STABILITY OF OPTIMAL STRUCTURES OF ELASTIC COMPOSITES

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ABSTRACT. We study instability of solution of the problem of optimal layout of the composites with arbitrary structures throughout the designed volume. Several reformulations of the problem are considered which improve the stability of the optimal solution.

1 Instabilities in problems of structural optimization

1.1 Preliminaries.

It was understood last decade that the problem of optimal layout of several elastic materials throughout a domain requires the consideration of formation of composites with optimal microstructures; such layouts correspond to extremal value of goal functionals such as the total rigidity of the body (see [15], [10], [14]). The rigidity I of an elastic body under fixed loading may be determined as a value inverse to the mechanical work produced by this loading. Therefore the problem is to minimize this work (or, equally, to minimize the total energy of the body) by choosing the appropriate layout of composites throughout the domain of construction.

Mathematically, the problem of optimal rigidity leads to the *relaxation* procedure [15], [18], [10], [14] which is the problem of

(i) determination of the "best" structures of composites provided the extremal coefficients of their effective tensors. Namely, we should find microstructures of composites, which store minimum specific energy under a given loading assuming that the volume fractions of materials in the composite are given. Such structures are clearly the only elements which may be used to build the totally optimal body.
(ii) the optimal layout of the optimal composites throughout a designing body.

The approach of relaxation was developed in many papers, see [15], [12], [18], [13, 14], [10]. The relaxation procedure replaces the initial ill-posed problem by the new one which definitely has a solution. However, now it becomes clear that the relaxation leads to the instabilities of solutions. The first reason for instability is that the optimal structures, being anisotropic, "concentrate" their capacity of resistance along the direction of the expected field; because of this they do not resist to a stress field directed orthogonal to the expected one. Therefore the optimal construction is extremely sensitive to the change of loading and to the orientation of the microstructures in each point of the body: small perturbations of these parameters could lead to essential reduction of its rigidity.

The other source of instability of optimally designed body is connected with the optimal distribution of the structures throughout the body. In order to find an optimal distribution of the structures throughout the construction we could express the optimal parameters of the structure in each point through the acting stresses in this point by using the necessary conditions of optimality. Doing this, we end up with a nonlinear equation for the state of optimal body. This equation, however, turns out to be on the boundary of ellipticity and its solution is clearly unstable.

We discuss here these two sources of the instabilities and we suggest reformulations of the problem which lead to more stable solutions.

1.2 The optimal bar

Consider first the simplest problem of the design of an elastic bar with maximum torsion rigidity $I(D)$, following [11, 9, 6]. The initial optimization problem is

$$\min_{D(x)} I(D(x)) \tag{1}$$

where

$$I(D(x)) = \min_{w:\ w=0\ on\ \partial\Omega} \int_\Omega [\nabla w \cdot D(x) \cdot \nabla w - 4w] \tag{2}$$

under conditions:

$$\int_\Omega D(x) = D_{fix}, \quad D(x) = D_1 \ or\ D_2. \tag{3}$$

Here $D(x)$ is the compliance of the material in the point x of the domain Ω of cross section of the bar, D_1 and D_2 are the compliance constants of the available materials, D_{fix} is a given mean compliance which fixes the amounts of the given materials, w is a potential of the problem of torsion. This problem does not have a solution [11, 9, 6] because of the nonconvexity of the set of admissible controls. The problem needs a relaxation which is the enlarging the class of admissible control $D(x)$ by adding to it all effective properties tensors of all composites made from initial materials.

The structures of elastic optimal composites with extremal rigidity against torsion were found in [11], they turn out to be the properly oriented laminates, namely an orientation of optimal layers should coincide with the level lines of the function ∇w. The functional $I(D(x))$ should be replaced after relaxation with the following one:

$$I_r(D(x)) = \int_\Omega \nabla w \cdot d(m) \cdot \nabla w = -2, \quad in\ \Omega, \quad w = 0 \ \ on\ \partial\Omega, \tag{4}$$

where $d(m)$

$$d(m) = [mD_1^{-1} + (1-m)D_1^{-1}]^{-1} \tag{5}$$

is the compliance of the anisotropic laminates across the layers.

The optimal choice of the volume fraction m could be made by studying the optimality conditions. This choice requires that a vector P equal to

$$P = d(m) \cdot \nabla w \tag{6}$$

remains constant [6], if the volume fraction does not reach limits.

$$|P| = \pi = const, \quad if \ \ m(1-m) \neq 0, \tag{7}$$

So we end up with the following first-order problem in a zone, occupied with optimal composites

$$div\ P = -2, \quad |P| = const, \tag{8}$$

This demonstrates that the problem of optimal design is reduced to a problem for a nonlinear isotropic material. This material corresponds to the energy function $W(\nabla w)$ equal to

$$W(\nabla w) = P(\nabla w)\nabla w = \pi \nabla w \quad if \quad m(1-m) \neq 0; \tag{9}$$

the corresponding Euler equations describe the state of the optimal body.

Note that the state equations of optimally designed body are proved to be on the boundary of ellipticity. Indeed, the equivalent nonlinear material for the torsion problem shows the properties of a plastic material if the volume fraction is between zero and one. In other words, the energy of the optimally designed body is on the "rim" of stability, and any infinite small perturbation of it could lead to the nonexistence of a solution of the Euler equation.

Clearly, any perturbation of the solution of the state equation (8) of optimally designed bar leads to change of the value of a solution along its characteristics.

This property is typical for the optimal media; it follows from the relaxation procedure.

1.3 Optimal structures in plane elasticity

The consideration of plane problem of elasticity leads to similar results: the state equation of optimally designed body is not strongly elliptic.

The two-dimensional elastic composites stored the minimum specific energy were described by Cherkaev and Gibiansky [4] ; they were investigated by Kohn and Strang [10], by Milton [16], by Avelaneda [2], by Bendsoe and Kikuchi [3] , by Allaire and Kohn [1] ; similar structures were used by Kikuchi and Suzuki [17] for numerical solution of a number of problems of practical importance. The structure of three-dimensional composites with maximum rigidity were determined by Avelaneda [2] , by Gibiansky and Cherkaev [5], they were investigated by Allaire and Kohn [1]; similar structures were also used by Kikuchi and Suzuki [17] for numerical solutions of optimal design problems.

These structures are but so-called matrix laminates of the second and the third rank [13] , they are formed by lamination of the strong material D_1 with the laminate structure of this material and the weak one D_2.

The effective compliance tensor $D^N(D_1, D_2)$ of the N-th rank laminate structure for the plane elasticity problem is equal to (see, for example [16, 2])

$$D^N(D_1, D_2) = D_1 + m_2[(D_2 - D_1)^{-1} + \frac{m_2}{m_1}\frac{1}{E_1}\sum_{i=1}^{N}[\alpha_i n_i \otimes n_i \otimes n_i \otimes n_i \otimes]^{-1} \quad (10)$$

where E_1 is the inverse Young modulus of the enveloping material, m_1, m_2 are the volume fractions of the first and second materials, n_1, n_2 are the normals to the i-th rank layers in the structure, and α_i

$$\sum_{i=1}^{N} \alpha_i = 1, \ \alpha_i \geq 0 \ \ \forall i \quad (11)$$

are the relative concentrations of the enveloping material in the i-th rank.

It was shown in [4, 5], that the optimal designed elastic body (plain problem of elasticity) consists only of second-rank laminates. One can check, that the minimum of a specific energy among all the N-rank laminates corresponds to a second-rank one. The normals to layers n_1^{opt}, n_2^{opt} in the optimal structure must be oriented along the directions of external principle stresses. Note, that effective compliance tensor D^{opt} has minimum compliance in the projection on the stress field σ; the projection of the tensor compliance on a external shear field $\sigma_{perp} = n_1 \otimes n_2 + n_2 \otimes n_1$ orthogonal to the given stress field σ is much higher and it is equal the average compliance of the components. This compliance is infinite if one of materials is infinitely soft (holes).

Similarly to the problem of torsion, it is easy to see that the problem of optimal design of elastic body can be reduced to an elasticity problem for a nonlinear isotropic material. To demonstrate this one could exclude the volume fractions m_1 and $m_2 = 1 - m_1$ by using the optimality conditions. Similarly to the previous case one ends up with the specific energy which is on the boundary of *quasiconvexity* (see [10]) which means, in particular, that the Euler equations of the problem are not strongly elliptic.

1.4 Discussion.

A numerical solution of an optimization problem requires a finite dimensional approximation of it. We face here the following problem. The minimum value of energy among the set of effective properties of *all* microstructures is proved to coincide with the quasiconvex envelope. However, if one restricts oneself with the minimum among a special class of microstructures or if parameters of microstructures such as orientation, volume fraction etc. are not absolutely precisely calculated *in each point* of the body than the homogenized energy is not quasiconvex similarly to the energy of the initial problem before its relaxation. The corresponding solution of the Euler equations therefore does not exist in this case and the new problem needs applying of relaxation again. We should expect here the absence of the convergence of a minimizing sequence, even if the speed of divergence may be small. In other words, we may expect that the minimizing sequence will consists of the degenerative structures with volume fractions equal to one or to zero, or of initial materials. However, the distribution of these "structures" should be again irregular, and the requirement of optimality will force them to form the inhomogeneous solution which imitate the real optimal structures.

On the contrary, one may use a priory estimates of the stored in a composite energy to build the model for the quasiconvex envelope. These estimates give inequalities for the density of stored energy valid for any microstructure. Even if these estimates are not exact the homogenized energy is quasiconvex. This time one could be sure that the solution of the corresponding variational problem exists and it is smooth, although this solution may not correspond to any structure. However, the convergence of an approximation of this solution is the worse the better the estimate is, because the constant of ellipticity of Euler equations tends to zero when one comes closer to the quasiconvex envelope.

Because of previous arguments it seems to be in principle impossible to make a general conclusion about the convergence of an arbitrary final dimensional approximation of the Euler equations and the result of numerical procedure strongly depends of the numerical scheme.

Probably the mentioned facts explain the different character of optimal solution obtained by Gibiansky and Cherkaev [4] and by Bendsoe and Kikuchi [3], Suzuki and Kikuchi [17].

2 Stable optimization problems

We suggest here several optimal design problems which have similar physical meaning as the problem discussed above but are free of the mentioned instabilities.

2.1 Optimization of the sum of energies

Let us consider the following problem. Find structures of composites which minimizes the total energy of the body under some given loading, with the restriction that the energy stored in the body under other loadings should be bounded by some constant. This restriction may express, for example, the degree of our knowledge of the loading conditions. On the other hand, it may correspond to the situation when the loading is changed in time.

We use here the results by Avelaneda [2] who has shown that the matrix composites of high rank provide the minimum value of the sum of energies, stored in the elastic media under several linearly independent loadings. The effective properties of this class of composites for the plane elasticity are described by formula (10), where one should take $N = 3$ instead of $N = 2$. Note that if none of the parameters $\alpha_1,\ \alpha_2,\ \alpha_3,$ is zero, than the compliance tensor of the structure is finite, even if the compliance of the enveloped material D_2 is infinite, therefore they are more stable on changing the loading and on the error in their orientation. Second-rank laminates as well as simple laminates clearly correspond to the degeneration of the high-rank composites. Physically speaking, the layers of strong material form triangles or tetrahedrons (instead of rectangles) which provide the uniform rigidity to microstructures. This way one may avoid the instability in structures, however the layout of the composites throughout the body should show instability similarly to the solution of the previous problem.

For example, suppose that we do not have any information of the nature of loading field, and we are asked for material which is the best resister for the "worst" loading;

$$\min_{structure} \max_{\sigma:\ ||\sigma||=1} W(\sigma), \tag{12}$$

This material is obviously isotropic, and it must possess the maximal effective bulk and shear moduli. These structures are known: example of it is given by an isotropic laminate composite of third rank with normals n_i uniformly oriented in the plane: $n_i \cdot n_j = -\frac{1}{2}$ and with $\alpha_i = \frac{1}{3}$. Both bulk and shear effective moduli of these structures coincide with the upper Hashin-Shtrikman estimate [8] and are therefore maximal.

Remark. The correspondent three-dimensional structure is the analogues laminates of sixth rank, see [2].

2.2 Optimal design of periodic structures

The other problem naturally arising is the optimal design of composites with a *periodic* structure of maximum rigidity. Suppose that we need to find the pointwise constant tensor of effective properties of a composite and its microstructure so that it corresponds to the maximum rigidity of the body.

This problem also could be reduced to the problem of finding of a microstructure with minimum value of the sum of energies. The physical reason for it is that an effective tensor is required to resist to different stresses in different points of the body and therefore it cannot have zero rigidity along any direction.

Indeed, the total energy W of a body is equal

$$W = \int_\Omega \sigma : C : \sigma = \sum c^*_{ijkl} \int \sigma_{ij}\sigma_{kl} = D^* :: A \tag{13}$$

where A is a positively defined fourth range tensor

$$A_{ijkl} = \int \sigma_{ij}\sigma_{kl}$$

Any tensor of this sort could be represented as a diadic sum of second-rank tensors:

$$A = \sum_{i=1}^{3} \lambda_i a_i \otimes a_i, \quad \lambda_i \geq 0. \tag{15}$$

where $a_i, \;\; i = 1, 2, 3$ form an orthonormal tensor basis, $\lambda_i \;\; i = 1, 2, 3$ are the eigenvalues of the fourth-order tensor A. The problem now is reduced to the optimization of the sum of the energy of three fields

$$s_i = \sqrt{\lambda_i} a_i, \quad i = 1, \; 2, \; 3 : \tag{16}$$

The minimum of the functional W is equal to

$$W = \min_{D^* \in D^3(D_1, D_2)} \sum_{i=1}^{3} s_i : D^* : s_i \tag{17}$$

This problem also has a solution within the class of three-rank laminates. The optimization problem within this class is a finite-dimensional one, so its solution exists and is stable under small perturbation of the structural parameters.

2.3 The stable iterative process for the body with the optimal rigidity

We suggest here an approximation procedure for solving the optimal design problem, which
(i) is stable under perturbation of the external parameters in each step,
(ii) tends to the optimal solution
(iii) corresponds to the known structures at each step.

Consider again the problem of minimization of the rigidity of the body. Suppose that the domain Ω is divided into N subdomains Ω_k with the areas v_k. Suppose also that the composite tensor should remain constant within each subdomain. We ask for the optimal distribution of the microstructures, assuming that the total volume M of the material D_1 is fixed. We come to the following problem:

$$W_N = \min_{m_s}[\sum_{s=1}^{N} \min_{structure} c^s_{ijkl}(structure,\ m_s) \int_{\Omega_s} \sigma_{ij}\sigma_{kl}] \tag{18}$$

$$\sum_{s=1}^{N} v_s m_s = M, \quad m_s \geq 0 \ \ \forall s \tag{19}$$

where $m_1, ...m_N$ is the vector of volume fractions of materials. Obviously, the solution of this problem is given by the structures described in the previous section, because the problem in each subdomain Ω_k coincides with the previous one. Note, that the optimal structures are the matrix laminates of high rank and that the second-rank laminates do not occur as optimal elements if only the stress field is not constant in the domain Ω.

The optimal distribution of the volume of strong materials among these domains should be determined as a solution of a finite-dimensional optimization problem. One can be sure that this problem has a solution and the solution is stable. Moreover, the sequence of the partitions of the domain Ω into subdomains corresponds to the sequence of the solutions of optimization problem which converges to the solution of the initial problem when the maximal diameter of subdomains Ω_k tends to zero. One should mention, of course, that the convergence becomes worse if the solution tends to its limit. However, the advantage of the proposed procedure is that any reasonable finite partition Ω_k leads to a stable solution which corresponds to known composite structures.

The numerical solution of the optimization problem requires a discretization of the initial continuum model. The described procedure corresponds to the following: we introduce two scales of discretization: the fine and the rough ones. The fine scale provides the discretization of the solution of the boundary value problem of elasticity with a piece-wise constant displacement of rigidity tensors $D(x)$. The rough scale corresponds to the problem of determination of the displacement $D(x)$ from the known solution $\sigma(x)$ of the elasticity problem and the fine scale is used to find the solution $\sigma(x)$. This numerical scheme requires to solve the local optimization problem of find a constant (within rough scale) microstructure which minimizes the energy of the varied solution, calculated with a fine scale. If these two scales do not coincide then the optimal solution is given by matrix laminate of third rank. If the fine scale degenerates into the rough one, then a optimal solution also degenerates into the second-rank laminates.

The physical arguments for justifying such a procedure are also quite clear:
(i) Piecewise periodic structures are searched instead of quasiperiodic ones i.e. instead of structures with different orientation of axis of periodicity in each point of the body,
(ii) the structures are stable under small perturbations of the loading,
(iii) they are less sensitive to errors in finding their parameters.

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