's modulus of a two-dimensional media with holes does not depend on the Poisson's ratio of the matrix material.

1 Introduction

The purpose of this paper is first to review some results for two-dimensional elasticity, which are not widely known but which may have wide application, and second to give an appropriate physical interpretation the formal mathematical statements embodied in these results. These results concern the invariance of the stress field in materials with different elastic moduli subject to the same loadings. In fact this property was noted many times in different contexts, beginning with the work of Michell (1899) who studied the behavior of media with holes. Dundurs (1967) used the complex representation of the Airy stress potential to show the invariance of the stress field for mixtures of two isotropic components. Lurie and Cherkaev (1984) studied the problem in the context of Kirchhoff plate theory which involves the same equations as two-dimensional elasticity. The interest in this problem increased when Day, Snyder, Garboczi and Thorpe (1991) observed, through numerical simulation, that the effective Young's modulus of a two-dimensional media with holes does not depend on the Poisson ratio of the matrix material. All these results, together with their applications, were recently reviewed by Thorpe and Jasiuk (1991).

In section 2 we fix notations and state the standard equations of two-dimensional elasticity [See, for example, (Atkin and Fox 1990) for a general reference]. In section 3 we explain how the material constants can be modified without altering the stress field, and in section 4 we discuss the implications for composite materials.

2 Equations of planar elasticity

Here we consider a planar simply-connected domain Ω filled by an elastic material and loaded on the boundary $\partial\Omega$ by a force $\mathbf{t}(\mathbf{x})$; this constrains the normal component of the stress field $\sigma(\mathbf{x})$ at $\partial\Omega$ according to

$$\sigma(\mathbf{x}) \cdot \mathbf{n} = \mathbf{t}(\mathbf{x}), \qquad \forall \mathbf{x} \in \partial\Omega \tag{1}$$

where $\mathbf{n}$ denotes the normal to the surface. For the body to be in equilibrium the net force and net torque acting on it must be zero:

$$\int_{\partial\Omega} \mathbf{t}(\mathbf{x}) = 0, \quad \int_{\partial\Omega} \mathbf{x} \times \mathbf{t}(\mathbf{x}) = 0. \tag{2}$$

The equilibrium equation of plane elasticity

$$\nabla \cdot \sigma = 0, \quad \sigma = \sigma^T \tag{3}$$

may be satisfied identically by means of the Airy potential function ϕ (Atkin and Fox 1990)

$$\sigma = \begin{pmatrix} \phi_{,22} & -\phi_{,21} \\ -\phi_{,12} & \phi_{,11} \end{pmatrix} = \mathbf{R}^T \cdot (\nabla\nabla\phi) \cdot \mathbf{R}, \tag{4}$$

where the comma means differentiation with respect to subsequent directions, $\nabla\nabla$ is the differential operator

$$\nabla\nabla = \begin{pmatrix} \frac{\partial^2}{\partial x_1 \, \partial x_1} & \frac{\partial^2}{\partial x_1 \, \partial x_2} \\ \frac{\partial^2}{\partial x_2 \, \partial x_1} & \frac{\partial^2}{\partial x_2 \, \partial x_2} \end{pmatrix} \tag{5}$$

and

$$\mathbf{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{6}$$

is the tensor of rotation by a right angle.

The requirement that Ω be simply-connected is needed to ensure that ϕ is single-valued. Multiply connected domains also have a single valued ϕ provided that the integrals of the type (2) vanish separately on each internal boundary.

We could also rewrite (4) using a fourth order tensor $\mathcal{R}$, representing the action of rotation by a right angle on second order tensor, defined as follows

$$\mathcal{R}_{ijkl} = \frac{1}{2}\delta_{ij}\delta_{kl} \; - \; \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}) \tag{7}$$

in the form

$$\sigma = \mathcal{R} : \nabla\nabla\phi \tag{8}$$

where here as elsewhere the symbol $\mathcal{A} : \mathcal{B}$ denotes a contraction of the last two indices of a fourth order tensor $\mathcal{A}$ with the first two indices of $\mathcal{B}$ in the same way that $\mathbf{A} \cdot \mathbf{B}$ denotes a contraction of the last index of a second order tensor $\mathbf{A}$ with the first index of $\mathbf{B}$.

The equilibrium condition for the stress is to be supplemented by the constitutive (Hooke's) law

$$\epsilon = \mathcal{S} : \sigma, \tag{9}$$

where ϵ is the strain tensor and $\mathcal{S}$ denotes the fourth- order compliance tensor of the elastic material possibly depending on the position $\mathbf{x}$.

If the material is isotropic, then the compliance tensor $\mathcal{S} = \mathcal{S}^I(\kappa, \mu)$ has the particular form

$$\mathcal{S}^I_{ijkl}(\kappa, \mu) = \frac{1}{4\kappa}\delta_{ij}\delta_{kl} \ + \ \frac{1}{4\mu}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}) \tag{10}$$

where κ denotes the two dimensional bulk modulus and μ denotes the shear modulus. It should be noted that κ is not the same as the customarily used three dimensional bulk modulus k; however they are related by the formula

$$\kappa = k + \mu/3 \tag{11}$$

The strain field $\epsilon(\mathbf{x})$ is connected with the displacement $\mathbf{u}(\mathbf{x})$ via

$$\epsilon = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T). \tag{12}$$

The last equation is equivalent to requiring that the strain satisfies the scalar differential constraint

$$\nabla(\nabla \cdot \mathbf{R^T} \cdot \epsilon \cdot \mathbf{R}) = 0 \tag{13}$$

or, in the notation (5), (7)

$$\nabla\nabla : \mathcal{R} : \epsilon = 0 \tag{14}$$

Indeed, we have

$$\mathbf{R^T} \cdot \epsilon \cdot \mathbf{R} = \begin{pmatrix} \epsilon_{22} & -\epsilon_{12} \\ -\epsilon_{12} & \epsilon_{11} \end{pmatrix} = \begin{pmatrix} u_{2,2} & -\frac{1}{2}(u_{1,2} + u_{2,1}) \\ -\frac{1}{2}(u_{1,2} + u_{2,1}) & u_{1,1} \end{pmatrix} \tag{15}$$

and so one can easily check that

$$\begin{gathered} \nabla\nabla : \mathcal{R} : \epsilon = \nabla \cdot (\nabla \cdot \mathbf{R}^T \cdot \epsilon \cdot \mathbf{R}) \\ = (u_{2,2})_{,11} - (u_{1,2} + u_{2,1})_{,12} + (u_{1,1})_{,22} = 0 \end{gathered} \tag{16}$$

Combining the equations (8) (9) (14) we arrive at the fourth order differential equation for the Airy potential

$$\nabla\nabla : \mathcal{S}_\perp : \nabla\nabla\phi = 0 \tag{17}$$

where $\mathcal{S}_\perp$ is $\mathcal{S}$ rotated by a right angle:

$$\mathcal{S}_\perp = \mathcal{R} : \mathcal{S} : \mathcal{R}. \tag{18}$$

Clearly, the isotropic tensor $\mathcal{S}^I$ coincides with the the rotated one:

$$\mathcal{S}^I(\kappa, \mu) = \mathcal{S}_\perp{}^I(\kappa, \mu) \tag{19}$$

and equation (17) takes the form

$$\begin{aligned}
&[(\frac{\partial^2}{\partial x_1\,\partial x_1} + \frac{\partial^2}{\partial x_2\,\partial x_2})(\frac{1}{\kappa} + \frac{1}{\mu})(\frac{\partial^2}{\partial x_1\,\partial x_1} + \frac{\partial^2}{\partial x_2\,\partial x_2}) \\
&\quad -(\frac{\partial^2}{\partial x_1\,\partial x_1})\frac{2}{\mu}(\frac{\partial^2}{\partial x_2\,\partial x_2}) - (\frac{\partial^2}{\partial x_2\,\partial x_2})\frac{2}{\mu}(\frac{\partial^2}{\partial x_1\,\partial x_1}) \\
&\qquad +(\frac{\partial^2}{\partial x_1\,\partial x_2})\frac{4}{\mu}(\frac{\partial^2}{\partial x_1\,\partial x_2})]\phi = 0.
\end{aligned} \tag{20}$$

We remark in passing that (17) also describes the bending of thin plates according to Kirchoff theory: ϕ represents the vertical deflection of a horizontal plate, $\nabla\nabla\phi$ is the tensor of curvature of the plate, $\mathcal{S}_\perp$ is the tensor of flexural rigidity dependent on the local thickness of the plate and $\mathcal{S}_\perp : \nabla\nabla\phi$ represents the tensor of bending moments, satisfying the equilibrium equation (17) (Timoshenko 1959). Thus all the ensuing analysis applies equally well to the plate equation.

3 Equivalent plane elasticity problems

It is remarkable that any potential ϕ satisfying the equation(17) is also a solution of

$$\nabla\nabla : \mathcal{S}'_\perp : \nabla\nabla\phi = 0 \tag{21}$$

where

$$\mathcal{S}'_\perp = \mathcal{R} : \mathcal{S}' : \mathcal{R}, \quad \mathcal{S}' = \mathcal{S} + \mathcal{S}^I(\lambda, -\lambda) \tag{22}$$

for any choice of the parameter λ, implying that the Airy potential and therefore the stress field remains unchanged when the material constants are modified from $\mathcal{S}$ to $\mathcal{S}'$. We will call such a pair of materials with compliance tensors $\mathcal{S}(\mathbf{x})$ and $\mathcal{S}'(\mathbf{x})$ equivalent. The result (21) can be established directly by substituting $\kappa = \lambda, \ \ \mu = -\lambda$ in (20) thereby showing that

$$\nabla\nabla : \mathcal{S}^I(\lambda, \ -\lambda) : \nabla\nabla\phi = 0 \tag{23}$$

for any function ϕ. Formally speaking the compliance tensor $\mathcal{S}^I$ is nothing other than a multiple of the fourth order rotation tensor $\mathcal{R}$: from (8) and (10) it follows that

$$\mathcal{S}^I(\lambda, \ -\lambda) = \frac{1}{2\lambda}\mathcal{R} \tag{24}$$

The physical explanation for the identity (23) is that the stress field σ when rotated by a right angle at each point results in a field (see (4))

$$\epsilon^0 = \mathbf{R}\cdot\sigma\cdot\mathbf{R}^T = \mathcal{R} : \sigma = \nabla\nabla\phi \tag{25}$$

satisfying the same differential constraints (13) as a strain field. Indeed, it is clear that

$$\nabla\nabla : \mathbf{R}^T \cdot \epsilon^0 \cdot \mathbf{R} = \nabla\nabla : \sigma = \nabla\cdot(\nabla\cdot\sigma) = 0 \tag{26}$$

Furthermore, from (25) it follows that

$$\epsilon^0 = \frac{1}{2}(\nabla\mathbf{u}^0 + (\nabla\mathbf{u}^0)^T), \ \ \mathbf{u}^0 = \nabla\phi. \tag{27}$$

So, this strain field is associated with a special vector displacement field $\mathbf{u}^0$ which is in turn the gradient of the Airy potential function. Note, that the displacements $\mathbf{u}$ and $\mathbf{u}'$ in the equivalent materials $\mathcal{S}$ and $\mathcal{S}'$ under the same surface loading $\mathbf{t}(\mathbf{x})$ differ by a multiple of $\mathbf{u^0}$

$$\mathbf{u}' = \mathbf{u} + \mathbf{u}^0/(2\lambda) = \mathbf{u} + \nabla\phi/(2\lambda) \tag{28}$$

In particular this gives a direct way of determination the gradient of the Airy stress potential $\nabla\phi$ from measurements of the displacements $\mathbf{u}$ and $\mathbf{u}'$.

Another explanation of the equivalence (21, 22)follows from consideration of the elastic energy variational principle

$$\min_{\sigma} \int_{\Omega} W(\mathcal{S}), \tag{29}$$

where the minimum is over all σ satisfying (1) and (3), and

$$W(\mathcal{S},\sigma) = \sigma : \mathcal{S} : \sigma = \nabla\nabla\phi : \mathcal{R} : \mathcal{S} : \mathcal{R} : \nabla\nabla\phi \tag{30}$$

The Euler-Lagrange equation associated with this minimization over σ, or equivalently ϕ, coincides with (17). It is easy to observe that the integrand $W(\mathcal{S}^{\mathcal{I}}(\lambda,-\lambda),\sigma)$ can be expressed as the divergence of a vector field

$$\begin{aligned} W(\mathcal{S}_I(\lambda,-\lambda),\sigma) &= \frac{1}{2\lambda}\nabla\nabla\phi : \mathcal{R} : \nabla\nabla\phi \\ &= (\phi_{,11}\phi_{,22} - \phi_{,12}^2)/\lambda = \nabla\cdot\mathbf{v} \end{aligned} \tag{31}$$

where

$$\mathbf{v} = \frac{1}{2\lambda}\nabla\phi\cdot\mathcal{R} : \nabla\nabla\phi = \frac{1}{2\lambda}\begin{pmatrix} \phi_{,1}\phi_{,22} - \phi_{,2}\phi_{,12} \\ \phi_{,2}\phi_{,11} - \phi_{,1}\phi_{,12} \end{pmatrix} \tag{32}$$

and therefore its integral depends only on the boundary terms. Such functions are called Null-Lagrangians [see, for example (Ball *et al.* 1981)] because their Euler-Lagrange equations vanish identically. They play an important role in the theories of quasiconvexity (Kohn and Strang 1986) and compensated compactness (Tartar 1979) used now in many applications including optimal design, homogenization, liquid crystals, and bounds on effective moduli of composite materials.

In fact these arguments parallel results of Dykhne (1970), and Stroud and Bergman (1984) [see also (Keller 1988) and (Milton 1988)] for conductivity in two dimensions in the presence of a magnetic field perpendicular to the plane of conduction, which generates through the Hall-effect a nonsymmetric conductivity tensor. They observed that any electrical potential $V(\mathbf{x})$ satisfying the equation

$$\nabla\cdot\Sigma(\mathbf{x})\cdot\nabla V = 0 \tag{33}$$

where $\Sigma(\mathbf{x})$ represents the conductivity of the body, also solves

$$\nabla\cdot(\Sigma(\mathbf{x}) + \lambda\mathbf{R})\cdot\nabla V = 0 \tag{34}$$

for any choice of λ. Just as a stress field rotated by a right angle produces a strain field so does an electric field, when rotated by a right angle produce a current (divergence free) field. (Note, however, that a strain field rotated by a right angle does not produce a stress field whereas a current field, when rotated by a right angle does produce an electric field.)

Now we investigate some particular physical consequences that follow from (21).

The representation (21) shows that the coefficients of the compliance tensor $\mathcal{S}$ and consequently the strain field ϵ cannot be determined from the solution σ of the boundary value problem (see (21), (1), and (4)). For example, instead of a locally isotropic elastic material with moduli κ and μ one can substitute into the equation (21) the moduli κ' and μ' :

$$\frac{1}{\kappa'(\mathbf{x})} = \frac{1}{\kappa(\mathbf{x})} + \frac{1}{\lambda}; \quad \frac{1}{\mu'(\mathbf{x})} = \frac{1}{\mu(\mathbf{x})} - \frac{1}{\lambda} \tag{35}$$

and it doesn't change the solution ϕ nor the stress field σ. We will call such a pair of materials equivalent. Note, that the substituted material does not necessarily have positive moduli κ' and μ' and therefore may not have a physical interpretation.

Let us consider the relation between the Young moduli E and E' and the Poisson ratios ν and ν' of equivalent materials. Comparing the constitutive law (9) with the defining equation

$$\epsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22}) \tag{36}$$

for the Young modulus E and the Poisson ratio ν we obtain the relations

$$\frac{1}{E} = \frac{1}{4\kappa} + \frac{1}{4\mu}, \quad \nu = \frac{\frac{1}{\mu} - \frac{1}{\kappa}}{\frac{1}{\kappa} + \frac{1}{\mu}} = \frac{\kappa - \mu}{\kappa + \mu}. \tag{37}$$

It is clear from (35) that the equivalent materials have the same Young moduli at each point in the body

$$E'(\mathbf{x}) = E(\mathbf{x}) \tag{38}$$

and Poisson ratios linked by

$$\nu'(\mathbf{x}) = \nu(\mathbf{x}) - E(\mathbf{x})/2\lambda \tag{39}$$

where λ is an arbitrary parameter constant throughout the body.

In particular this implies that the stress field in a loaded body with constant Young modulus does not react when the Poisson ratio $\nu(\mathbf{x})$ is shifted uniformly.

4 Applications to composites

Finally, there are useful implications of the identity (23) to the theory of composites. Suppose now that the body Ω is a statistically homogeneous (or periodic) composite with grain sizes much smaller than the size of Ω. This composite can be replaced by an equivalent homogenious effective medium with compliance tensor $\mathcal{S}^*$, connecting the locally averaged stress and strain tensor fields,

$$< \epsilon >= \mathcal{S}^* :< \sigma > \tag{40}$$

where $< . >$ denotes local averaging over a test sphere $\Theta(\mathbf{x})$ centered at $\mathbf{x}$, which is larger than the grain sizes but smaller than all other characteristic lengths such as the size of Ω and the length scale over which $\mathbf{t}(\mathbf{x})$ varies. To obtain the effective compliance $\mathcal{S}^*$ it suffices to consider an infinite body of the composite loaded uniformly, and the averages $< . >$ can then be taken over the whole body, or equivalently over a period cell.

Taking averages of the relations

$$\epsilon = \mathcal{S} : \sigma \tag{41}$$

$$\epsilon' = \mathcal{S}' : \sigma \tag{42}$$

between the fields in the material $\mathcal{S}(x)$ and $\mathcal{S}'(x)$ we deduce that the effective tensors are linked through the equation

$$\mathcal{S}'^* = \mathcal{S}^* + \mathcal{S}^I(\lambda, -\lambda) \tag{43}$$

In other words the effective compliance tensor is translated by the same tensor as the compliance tensor of the initial inhomogeneous medium. Applied to an isotropic composite of isotropic components it implies that when the moduli $1/\kappa(\mathbf{x})$ and $1/\mu(\mathbf{x})$ are shifted according to (35) their effective moduli are shifted in the same way

$$\frac{1}{\kappa'^*} = \frac{1}{\kappa^*} + \frac{1}{\lambda}; \quad \frac{1}{\mu'^*} = \frac{1}{\mu^*} - \frac{1}{\lambda} \tag{44}$$

This observation explains the numerical results of Day, Snyder, Garboczi and Thorpe (1991). They found that the relative Young's Modulus E^*/E of a two-dimensional sheet containing circular holes, overlapping or not is independent of the Poisson's ratio of the sheet. The reason for this is clear: plates with the same Young's modulus are equivalent: (44) and (37) implies $E'^* = E^*$.

An interesting consequence of (44) is that it leads to a very simple proof of the Hashin-Shtrikman-Hill bound on the bulk modulus κ_*. We begin with the Voigt bound on κ'^* (Hill, 1952)

$$\kappa'^* \leq < \kappa' > \tag{45}$$

which holds for all choices of λ such that κ' and μ' given by (35) remain nonnegative. In particular, when λ takes its extreme value,

$$\lambda = \mu^+ = \max_{\mathbf{x}} \mu(\mathbf{x}), \tag{46}$$

(45) implies

$$(\frac{1}{\kappa^*} + \frac{1}{\mu^+})^{-1} \leq < (\frac{1}{\kappa(\mathbf{x})} + \frac{1}{\mu^+})^{-1} > \tag{47}$$

which coincides with the Hashin-Shtrikman bound. This is one of the simplest examples of the so-called translation method to bounding the effective moduli of composites. The method generalizes the idea of equivalence, and it not only provides an alternative derivation of the Hashin-Shtrikman bounds on the effective conductivity, bulk and shear moduli (see Tartar, 1985, Lurie and Cherkaev 1982, 1984 1986, Frankfort and Murat, 1991, and Milton, 1990) but also generates coupled estimates on the possible $(\kappa^*,\ \mu^*)$ pairs (Cherkaev and Gibiansky, 1991). It has been successfully applied to many problems, too numerous to list here.

The equivalence (43) was the key point used in the paper by Lurie and Cherkaev (1984) to get the bounds for the set of all possible effective tensors $\mathcal{S}^*(\mathbf{x})$ when $\mathcal{S}(\mathbf{x})$ is isotropic and either the bulk or shear modulus of the material is constant. Then the effective tensor is necessarily isotropic and shares the same shear modulus as the components (Hill 1952). To obtain the effective bulk modulus one can choose $\lambda = \mu$ giving

$$\frac{1}{\kappa'(\mathbf{x})} = \frac{1}{\kappa(\mathbf{x})} + \frac{1}{\mu}, \quad \frac{1}{\mu'} = 0. \tag{48}$$

From (21) it is immediately clear that the Airy potential satisfies

$$\Delta(\frac{1}{\kappa(\mathbf{x})} + \frac{1}{\mu})\Delta\phi = 0 \tag{49}$$

where Δ denotes the Laplacian, implying that the scalar field

$$Tr\epsilon'(\mathbf{x}) = \frac{1}{2}(\frac{1}{\kappa(\mathbf{x})} + \frac{1}{\mu})\Delta\phi(\mathbf{x}) \tag{50}$$

is harmonic (see (49)).

The physical explanation of this is quite clear. The equivalent material has infinite shear modulus and so its only possible deformations are conformal ones since a change in the angle between two lines under deformation indicates shear. In particular, if the deformation in the equivalent media is on average uniform then it must be necessarly be a uniform dilation. The associated strain field in the equivalent media is independent of $\mathbf{x}$, and proportional to the identity tensor

$$\epsilon'_{ij}(\mathbf{x}) =< \epsilon'_{ij} >= \alpha\delta_{ij} \quad \forall\mathbf{x} \tag{51}$$

and the trace of the average stress field is

$$\begin{aligned} Tr < \sigma' > =< \Delta\phi(\mathbf{x}) >= 2 < (\frac{1}{\kappa(\mathbf{x})} + \frac{1}{\mu})^{-1} Tr\epsilon' > \\ = 2 < (\frac{1}{\kappa(\mathbf{x})} + \frac{1}{\mu})^{-1} > Tr < \epsilon' > . \end{aligned} \tag{52}$$

This in conjunction with (44) shows that the effective bulk modulus κ^* satisfies

$$(\frac{1}{\kappa^*} + \frac{1}{\mu})^{-1} =< (\frac{1}{\kappa(\mathbf{x})} + \frac{1}{\mu})^{-1} > \tag{53}$$

independent of the microstructure of the composite. The same conclusion was also reached by Hill (1952), and by Frankfort and Tartar (1991) who showed the relation can be generalized to three dimensional composites with constant shear modulus. The result also follows from the Hashin-Shtrikman bounds (Hashin and Shtrikman 1963) which collapse to the single relation (53) when the shear modulus is constant.

Since the equivalent medium has infinite shear modulus its Poisson ratio given by (37) is -1) (Materials with Poisson's ratios close to -1 although not yet found in nature can nevertheless be constructed: see for example Lakes (1991) and Milton (1991).

In a similar way, supposing that the bulk modulus is constant and choosing $\lambda = -\kappa$ one can get the equivalent incompressible medium. Clearly equivalent media differ very much in their elastic behavior.

The equivalence means that when subject to the same loading conditions they (as well as infinitely many other equivalent materials) possess the same stress tensor in each point of the body Ω.

Note finally, that (43) also provides a useful test for numerical codes for evaluating effective moduli, no matter if the component phases are isotropic or polycrystalline.

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