EQUALLY STRESSED STRUCTURES: TWO-DIMENSIONAL PERSPECTIVE

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Keywords: Elasticity Fields, Necessary Conditions, Equally Stressed Structures

Abstract The widely used engineering principle of optimal structure requires that a material in a design should not be *understressed*. Here we extend this principle to two-dimensional designs made of two materials. We show that a norm of the stress tensor in a strong material is bounded from below, and a norm in a weak but cheap material is bounded from above. We also found an analog of this principle for badly ordered materials, and we apply it to suboptimal designs.

1. INTRODUCTION

Preliminaries. In this paper, we investigate the stress field in optimal elastic structures and we show that a norm of this tensor field is uniformly bounded in each material. An application of the method of necessary conditions to structural optimization is discussed: Briefly, an infinitesimal amount of one of the two available materials, *nuclei*, is added into a zone occupied with the other one. This adding generates an increment in energy and cost. It depends on the shape of the nuclei. We adjust this shape to obtain the most sensitive variation. Then we find a region of the elastic fields in which the tested material is optimal. This technique enables us to characterize the fields in the optimal design, and to designate quantitatively suboptimal projects, (see the details in Cherkaev, 2000.)

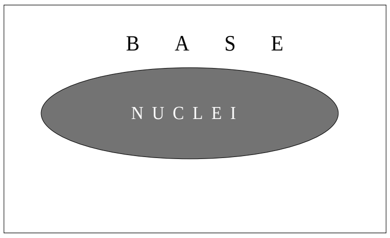


Figure 1 An inclusion in a base material.

To demonstrate the technique we use the well-known problem of the minimization of the complementary energy in two-dimensional elasticity, studied in Gibiansky and Cherkaev, 1997; Allaire and Kohn, 1993; Bendsøe, 1995; Grabovsky and Kohn, 1995, *et al*. However, this technique can be applied to more general problems: The optimal project for more-than-two available materials or minimization of various functionals.

The discussed problem was already well investigated by various means. The lower bound of the energy among all structures was calculated by means of the *translation method.* The necessary condition method that we are developing requires a comparison of nearby configurations, rather than a comparison of all configurations, as sufficient methods do.

It was shown that the optimal structures are either second-rank laminates Gibiansky and Cherkaev, 1997 or, in some special cases, the *Vigdergauz structures* Vigdergauz, 1989; Grabovsky and Kohn, 1995 or *coated spheres* Hashin and Shtrikman, 1962. In contrast to these techniques that deal with an a priori chosen geometric configurations, the necessary conditions analyze the fields in an optimal structure.

Formulation of the Elasticity Problem. We consider standard formulation of a problem of optimizing the structure of a composite of two elastic materials in a two-dimensional domain for simplicity. A periodic cell Ω is divided into two subdomains $\Omega_i,\ i = 1, 2$. The subdomains Ω_i are occupied by isotropic materials with bulk and shear moduli, κ_i and μ_i, respectively. For the given applied load and the given amount of the two materials, the layout over the structure is sought. Using the principle of minimum total complementary energy observes the following formulation for the minimum of the total complementary energy (or maximum stiffness problem)

$$\mathcal{J} = \min_{\chi(\mathtt{x})} \mathcal{W}(\chi), \quad \mathcal{W}(\chi) = [\mathtt{E}(\chi) + \gamma_1\chi + \gamma_2(1-\chi)] \tag{1.1}$$

where $\chi(\mathtt{x})$ is the characteristic function defined as follows

$$\chi(\mathtt{x}) = \begin{cases} 0 & \text{if } \mathtt{x} \in \Omega_1 \\ 1 & \text{if } \mathtt{x} \in \Omega_2, \end{cases} \tag{1.2}$$

and $\mathtt{E}(\chi)$ is the elastic energy

$$\mathtt{E}(\chi) = \min_{\boldsymbol{\sigma}(\mathtt{x}) \in \mathcal{S}(\mathtt{x})} \quad \frac{1}{2}\int_\Omega \boldsymbol{\sigma}(\mathtt{x}) : \mathcal{C}(\mathtt{x}) : \boldsymbol{\sigma}(\mathtt{x}) d\Omega. \tag{1.3}$$

Here $\mathcal{C}(\mathtt{x})$ is a fourth-rank material compliance tensor, $\mathcal{C}_{ad}$ is a set of admissible compliance tensors described as follows,

$$\mathcal{C}(\mathtt{x}) = \boldsymbol{\chi}(\mathtt{x})\mathcal{C}_1(\mu_1, \kappa_1) + (1 - \boldsymbol{\chi}(\mathtt{x}))\mathcal{C}_2(\mu_2, \kappa_2) \tag{1.4}$$

μ_i and κ_i are shear and bulk moduli of the materials, respectively. The second-rank stress tensors $\boldsymbol{\sigma}(\mathtt{x})$ belong to the set $\mathcal{S}(\mathtt{x})$ of statically admissible stress fields

$$\mathcal{S}(\mathtt{x}) = \{\boldsymbol{\sigma}(\mathtt{x}) \mid \nabla \cdot \boldsymbol{\sigma}(\mathtt{x}) + \mathbf{p}(\mathtt{x}) = 0 \text{ in } \Omega, \boldsymbol{\sigma}(\mathtt{x})\mathbf{n}(\mathtt{x}) = \mathbf{t}(\mathtt{x}) \text{ on } \Gamma_T.\}$$

Here $\mathbf{p}(\mathtt{x})$ and $\mathbf{t}(\mathtt{x})$ are the given body forces and surface tractions, respectively, $\mathbf{n}(\mathtt{x})$ is the normal vector to the surface, and Γ_T is the traction part of the surface.

In problem (1.1), χ should be chosen to minimize the sum of the energy $\mathtt{E}(\chi)$ and the total cost $\gamma_1\chi + \gamma_2(1-\chi)$. It is known that the optimal distribution corresponds to a fastly oscillating sequence of the domains Ω_i; this makes the variational problem unstable , see Cherkaev, 2000.

Necessary Conditions of Weierstrass Type. To investigate an unstable variational problem (1.1), we introduce the variations $\delta\chi$. The variation detects the permitted regions of $\boldsymbol{\sigma}(\mathtt{x})$ where the materials $\mathcal{C}_1$ and $\mathcal{C}_2$ are optimal, then it detects the forbidden region Ω_f where the condition,

$$\delta\mathcal{J} = \min_{\delta\chi}[\mathcal{W}(\chi + \delta\chi) - \mathcal{W}(\chi)] \geq 0, \tag{1.5}$$

is violated for the problem, (1.1).

The choice of the class of variations is nontrivial. The variations must be sufficiently simple to permit explicit calculation of the increment of the functional and must also be sensitive enough to detect instabilities.

Remark 1 *The method is a multidimensional analogue of the Weierstrass variations in classical calculus of variations. This type of optimality condition (strong variations of the material's properties) was suggested in Lurie, 1963; Lurie et al., 1982 and summarized in Lurie, 1993 where the variation in the strip was used and the forbidden regions were detected. This way, the problem was proven to be ill-posed. The paper by Petukhov, 1995 considers more complicated variations for problems in elasticity.*

Matrix Form of Tensors. Complexity of the calculations with tensors is eased by transforming them into a matrix or a vector forms in the following base, $\mathbf{b}^{(i)} = \{b^{(i)}_{\alpha\beta}\}$ where α and β are indexes of the $\mathbf{b}$,

$$\mathbf{b}^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{b}^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{b}^{(3)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.6}$$

Using this base, any forth-rank tensor can be rewritten as three by three matrix while any second-rank tensor can be represented by a vector. In this basis, a second-rank stress tensor has the following representation,

$$\sigma_{ij} = \sigma_{\kappa\eta} b^{(i)}_{\kappa\eta} b^{(i)}_{\alpha\beta} = \mathtt{s}_1 b^{(1)}_{\alpha\beta} + \mathtt{s}_2 b^{(2)}_{\alpha\beta} + \mathtt{s}_3 b^{(3)}_{\alpha\beta} \tag{1.7}$$

where $(\mathtt{s}_1\ \mathtt{s}_2\ \mathtt{s}_3)^T = \mathtt{s}$ is a vector with the components,

$$\mathtt{s}_1 = \frac{\sigma_{11} - \sigma_{22}}{\sqrt{2}}, \quad \mathtt{s}_2 = \frac{2}{\sqrt{2}}\sigma_{12} \ \ and \ \ \mathtt{s}_3 = \frac{\sigma_{11} + \sigma_{22}}{\sqrt{2}}, \tag{1.8}$$

and an anisotopic fourth-rank compliance tensor $\mathfrak{C}$ is replaced by a symmetric three by three matrix $\mathfrak{D} = (\mathcal{D}_{ij})$. Components of $\mathfrak{D}$ are determined by the following transformation,

$$\mathcal{D}_{ij} = b^{(i)}_{\alpha\beta} \mathcal{C}_{\alpha\beta\kappa\eta} b^{(j)}_{\kappa\eta}, \quad i,j = 1,2. \tag{1.9}$$

Term-wise, the components of $\mathfrak{D}$ can be written as follows

$$\begin{aligned} \mathcal{D}_{11} &= \frac{1}{2}(\mathcal{C}_{1111} + \mathcal{C}_{2222}) - \mathcal{C}_{1122}, \\ \mathcal{D}_{12} &= \mathcal{C}_{1112} - \mathcal{C}_{2221}, \quad \mathcal{D}_{13} = \frac{1}{2}(\mathcal{C}_{1111} + \mathcal{C}_{2222}), \\ \mathcal{D}_{22} &= \frac{1}{2}(\mathcal{C}_{1111} + \mathcal{C}_{2222}) + \mathcal{C}_{1122}, \\ \mathcal{D}_{12} &= \mathcal{C}_{1112} + \mathcal{C}_{2221}, \quad \mathcal{D}_{33} = 2\mathcal{C}_{1212}. \end{aligned} \tag{1.10}$$

Particularly, a fourth-rank isotropic compliance tensor $\mathfrak{C}$ is given by a diagonal matrix,

$$\mathfrak{D}_i = diag\left(\frac{1}{2\mu_i}, \frac{1}{2\mu_i}, \frac{1}{2\kappa_i}\right). \tag{1.11}$$

2. VARIATIONS

2.1. STRUCTURAL VARIATION

In this section, we describe the Weierstrass-type *structural variation* that is used to check the optimality of a structure. To perform this variation we implant infinitesimal inclusions (nuclei) of an admissible material $\mathfrak{D}_n$ at a point $\mathbf{x}$ in the domain Ω occupied by a host (base) material $\mathfrak{D}_b$ and compute the difference in energies and cost, see Figure(1). If the examined structure is optimal, then the increment of the cost is nonnegative.

Specifically, we replace material $\mathfrak{D}_b$ in a neighborhood of a point $\mathbf{x}$ by a quasiperiodic dilute composite of second-rank laminates. In these laminates, the envelope is made of the host material $\mathfrak{D}_b$, and the inclusions are made of material $\mathfrak{D}_n$. In other words, we construct a second-rank laminate from the isotropic host medium in the envelope. This matrix laminate composite of second rank is characterized by its effective tensor $\mathfrak{D}_*$, see Cherkaev, 2000,

$$\mathfrak{D}_* = \mathfrak{D}_n - (1-m)\left((\mathfrak{D}_n - \mathfrak{D}_b)^{-1} - m\mathbf{\Delta}(\alpha)\right)^{-1}, \tag{1.12}$$

where m is the volume fraction of the nuclei and matrix $\mathbf{\Delta}$ determines the degree of anisotropy. In the base (1.6), $\mathbf{\Delta}(\alpha)$ is

$$\mathbf{\Delta}(\alpha) = \frac{\kappa_n \mu_n}{\kappa_n + \mu_n} \begin{pmatrix} 1 & 0 & 2\alpha - 1 \\ 0 & 0 & 0 \\ 2\alpha - 1 & 0 & 1 \end{pmatrix}, \quad 0 \le \alpha \le 1. \tag{1.13}$$

Here the inner parameter α is the relative elongation of the array of inclusions that determines the anisotropy of $\mathfrak{D}_*$. If $\alpha = 0$ or $\alpha = 1$, then $\mathfrak{D}_*$ becomes a tensor of laminates. The eigenvectors of $\mathbf{\Delta}$ coincide with the directions of lamination.

Let us compute the increment $\delta\mathfrak{D}$ of the tensor properties caused by the array of infinitely dilute nuclei with material $\mathfrak{D}_n$ and infinitesimal volume fraction δm. Replacing m with $m + \delta m$ and $\mathfrak{D}_*$ with $\mathfrak{D} + \delta\mathfrak{D}$ in (1.12), we obtain

$$\delta\mathfrak{D} = \mathcal{V}\delta m + \vartheta(\delta m), \quad \mathcal{V} = \left((\mathfrak{D}_n - \mathfrak{D}_b)^{-1} - \mathbf{\Delta}\right)^{-1}. \tag{1.14}$$

Notice that the term $\mathbf{\Delta}$ contains all information about the geometry of the variation.

Remark 2 *The increments caused by a single elliptical inclusion can be calculated using methods of Eshelby, 1975; Movchan and Movchan,* *1995. However, the array of inclusions generally leads to more sensitive variation than a single inclusion of the same area; see Cherkaev et al., 1998. Namely, one elliptical inclusion is equivalent to the array of dilute second-rank laminates if $Det(\boldsymbol{\sigma}) \geq 0$, but the array leads to the stronger variation than a single inclusion if $Det(\boldsymbol{\sigma}) < 0$.*

Remark 3 *The use of homogenization procedure in Equation (1.12) reduces the Weierstrass test to a small variation of the effective properties. All discontinuities of the field $\boldsymbol{\sigma}$ and the geometry of the variation are already accounted in Equation(1.12).*

2.2. INCREMENT

The increment of energy depends on the shape of the region of variation; this shape must be adjusted to the field so that the increment reaches its minimal value which must remain nonnegative. Let us compute the variation of energy caused by the variation (1.14) of properties $\delta\mathfrak{D}$. For simplicity, we assume that the main axes of $\delta\mathfrak{D}$ are codirected with the principal axes of the matrix $\boldsymbol{\sigma}$ (later we will see that the obtained necessary conditions are the strongest ones, which justifies this assumption).

The cost consists of the increment of energy due to the variation of $\delta\mathfrak{D}$ and the *direct cost* of the variation, that is, the change in the total cost due to change of quantities of the materials used. The direct cost of the variation is determined by its type: We replace the base material $\mathfrak{D}_b$ (with the specific cost γ) by the nuclei material $\mathfrak{D}_n$ (with the specific cost γ'). The change in the total cost is $(\gamma' - \gamma)\,\delta m$.

The increment of the energy $\delta\mathcal{W}$ is equal to

$$\delta\mathcal{W}(\alpha) = \mathbf{s}^{\mathbf{T}}\delta\mathfrak{D}(\alpha)\mathbf{s}.$$

It depends on the shape of the variation, specifically, on the parameter α of anisotropy (elongation) of the inclusions. Using the formulas given in (1.8) and (1.14) for $\delta\mathfrak{D}$, the latter equation becomes

$$\delta\mathcal{W}(\alpha) = \Big(\mathcal{V}_{11}\mathbf{s}_1^2 + \mathcal{V}_{22}\mathbf{s}_2^2 + \mathcal{V}_{33}\mathbf{s}_3^2 + 2\mathcal{V}_{12}\mathbf{s}_1\mathbf{s}_2 + 2\mathcal{V}_{13}\mathbf{s}_1\mathbf{s}_3 + 2\mathcal{V}_{23}\mathbf{s}_2\mathbf{s}_3\Big)\delta m; \quad (1.15)$$

and the components of $\boldsymbol{\mathcal{V}}$ depend on α as follows

$$\mathcal{V}_{11} = -\frac{1}{2}\frac{(\mu_n - \mu_b)(\kappa_b(2\kappa_n + \mu_n) + \kappa_n\mu_n)}{\mu_n K},$$

$$\mathcal{V}_{13} = \frac{1}{2}\frac{(\mu_n - \mu_b)(\kappa_n + \mu_n)(2\alpha - 1)}{K},$$
$$\mathcal{V}_{22} = -\frac{1}{2}\frac{\mu_b - \mu_n}{\mu_n \mu_b}, \quad \mathcal{V}_{12} = 0, \quad \mathcal{V}_{23} = 0$$
$$\mathcal{V}_{33} = -\frac{1}{2}\frac{(\mu_b(2\mu_n + \kappa_n) + \kappa_n \mu_n)(\kappa_b - \kappa_n)(\kappa_n + \mu_n)}{\kappa_n K},$$

where

$$K = K_1\alpha(\alpha - 1) + \mu_b\kappa_b\kappa_n(\kappa_n + 2\mu_n) + \mu_n\kappa_n(\kappa_n + \mu_n)(\mu_b + \kappa_b) + \mu_b\kappa_b\mu_n^2,$$

and $K_1 = 2\kappa_n\mu_n(\mu_n - \mu_b)(\kappa_b - \kappa_n)$.

The total cost of the variation, $\Delta\mathfrak{J}\delta m$, is

$$\Delta\mathfrak{J}\delta m = (\gamma' - \gamma + \delta\mathtt{E}(\alpha))\, \delta m. \tag{1.16}$$

If the structure is optimal, then all variations (including the most sensitive one) lead to the nonnegative increment $\Delta\mathfrak{J}$ or to the condition

$$\frac{\Delta\mathfrak{J}}{\delta m} = \delta_b\mathfrak{J}(\mathfrak{D}_b, \mathfrak{D}_n, \boldsymbol{\sigma}_b) = \gamma' - \gamma + \min_{\alpha} \delta\mathtt{E}(\alpha) \geq 0, \tag{1.17}$$

where $\delta_b\mathfrak{J}(\mathfrak{D}_b, \mathfrak{D}_n, \boldsymbol{\sigma}_b)$ is a variation caused by adding the material $\mathfrak{D}_n$ into the material $\mathfrak{D}_b$, subscript $(_b)$ shows the host material, and σ_b is the field in it. If the condition (1.17) is violated, then the cost will be reduced by a variation, and the structure fails the test.

Minimizing the Increment: The Most Sensitive Variations. In the equation (1.17), the optimal increment energy function $\mathfrak{F}_0$ is introduced as

$$\mathfrak{F}_0 = \min_{\alpha \in [0,1]} \delta\mathtt{E}(\alpha).$$

Geometric shape of the structure is adjusted to the acting field over the domain by the inner parameter α. To compute an optimal variation, we take the derivative of $\delta\mathtt{E}(\alpha)$ with respect to α, and obtain the following α's as zeros of $\dfrac{\partial\delta\mathtt{E}(\alpha)}{\partial\alpha}$:

$$\alpha_1 = -\frac{\sigma_{11}(\kappa_n + \mu_b)\mu_n + \sigma_{22}(\kappa_n + \mu_n)\mu_b}{\kappa_n(\sigma_{11} - \sigma_{22})(\mu_b - \mu_n)}, \tag{1.18}$$
$$\alpha_2 = \frac{\sigma_{11}(\mu_n + \kappa_b)\kappa_n - \sigma_{22}(\kappa_n + \mu_n)\kappa_b}{\mu_n(\sigma_{11} + \sigma_{22})(\kappa_n - \kappa_b)}. \tag{1.19}$$

The minimum of $\delta\mathtt{E}(\alpha)$ depends on the values of optimal values α_1 and α_2, $\alpha_i \in [0,1]$, and the parameters κ_i's and μ_i's.

There are two cases to consider depending on the sign of $\kappa_n - \kappa_b$;

Case a: If $\mu_n - \mu_b < 0$, then $\mathfrak{F}_0$ is $\mathfrak{F}_a$ where $\mathfrak{F}_a$ is

$$\mathfrak{F}_a = \begin{cases} \mathtt{E}(0) & \text{if } \dfrac{\sigma_{22}}{\sigma_{11}} < K, \\ \mathtt{E}(\alpha_2) & \text{if } K \leq \dfrac{\sigma_{22}}{\sigma_{11}} \leq \dfrac{1}{K}, \\ \mathtt{E}(1) & \text{if } \dfrac{\sigma_{22}}{\sigma_{11}} > \dfrac{1}{K}, \end{cases} \tag{1.20}$$

when $\kappa_n - \kappa_b < 0$. Here

$$K = \frac{(\kappa_b + \mu_n)\kappa_n}{(\kappa_n + \mu_n)\kappa_b} < 1.$$

Case b: If $\mu_n - \mu_b > 0$, then α_0 always belongs to the boundary of the interval $[0, 1]$. The minimal $\delta\mathtt{E}$ is reached at the end points, *i.e.*, α_0 is either 0 or 1 and $\mathfrak{F}_0$ is $\mathfrak{F}_b$ where $\mathfrak{F}_b$ is

$$\mathfrak{F}_b = \begin{cases} \mathtt{E}(1) & \text{if } \sigma_{11} \leq \sigma_{22}, \\ \mathtt{E}(0) & \text{if } \sigma_{11} \geq \sigma_{22}, \end{cases} \tag{1.21}$$

when $\kappa_n - \kappa_b > 0$.

Calculations in Case a and Case b are similar for opposite signs of $\kappa_n - \kappa_b$.

2.3. NECESSARY CONDITIONS

Based on the results obtained in the previous section, the range of admissible fields, $\boldsymbol{\sigma}$, can be determined in an optimal structure.

Consider a specific case, $well-ordered$ case, in which the moduli κ, μ of the isotropic materials (see (1.11)) are ordered so that $\mu_1 < \mu_2$ and $\kappa_1 < \kappa_2$, (otherwise the problem will be called *badly ordered* case.) Let us find the region of permitted fields by the described variations for an optimal composite made of these given materials by using (1.17) and the results given in (1.20) - (1.21). (*Badly ordered* case is considered similarly.)

1 Suppose that the second material $\mathfrak{D}_2$ is inserted into the domain Ω_1. Then the increment $\delta_1\mathfrak{I}(\mathfrak{D}_1, \mathfrak{D}_2, \boldsymbol{\sigma}_1)$ of the functional is (see (1.17))

$$\delta_1\mathfrak{I}(\mathfrak{D}_1, \mathfrak{D}_2, \boldsymbol{\sigma}_1) = \gamma_2 - \gamma_1 + \mathfrak{F}_a(\mathfrak{D}_1, \mathfrak{D}_2, \boldsymbol{\sigma}_1) \geq 0, \tag{1.22}$$

where $\boldsymbol{\sigma}_1$ is the field in the domain Ω_1 of the first material. The inequality (1.22) depends only on the stress field $\boldsymbol{\sigma}_1$ since $\mathfrak{D}_1$, and

$\mathfrak{D}_2$ are given. The set of $\boldsymbol{\sigma}_1$ that satisfies the condition (1.22) is called $\mathtt{F}_1$; if $\boldsymbol{\sigma}_1 \in \mathtt{F}_1$, then the material $\mathfrak{D}_1$ is optimal.

2 The increment $\delta_2\mathfrak{J}(\mathfrak{D}_2, \mathfrak{D}_1, \boldsymbol{\sigma}_2)$ of the functional due to the inclusion of the first material $\mathfrak{D}_1$ into the domain Ω_2 is

$$\delta_2\mathfrak{J}(\mathfrak{D}_2, \mathfrak{D}_1, \boldsymbol{\sigma}_2) = \gamma_1 - \gamma_2 + \mathfrak{F}_b(\mathfrak{D}_2, \mathfrak{D}_1, \boldsymbol{\sigma}_2) \geq 0, \tag{1.23}$$

where $\boldsymbol{\sigma}_2$ is the field at a point of the domain Ω_2 of the second material. The set of $\boldsymbol{\sigma}_2$ that satisfies the condition (1.23) is called $\mathtt{F}_2$; if $\boldsymbol{\sigma}_2 \in \mathtt{F}_2$, then the material $\mathfrak{D}_2$ is optimal.

The union, $\mathtt{F} = \mathtt{F}_1 \cup \mathtt{F}_2$, of these two sets does not coincide with the whole range of $\boldsymbol{\sigma}$. The remaining part $\mathtt{F}_f$ is the forbidden region; none of the materials is optimal in $\mathtt{F}_f$. An optimal structure should be constructed in such a way that the fields never belong to $\mathtt{F}_f$, no matter what the loading is.

Let us show details of the calculations for $\mathtt{F}$. Assume that the axes are oriented so that $\sigma_{12} = 0$ and $0 < \sigma_{11} \leq \sigma_{22}$; σ_{11} and σ_{22} become the eigenvalues of $\boldsymbol{\sigma}$. Then the fields $\mathtt{F}_1$ and $\mathtt{F}_2$ in each material $\mathfrak{D}_1$ and $\mathfrak{D}_2$ determined by (1.22) and (1.23), respectively, are as follows:

$$\mathtt{F}_1 = \left\{ \boldsymbol{\sigma}_1 \mid \delta_1\mathfrak{J}(\mathfrak{D}_1, \mathfrak{D}_2, \boldsymbol{\sigma}_1) = \gamma_2 - \gamma_1 + \left\{ \begin{array}{ll} \mathtt{E}(0) & \text{if } \dfrac{\sigma_{22}}{\sigma_{11}} < K, \\ \mathtt{E}(\alpha_2) & \text{if } \dfrac{\sigma_{22}}{\sigma_{11}} \geq K \end{array} \right\} \geq 0 \right\}, \tag{1.24}$$

where

$$\mathtt{E}(\alpha_2) = \frac{1}{4} \frac{(\kappa_n + \mu_n)(\kappa_n - \kappa_b)}{\kappa_n(2\kappa_b\kappa_n + \kappa_b\mu_n + \kappa_n\mu_n)} (\sigma_{11} + \sigma_{22})^2, \tag{1.25}$$

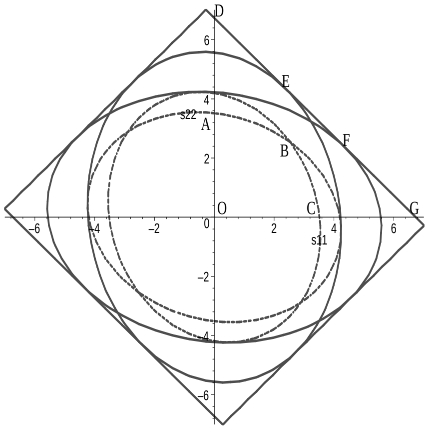
and

$$\mathtt{F}_2 = \{\boldsymbol{\sigma}_2 \mid \delta_2\mathfrak{J}(\mathfrak{D}_2, \mathfrak{D}_1, \boldsymbol{\sigma}_2) = \gamma_1 - \gamma_2 + \mathtt{E}(1) \geq 0\}. \tag{1.26}$$

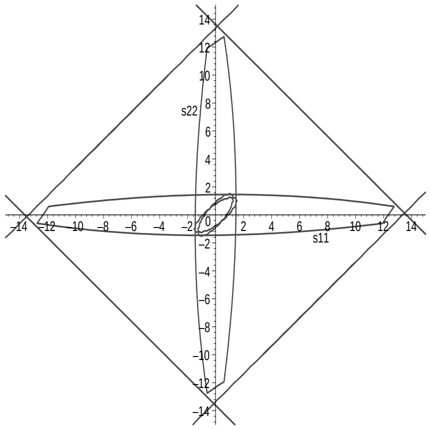
Let us comment on the obtained results:

- In Formula (1.24), a *weak material* is added to a *strong material*. The best shape of the nuclei is either the family of nucleus in the second rank lamination (if the ratio between σ_{11} and σ_{22} is small) or a fillet (if the ratio between σ_{11} and σ_{22} is large.) The *weak material* hides itself to reduce the effects of the variation.

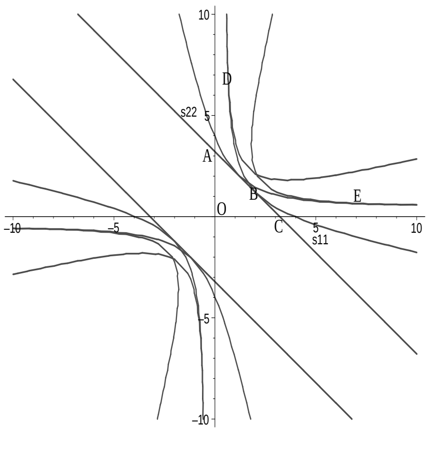
- In Formula (1.26), a *strong material* is added to a *weak material*. The best shape of the nuclei is a fillet which is placed along the direction of maximal field by doing that it exposes itself to increase the effect of the variation.



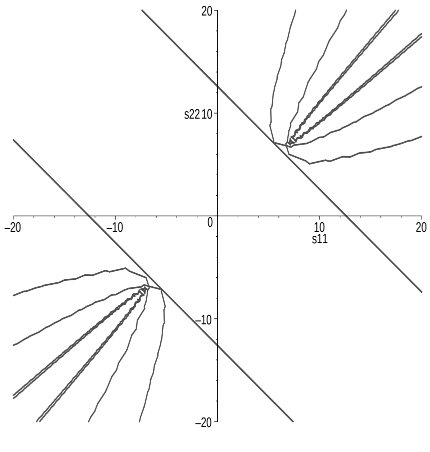
(a) Region $\mathtt{F}_1$ lies outside $DEFG$, region $\mathtt{F}_2$ lies inside $OABC$, and in between lies the forbidden region $\mathtt{F}_f$.



(b) The asymptotic case when the ratio between the material properties is big.



(c) Region $\mathtt{F}_1$ lies outside DBE, region $\mathtt{F}_2$ lies inside $OABC$, and in between lies the forbidden region $\mathtt{F}_f$.



(d) The asymptotic case when the ratio between the material properties is big.

Figure 2 Optimal Fields for well-ordered cases $\kappa_1 < \kappa_2$ and $\mu_1 < \mu_2$ (a) and (b); *Badly ordered* cases $\kappa_1 > \kappa_2$ and $\mu_1 < \mu_2$ (c) and (d).

3. RESULTS

Range of admissible Fields. Notice that a component of the boundary of $\mathtt{F}$ has the following form

$$\mathtt{E}(\alpha) = A\sigma_{11}^2 + B\sigma_{22}^2 = 1 \tag{1.27}$$

where the constant coefficients A, and B are determined by the material constants and it corresponds to optimal values of $\mathtt{E}(\alpha)$ at $\alpha = 0$ or $\alpha = 1$.

Another component of the boundary of $\mathtt{F}$ has the following from

$$\mathtt{E}(\alpha_i) = C(\sigma_{11} \pm \sigma_{22})^2 = 1 \tag{1.28}$$

where the constant coefficient C is determined by the material constants and it corresponds to stationary values, Equation (1.19) of α. The boundaries of the forbidden region consist of either the straight segment $\mathtt{E}(\alpha_2) = 1$, or the elliptic segment in the $well - ordered$ case, or the hyperbolic segment in the *badly ordered* case $\mathtt{E}(\alpha) = 1$. Similar results are obtained for the case $\sigma_{11} \leq \sigma_{22} < 0$ by replacing α_2 with α_1, Equation (1.18).

In figures (2(b)) and (2(d)), the asymptotic cases are considered,*i.e.*, one of the materials has infinite moduli. This is also called as a problem of *topology optimization* (Bendsøe, 1995.) In this case, the elliptic component disappears.

In the *badly ordered* case, the situation is more tedious. Each material plays both roles as *strong* or *weak* one, depending on the field $\boldsymbol{\sigma}$. The elliptical components of the boundaries in the $well - ordered$ case become hyperbolic; they correspond to optimal fillet type inclusions. The boundaries are given either by intersection of two hyperbolas $\mathtt{E}(0) =$ *constant* and $\mathtt{E}(1) =$ *constant*, or by linear tangent envelope to two hyperbolas ($\overline{ABC}$, in Figure (2(c))) In the last case, the straight part of the envelope correspond to optimal inclusions as the nuclei of a second-rank composites. The cheapest material is optimal in the proximity of the origin. One can check that the boundaries become straight lines passing through the origin if the cost of both material is the same.

The graph of the permitted fields is presented in Figure (2(a)) and (2(c)). In the $well - ordered$ case, Figure (2(a)), region $\mathtt{F}_1$ lies outside $DEFG$, region $\mathtt{F}_2$ lies inside $OABC$, and the region between $\mathtt{F}_1$ and $\mathtt{F}_2$ is the forbidden region $\mathtt{F}_f$.

Now let us analyze the fields in an optimal structure: Suppose that an optimal structure is submerged into an external field $\boldsymbol{\sigma}$. If the external field belongs to $\mathtt{F}_1$ or $\mathtt{F}_2$, then the optimal design consists of one material $\mathfrak{D}_1$ or $\mathfrak{D}_2$. The field must jump over the forbidden region along the boundary line between these zones. It turns out, for example Cherkaev, 2000, that such jumps are possible only if the boundary becomes a fractal curve with infinitely many wiggles. As a result, the optimal structure becomes a composite in which the fields belong to the boundaries of $\mathtt{F}_1 = constant$ or $\mathtt{F}_2 = constant$ at each point. The mentioned optimal structures (second-rank laminates, *etc*) are geometrically different but they all carry this feature.

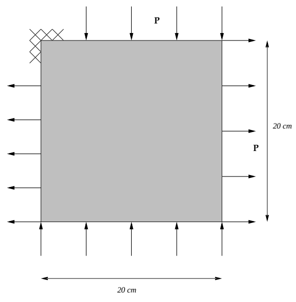


Figure 3 Plate design domain, load and boundary conditions.

Norms. The boundaries of $\mathtt{F}$ corresponds to constancy of two norms of $\boldsymbol{\sigma}$

$$\mathtt{N}_1 = \sqrt{\mathtt{F}_1}, \text{ and } \mathtt{N}_1 = \sqrt{\mathtt{F}_2}. \tag{1.29}$$

Notice that $\mathtt{N}_1$ and $\mathtt{N}_2$ are indeed rotationally invariant norms of two-by-two symmetric tensors. In one-dimensional case, both $\mathtt{N}_1$ and $\mathtt{N}_2$ degenerate to absolute value of the stress.

These norms are constant in each material for an optimal structure,*i.e*, $\mathtt{N}_1 \geq constant$, or $\mathtt{N}_2 \leq constant$. The boundary curve is dense in a region that corresponds to the mixture of the materials. In this mixture, the pointwise norms of $\boldsymbol{\sigma}$ fields are constant and the mean field varies due to the variation of concentrations. The mixtures should be fine otherwise one cannot preserve the constancy of the norm inside the solid region of a material.

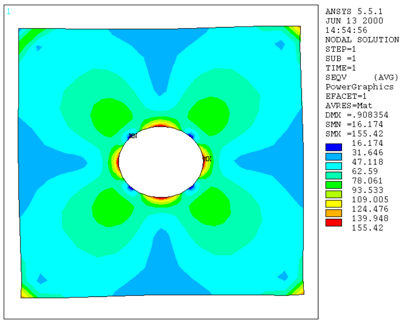
Numerics. The optimal layout of laminates was developed by Díaz et al., 1983; Gibiansky and Cherkaev, 1984; Bendsøe and Kikuchi, 1988; Haber et al., 1996; Díaz and Lipton, 1997; Allaire et al., 1997. Here we use necessary conditions for suboptimal projects.

In this section, the asymptotic cases are considered in the material properties when $\kappa_2 \to \infty$ and $\eta_2 \to \infty$. In this asymptotical case, $\mathtt{N}_2 = 0, \mathtt{N}_1 = |\sigma_{11}| + |\sigma_{22}|$; the norm $\mathtt{N}_1$ must be constant everywhere in an optimal projection if $\sigma_{11}\sigma_{22} > 0$, that trace of tensor $\boldsymbol{\sigma}$ is constant, and if $\sigma_{11}\sigma_{22} < 0$, then Von Mises stress is constant. This corresponds to the earlier results of Bendsøe, 1995; Grabovsky, 1996. We consider examples of optimal design for planar structures with constant material property. The analysis of Cherkaev et al., 1998 suggests a parameterization of suboptimal shapes. We compare the shapes of a circle, square and two rectangles with the same areas, and solve an elasticity problem for the inner minimization in Problem (1.1).

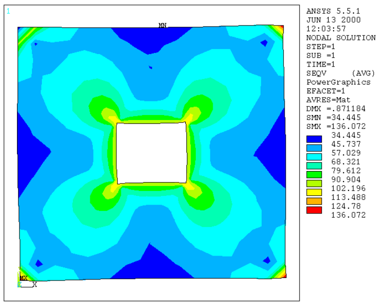
A plane design domain, a 20×20 plate with Poission's ratio and Young's moduli are 0.32 and and 2×10^5, respectively, is loaded and supported as shown in Fig (3). The voids will be placed into the plate in order to minimize the total complementary energy. The plate is discretized into 8-node isotropic finite element.

The design variables characterize that two fillet type of inclusions are better than one fillet with the same area in terms of the total complementary energy; we obtain the following normalized energies for each structure in Figure (4); (4(a)) 0.5121, (4(b)) 0.5033, (4(c)) 0.4956, and (4(d)) 0.4907.

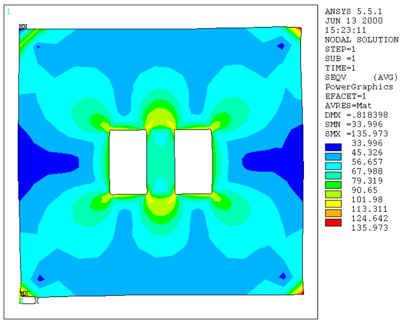
This approach allows us to describe quantitatively suboptimal structures; In the *true optimal* solution the norm $\mathtt{N}(\boldsymbol{\sigma})$ is expected to be constant. We compare this norm in Figure (4) for various structures, and we observe that the deviation from the constancy of the norm $\mathtt{N}(\boldsymbol{\sigma})$ is smaller in Figure (4(d)) than the others.



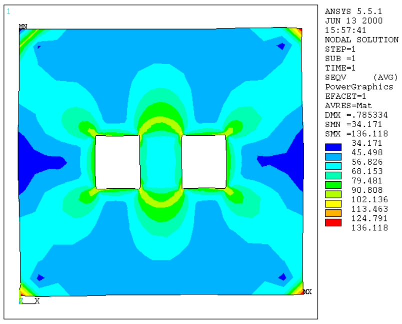
(a) Circle



(b) One quare



(c) Two rectangles



(d) Sub-optimal Structure

Figure 4 Von Mises stress distribution for various deformed structures.

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