THE EXACT COUPLED BOUNDS FOR EFFECTIVE TENSORS OF ELECTRICAL AND MAGNETIC PROPERTIES OF TWO-COMPONENT TWO-DIMENSIONAL COMPOSITES (DRAFT) [1]

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Contents

Abstract

The paper describes the effective properties of a two-dimensional two-component dielectric composite. Each component is supposed to be isotropic and is characterized by a pair of constants of electrical and magnetic permeabilities. Effective properties of an arbitrary anisotropic composite depend on its microstructure and are characterized by a pair of symmetric tensors of electrical and magnetic permeabilities. Here we construct the so called G_m-closure which is the set of all possible values of these pairs, corresponding to all possible microstructures with prescribed volume fractions of the given components.

The exact bounds of the G_m-closure are obtained and the microstructure corresponding to each point of the G_m-closure set is determined.

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1 INTRODUCTION

The paper is devoted to the problem of the exact description of a set of the effective properties tensors of all possible mixtures which are assembled in an arbitrary way from two isotropic materials taken in the prescribed proportions. This set is called [**23,25**] the G_m-closure of the set U of the initial materials and is denoted by G_mU. It depends only on the volume fractions of the materials in a composite and on their properties.

This problem is rather complicated; the first nontrivial example was done by Lurie, Cherkaev [**23,24**] and Murat, Tartar [**44**]. In those papers the G_m-closure for the conductivity problem was constructed. Another known result, concerning the G_m-closure of a two dimensional mixture of two elastic materials with equal bulk moduli is an analog of the first one, as was shown by Lurie, Cherkaev [**22**] and Lipton [**20**].

Here we will consider a mixture of two isotropic dielectrics in the two-dimensional space. The problem is richer than the previous ones because each material possesses two different scalar properties. Many inequalities which estimate the electrical and magnetic properties of materials separately were obtained. Namely, the interval of possible isotropic electrical or magnetic permeabilities was given by the Hashin-Shtrikman estimates [**13**] and the complete description of the set of the anisotropic permeability tensors follows from the exact estimates of the G_m-closure for conducting mixtures [**23,24,44**].

The first connected estimates of the dielectric and magnetic properties were obtained by Prager [**40**] and the optimal structures were studied by Schulgasser [**43**]. The exact estimates for an isotropic mixture (2D problem) with an arbitrary number of properties such as dielectric, magnetic, thermal etc. were obtained by Milton [**29,30**] using the "analytical method" (see Bergman [**3**], Milton [**29,30,34**]). The optimal microstructures were found to be the sequences of "coated spheres".

Later Milton showed [**34**] that any mixture with the complex conductivity function has the same effective properties as the special microstructure which we call here "multicoated matrices" and describe below. This result is in fact an implicit description of the G_mU-closure set because the properties of the "multicoated matrices" can be calculated analytically.

The variant of the translation method developed in this paper gives us the explicit and rather simple analytical expressions for all boundary surfaces of the G_m-closure set for the problem considered.

This method was suggested by Lurie, Cherkaev in the papers [**23,24,25**] for estimating the conductivity of a mixture. It is based on the theory of quasiconvexity and the theory of "compensated compactness" (see Morrey [**6**], Murat, Tartar [**38**], Tartar [**45**], Kohn, Strang [**18**]).

The problem is much more rich than the conductivity problem and so it is necessary to modify the method and to introduce some new features in it. In particular we use here a convenient transformation of the basic inequalities (so called Y transformation) similar to one which was introduced by Milton and Golden [**32**] (see also [**37**]).

The results presented deal with the dielectric problem for a two dimensional composite but the method is much more general. As it will be shown it does not use any particular features of the problem except isotropy of both initial materials.

In the last part of this paper we discuss the direct generalization of the obtained result for the problem of coupling properties. As we learned recently the procedure described here is similar to that studied by Milgrom and Shtrikman [**28**].

2 THE STATEMENT OF A G_m-CLOSURE PROBLEM FOR DIELECTRICS.

We begin with the formulation of the problem of the G_mU set construction for two isotropic dielectric materials. To study the effective properties of a mixture it is sufficient to deal with space periodic structures. We assume that each element of periodicity S is divided into the parts S_1 and S_2 with the prescribed volume fractions m_1 and m_2 respectively.

$$\begin{gathered} S_1 \cup S_2 = S, \\ (volS_1)/(volS) = m_1, \\ (volS_2)/(volS) = m_2, \\ m_1 + m_2 = 1 \end{gathered} \tag{2.1}$$

Suppose that these two parts are occupied by two isotropic dielectric materials with different dielectric permeabilities $\mathcal{E}_1 = \epsilon_1 I$ and $\mathcal{E}_2 = \epsilon_2 I$ and magnetic permeabilities $\mathcal{M}_1 = \mu_1 I$ and $\mathcal{M}_2 = \mu_2 I$ respectively. The state of the media is described by the linear elliptic system of differential equations [**26**] (the stationary Maxwell's equations for the dielectric).

$$\nabla \cdot D = f, \quad D = \mathcal{E} \cdot U, \quad U = -\nabla \phi, \tag{2.2}$$

$$\nabla \cdot B = 0, \quad B = \mathcal{M} \cdot H, \quad H = -\nabla \psi. \tag{2.3}$$

where ϕ and ψ are the electrical and magnetic potentials, f is the density of free charges, U and H are the intensities, D and B are the inductions of the electrical and magnetic fields and $(\cdot)$ is the sign of matrix multiplication. The tensors $\mathcal{E}$ and $\mathcal{M}$ of the electrical and magnetic permeabilities have the form

$$\mathcal{E}(x) = (\epsilon_1 \chi_1(x) + \epsilon_2 \chi_2(x))\ I, \tag{2.4}$$

$$\mathcal{M}(x) = (\mu_1 \chi_1(x) + \mu_2 \chi_2(x))\ I. \tag{2.5}$$

where I is the (2×2) unit matrix, $\chi_i(x),\ i = 1, 2$ are the characteristic functions of the subdomains S_1, S_2

$$\chi_i(x) = \begin{cases} 1, & \text{if } x \in S_i \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

We will denote also $\mathcal{E}_i = \epsilon_i\ I,\ \mathcal{M}_i = \mu_i\ I, \quad i = 1, 2$

The described periodical structure acts in a smooth external field as a homogeneous anisotropic dielectric. Namely, the averaged upon the element of periodicity values $< D >$, $< U >$ and $< B >$, $< H >$, of acting fields D, U and B, H, are connected by the linear relationships

$$< D >= \mathcal{E} \cdot < U >, \tag{2.7}$$

$$< B >= \mathcal{M} \cdot < H > . \tag{2.8}$$

Here the symbol $< . >$ denotes the average value of (.)

$$< (.) >= \int_S (.) dS / \ volS, \tag{2.9}$$

$\mathcal{E}_0$ and $\mathcal{M}_0$ are symmetric, positive definite (2×2) tensors called the tensors of effective electrical and magnetic permeabilities respectively.

The tensors $\mathcal{E}_0$ and $\mathcal{M}_0$ are independent of external fields, they are determined only by the properties of the components and the geometrical structure of the composite.

Our goal is to describe the set of possible values of the tensor pair $\mathcal{E}_0$ and $\mathcal{M}_0$. Any pair of symmetric tensors in two dimensions is characterized by an angle of orientation and by 5 rotationally invariant parameters: two pairs of the eigenvalues λ_1^ϵ, λ_2^ϵ and λ_1^μ,λ_2^μ of the tensors $\mathcal{E}_0$ and $\mathcal{M}_0$ respectively and the angle ω_0 between the eigenvectors of these tensors. The set of these parameters can be represented as a point in the 5 dimensional space of the invariants.

Each microstructure corresponds to a point in this space, and the G_mU set corresponds to a body L in it. We give here the description of the G_mU set that is the set of the tensors pairs ($\mathcal{E}_0$, $\mathcal{M}_0$) corresponding to the all possible microstructures.

It is easy to find that L is a closed singly connected bounded body.

Here we will follow the method of construction of the G_mU set suggested in [**20,23**]. We modify this method in the next section, introduce new notations after which we formulate the results and prove them.

3 THE METHOD OF CONSTRUCTION OF THE G_m-CLOSURES

3.1 THE NEW FORM OF THE ESTIMATES OF THE G_mU SET

In this part we give a brief account of a variational method for obtaining the optimal estimates of the G_m-closure for the electrostatic problem. In this case the G_m-closure is determined as a domain in the two dimensional space of the eigenvalues of the effective electrical permeability tensor $\mathcal{E}_0$. This problem is solved [**23,25,44**] but here we will use this simplest example to introduce new notations and describe the ideas of the translation method. The new notations which we introduce here simplify the form of the known inequalities and will be used later for the electromagnetic problem.

The bounds of the G_m-closure for the electrostatic problem described by the equations (2.2) were obtained in [**23,25**] by the procedure of estimation of a functional

$$J_2 = \int_S [U_1 \cdot \mathcal{E}(x) \cdot U_1 + U_2 \cdot \mathcal{E}(x) \cdot U_2] dS = \int_S Z \cdot A(x) \cdot Z dS, \qquad U_i \in (2.2), \quad < U_i >= U_{i0}, \quad i = 1, 2, \tag{3.1}$$

where

$$A(x) = \chi_1(x) A_1 + \chi_2(x) A_2 \tag{3.2}$$

and

$$A_i = \begin{pmatrix} \mathcal{E}_i & 0 \\ 0 & \mathcal{E}_i \end{pmatrix}, \quad i = 1, 2 \tag{3.3}$$

are the block-diagonal (4×4) matrices, 0 is the (2×2) null matrix and Z is the four dimensional vector

$$Z = (U_1, U_2), \tag{3.4}$$

The functional J_2 is equal to the sum of the two values of energy stored in a composite medium which is exposed into two linearly independent external electric fields.

Remark 3.1: *Instead of the requirement $U_i \in (2.2)$ in the relationships (3.1) one could use the equivalent variational statement (Dirichlet principle [**27**])*

$$J_2 = \inf_{Z \in \mathcal{Z}} \int_S Z \cdot A(x) \cdot Z dS \quad , \tag{3.5}$$

where

$$\mathcal{Z} = \{ Z : \ Z = (U_1, U_2), \quad curl\ U_i = 0, \quad < U_i >= U_{i0}, \quad i = 1, 2. \} \tag{3.6}$$

It is the known result of the homogenization theory (see [**42**]) that the functional J_2 represents the energy stored in the equivalent homogeneous medium which is exposed into the homogeneous electrical fields U_{10}, U_{20}. This media is characterized by the tensor of the effective properties $\mathcal{E}_0$

$$J_2 = (U_{10} \cdot \mathcal{E}_0 \cdot U_{10} + U_{20} \cdot \mathcal{E}_0 \cdot U_{20})\ volS = Z_0 \cdot A_0 \cdot Z_0\ volS. \tag{3.7}$$

(here A_0 is the block matrices composed similar to (3.3))

On the other hand the functional J_2 could be estimated [**5**] by the expressions (the Reuss estimate):

$$J_2 \geq Z_0 \cdot [m_1 A_1^{-1} + m_2 A_2^{-1}]^{-1} \cdot Z_0 \; volS \tag{3.8}$$

if A_1 and A_2 are positive semidefinite matrices:

$$A_1 \geq 0, \quad A_2 \geq 0 \tag{3.9}$$

Comparing (3.7) and (3.8) and using arbitrariness of the vectors U_{10}, U_{20} we come to the Reuss inequality

$$A_0 \geq [m_1 A_1^{-1} + m_2 A_2^{-1}]^{-1}. \tag{3.10}$$

The other known estimate (the Voigt inequality)

$$A_0 \leq [m_1 A_1 + m_2 A_2] \tag{3.11}$$

can be obtained in the same way by considering the functional J_2^*

$$J_2^* = \int_S Z^* \cdot A^{-1}(x) \cdot Z^* dS, \tag{3.12}$$

$$Z^* = (D_1, D_2), \quad D_i \in (2.3), \quad < D_i >= D_{i0} \tag{3.13}$$

instead of the functional J_2. This functional depends on the inductions D_1, D_2 instead of the intensities U_1, U_2, it arises in the Thompson variational principle which is conjugate to the Dirichlet principle.

These estimates show that each pair of the eigenvalues λ_1^ϵ, λ_2^ϵ of the tensor $\mathcal{E}_0$ belong to the square $[\epsilon_a, \epsilon_h] \times [\epsilon_a, \epsilon_h]$ where

$$\begin{gathered} (.)_h = (m_1 (.)_1^{-1} + m_2 (.)_2^{-1})^{-1}, \\ (.)_a = m_1 (.)_1 + m_2 (.)_2. \end{gathered} \tag{3.14}$$

They form the minimal square domain in the space of eigenvalues of the tensors $\mathcal{E}_0$ which contains the G_m-closure for the electricity problem.

It is convenient to introduce a transformation of the tensor of the effective properties $\mathcal{E}_0$ – the $Y(\mathcal{E}_0)$-transformation:

$$Y(\mathcal{E}_0) = Y(\mathcal{E}_0, \epsilon_1, \epsilon_2) = -(\mathcal{E}_0 - \epsilon_h I)(\mathcal{E}_0 - \epsilon_a I)^{-1} \epsilon_1 \epsilon_2 \epsilon_h^{-1}. \tag{3.15}$$

It maps the square $[\epsilon_h, \epsilon_a] \times [\epsilon_h, \epsilon_a]$ into the first quadrant of the plane of eigenvalues of the tensor $Y(\mathcal{E}_0)$.

It is immediately clear that the tensor $Y(\mathcal{E}_0)$ is positive definite if and only if the Reuss-Voigt inequalities (3.10) and (3.11) are satisfied. These inequalities can be transformed into the form:

$$Y(\mathcal{E}_0) \geq 0. \tag{3.16}$$

Remark 3.2 *It is easy to verify that the tensor $Y(\mathcal{E}_0)$ possesses the following properties:*

Due to isotropy of the components ϵ_1 and ϵ_2, the tensor $Y(\mathcal{E}_0)$ has the same eigenvectors as $\mathcal{E}_0$; isotropy of $\mathcal{E}_0$ leads to isotropy of $Y(\mathcal{E}_0)$.

It also satisfies the equalities:

$$Y(\mathcal{E}_1) = -\mathcal{E}_1, \tag{3.17}$$

$$Y(\mathcal{E}_2) = -\mathcal{E}_2, \tag{3.18}$$

$$Y(\mathcal{E}_0^{-1}, \mathcal{E}_1^{-1}, \mathcal{E}_2^{-1}) = Y^{-1}(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2). \tag{3.19}$$

The last equality means that the substitution of the inverse values of the tensors $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_0$ in (3.15) is equivalent to the inversion of Y

Remark 3.3 *The transformation similar to (3.15) naturally arises in the variational method by Berryman [**4**] and Milton [**31**]); it was used also in the analytical method (see Bergman [**3**], Milton [**29,30,34**]). This transformation was studied in the scalar case by Milton and Golden [**32**]. In the recent papers [**35,37**] Milton introduces the tensor fractional-linear transformation Y defined by the formula*

$$\mathcal{E}_0^* = m_1\mathcal{E}_1 + m_2\mathcal{E}_2 - m_1 m_2(\mathcal{E}_1 - \mathcal{E}_2)(m_2\mathcal{E}_1 + m_1\mathcal{E}_2 + Y)^{-1}(\mathcal{E}_1 - \mathcal{E}_2) \tag{3.20}$$

When the matrices $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_0$ commute this transformation coincides with (3.15).

However, we also could take into account (see [**23,24**]) the potentiality of the fields U_i to make the estimates more restrictive :

$$curl\ U_i = 0, \quad i = 1, 2.... \tag{3.21}$$

The integral corollary of these equalities is the existence of the quasiaffine quadratic form

$$\Phi(U_1, U_2) = [U_1, U_2] \cdot T(t) \cdot [U_1, U_2] \tag{3.22}$$

possessing the property

$$< \Phi(U_1, U_2) >= \Phi(< U_1 >, < U_2 >) \tag{3.23}$$

for any two potential fields [**6**] . Here the matrix T has the form

$$T(t) = \begin{pmatrix} 0 & tR \\ tR^t & 0 \end{pmatrix}, \tag{3.24}$$

where R is a tensor of rotation by a right angle.

Taking into account the equality (3.22) as a restriction on the set of the fields U_i we may use the procedure of the translation method [**24,25,36,37**] to estimate the functional J_2. We estimate the value

$$J_2 + < \Phi(U_1, U_2) > \tag{3.25}$$

by an expression similar to (3.8) keeping in mind the equalities (3.4), (3.7), (3.22). The corresponding estimate has the form

$$(A_0 - T) \geq [m_1(A_1 - T)^{-1} + m_2(A_2 - T)^{-1}]^{-1} \tag{3.26}$$

subject to the restriction

$$t \in \mathcal{T} = \{t : A_i - T(t) \geq 0, \quad i = 1, 2\} \tag{3.27}$$

which is analogous to (3.9).

The estimate (3.26) depends on the choice of the parameter t. For this problem it is possible to show [**23,25**] that this parameter t is to be chosen so as to attain

$$\min_t \{rank(A_1 - T(t)) + rank(A_2 - T(t))\}. \tag{3.28}$$

The appropriate value of t is equal to ϵ_1 for the electrostatic problem [**23**] (it is assumed here that $\epsilon_1 \leq \epsilon_2$). After some calculations the estimate (3.26) gets the following form [**25**]

$$TrF(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2) \geq 1 \tag{3.29}$$

where the tensor $F(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$ is defined as

$$F(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2) = (m_2 \mathcal{E}_1 (\mathcal{E}_0 - \mathcal{E}_1)^{-1} - \mathcal{E}_1 (\mathcal{E}_2 - \mathcal{E}_1)^{-1})/m_1. \tag{3.30}$$

The other estimate of the $G_m U$ set has been obtained [**23,25**] in a similar way by using the conjugate J_2^* to the functional J_2 and has the form

$$Tr F(\mathcal{E}_0, \mathcal{E}_2, \mathcal{E}_1) \leq 1 \ , \tag{3.31}$$

where the tensor $F(\mathcal{E}_0, \mathcal{E}_2, \mathcal{E}_1)$ is defined similar to (3.30).

Now we rewrite the results in the new notation. The tensor Y introduced above can be easily expressed through the tensor $F(\mathcal{E}_0, \mathcal{E}_2, \mathcal{E}_1)$

$$Y(\mathcal{E}_0) = \mathcal{E}_1 \cdot [F^{-1}(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2) - I]. \tag{3.32}$$

The estimates (3.29), (3.31) can be transformed into the following form

$$det \ Y(\mathcal{E}_0)/\epsilon_1 \epsilon_2 \in [(\epsilon_1/\epsilon_2)^2, (\epsilon_2/\epsilon_1)^2]. \tag{3.33}$$

Notice, that the representations of the G_m-closure (3.33) are independent of the parameters m_1, m_2, and are determined only by the ratio $\epsilon_1 \epsilon_2^{-1}$.

We see that the value of the tensor Y of a mixture determines the proximity of the effective tensor to the boundary of the $G_m U$ set.

Remark 3.4 *It is interesting to observe also that the conductivity of a random media assembled from the isotropic materials $\mathcal{E}_1, \mathcal{E}_2$ mixed in the volume fractions $m_1 = m_2 = 1/2$ is equal to $\sqrt{\epsilon_1 \epsilon_2}$ [**7**], i.e. the tensor $Y(\mathcal{E}_0)/\sqrt{\epsilon_1 \epsilon_2}$ of such mixture is the identity matrix.*

The tensor of a structure Y can be also introduced for a more complex system of equations than the electrostatic. The algebraic inequality (3.26) can be transformed for any positive definite matrices $A_i, i = 0, 1, 2$ and any matrix T (see Appendix for details) into the form

$$Y(A_0) \geq \ - \ (A_1 - A_2) \cdot T \cdot (A_1 - A_2)^{-1}. \tag{3.34}$$

If the matrices $Y(A_0)$ and $(A_1 - A_2)$ commute the estimate (3.34) becomes:

$$Y(A_0) + T \geq 0 \tag{3.35}$$

for any matrix T of a quasiaffine quadratic form satisfying the inequalities

$$A_i - T \geq 0, \quad or \quad T \leq A_i, \quad i = 1, 2. \tag{3.36}$$

The last inequalities can be also transformed by using the tensor Y into the form

$$Y(A_i) + T \leq 0, \quad or \quad T \leq -Y(A_i), \quad i = 1, 2, \tag{3.37}$$

because one can check easily that $Y(A_i) = -A_i, \quad i = 1, 2.$

Remark 3.5 *As was shown by Milton [**37**], the transformation $Y(A_0)$ defined by (3.20) instead of (3.15) allows us to avoid the conditions of commutability of the matrices $A_i, \quad i = 0, 1, 2$. Also, in [**37**], the simple variational proof of the inequality (3.35) was obtained and the variational interpretation of the tensor $Y(A_0)$ was given.*

The inequality (3.35) shows that all eigenvalues of the matrix $(Y(A_0)+T)$ are nonnegative. Therefore the determinant of this matrix vanishes at the boundary of the admissible set

$$det(Y(A_0) + T) \geq 0. \tag{3.38}$$

We can see that in general case the estimates (3.38) also have a simple and convenient form in the proposed Y notation. They don't depend on the volume fractions m_1 and m_2. We will use the inequalities (3.37), (3.38) to obtain the G_mU set estimates for the problem under study.

3.2 GEOMETRY OF THE OPTIMAL STRUCTURES

The simplest types of the composites are the simple layered microstructures. Their effective properties are easy to determine: they correspond to the corner points of the Reuss-Voigt square. The eigenvalue ϵ_a corresponds to the conductivity along the layers and the eigenvalue ϵ_h corresponds to the conductivity across them. The eigenvalues of the effective properties tensor of the simple layered structure are uniquely determined by the volume fractions of initially given materials.

More complex types of structures for which the effective properties can also be expressed in an analytical form, have recently been constructed. One type is the matrix layered composites (MLC) (see Fig.1). These structures were introduced in [**23**] and the convenient analytical form of their effective properties was given in [**25,44**]. The tensor $\mathcal{E}_m$ of their effective properties satisfies the equation

$$\begin{gathered} \mathcal{E}_m = \mathcal{E}_1 + m_2((\mathcal{E}_2 - \mathcal{E}_1)^{-1} + m_1\mathcal{E}_1^{-1} \cdot N)^{-1}, \\ N = \alpha n_{\alpha 1} \otimes n_{\alpha 1} + (1-\alpha) n_{\alpha 2} \otimes n_{\alpha 2}, \\ \alpha \geq 0, \quad (1-\alpha) \geq 0. \end{gathered} \tag{3.39}$$

Here $n_{\alpha 1}$ and $n_{\alpha 2}$ are the normal vectors , α and $(1-\alpha)$ are the relative concentrations of the external material $\mathcal{E}_1$ in the layers of the first and second ranks (see Fig.1). These last parameters define the degree of anisotropy of a homogenized composite. The symbol $\otimes$ denotes the tensor product.

The system of coated ellipses has the same properties as MLC [**44**].

These structures are anisotropic in general and cover all bounds (3.33) of the G_mU set for the electrostatic problem.

However, they possess (in two dimensional problems) only one free parameter - the relative width of the first rank layers. Therefore they cannot be sufficient to describe the G_mU set boundary for the electromagnetic problem (2.2)- (2.3) where the boundary of the G_mU-set is represented by a four dimensional surface.

Another type of microstructures was introduced by Milton [**29**] and is called multicoated spheres. (Similar structures were also studied earlier by Schulgasser [**43**].) Such isotropic mixtures belong to the G_mU set boundary for the problem of complex conductivity (for details see [**29**]). This type of the structures allows us to change the order of the materials in the structure but preserve isotropy.

Let us combine the ideas of both MLC and coated spheres. In particular, we study a type of microstructures with enough control parameters (similar to the one introduced by Milton [**34**]). The following iterative procedure is considered. At the first step we assemble a MLC from the initial materials taken in the volume fractions c_{11} and c_{12} respectively, the second material $\mathcal{E}_2$ is wrapped around the first one (see Fig 2.). The effective properties tensor $\mathcal{E}^*$ of such a structure is given by the relationship (3.39), which has the form

$$\mathcal{E}^* = \mathcal{E}_2 + c_{12}[(\mathcal{E}_1 - \mathcal{E}_2)^{-1} + c_{11}\mathcal{E}_2^{-1}(\alpha n_{\alpha 1} \otimes n_{\alpha 1} + (1-\alpha) n_{\alpha 2} \otimes n_{\alpha 2})]^{-1}. \quad (3.40)$$

Here α , $(1-\alpha)$ are the relative concentrations of the enveloping material $\mathcal{E}_2$ in the layers of the first and the second rank. We also assume here that the normals $n_{\alpha 1}$ and $n_{\alpha 2}$ are orthogonal. At the second step we assemble the new MLC in such way that the first material is wrapped around the homogenized composite already obtained (see Fig. 2). The volume fractions of the materials in this step are c_{21} (inclusions) and c_{22} (envelope). The effective properties tensor $\mathcal{E}_0$ of such structure is also given by relationship (3.39) which has the form

$$\mathcal{E}_0 = \mathcal{E}_1 + c_{21}[(\mathcal{E}^* - \mathcal{E}_1)^{-1} + c_{22}\mathcal{E}_1^{-1}(\beta n_{\beta 1} \otimes n_{\beta 1} + (1-\beta) n_{\beta 2} \otimes n_{\beta 2})]^{-1} \quad (3.41)$$

where β is the parameter defined analogous to α; normals $n_{\beta 1}$ and $n_{\beta 2}$ are orthogonal. The total amounts of initial materials are fixed (see Fig. 2):

$$m_1 = c_{11}c_{22} + c_{21}, \quad m_2 = c_{12}c_{22}. \tag{3.42}$$

We will call the described structures "the Multi Coated Matrices of type 1–2–1" or the MCM(1,2,1).The sequence of indices shows the order of enveloping of materials in the microstructure, see Fig.2. The already known matrix layered composites are denoted by MCM(1,2) (inclusions of the first material) or MCM(2,1) (inclusions of the second material)

Using (3.40) - (3.42), we get the representation for the MCM(1,2,1):

$$F(\mathcal{E}_0, \mathcal{E}_1\mathcal{E}_2) = (1-\gamma)(\beta n_{\beta 1}\otimes n_{\beta 1} + (1-\beta)n_{\beta 2}\otimes n_{\beta 2}) + \gamma\mathcal{E}_1[\alpha n_{\alpha 1}\otimes n_{\alpha 1} + (1-\alpha)n_{\alpha 2}\otimes n_{\alpha 2}][\mathcal{E}_2 + (\mathcal{E}_1 - \mathcal{E}_2)(\alpha n_{\alpha 1}\otimes n_{\alpha 1} + (1-\alpha)n_{\alpha 2}\otimes n_{\alpha 2})]^{-1} \tag{3.43}$$

where the parameter γ is equal to

$$\gamma = c_{11}/m_1, \quad (1-\gamma) = c_{12}c_{21}/m_1, \quad \gamma \in [0,1]) \tag{3.44}$$

The effective properties tensor of the described structure depends on four invariant parameters: the scalar product $n_{\alpha 1}\cdot n_{\beta 1}$ (remember that $n_{\alpha 1}\cdot n_{\alpha 2} = 0$, $n_{\beta 1}\cdot n_{\beta 2} = 0$) and on three parameters α, β, γ . The parameters α, β determine the degree of anisotropy of the first and the second envelopes respectively and the parameter γ controls the distribution of the first material between inclusions and the external envelope. The vectors $n_{\alpha 1}$ and $n_{\beta 1}$ determine the orientation of the principal axis of the effective properties tensor and the scalar value $n_{\alpha 1}\cdot n_{\beta 1} = \cos\omega$ affects its invariant characteristic.

Let us bound the set of admissible values for the parameters $\alpha, \beta, \gamma, \omega$:

$$\begin{array}{c} \alpha \in [0,1/2], \quad \beta \in [0,1/2], \\ \gamma \in [0,1], \quad \omega \in [0,\pi]. \end{array} \tag{3.45}$$

These restrictions provide one to one correspondence between the set of the structures and the set of the parameters. Indeed, the MCM corresponding to $\alpha \geq 1/2$ is equivalent to the one determined by the parameters

$$\begin{array}{c} \alpha' = (1-\alpha) \in [0,1/2], \\ n'_{\alpha 1} = n_{\alpha 2}, \quad n'_{\alpha 2} = n_{\alpha 1} \end{array} \tag{3.46}$$

and so on.

The introduced structures MCM can degenerate into the more simple types described above. Indeed, if $\alpha = \beta = 1/2$ the structure becomes isotropic and its effective properties coincide with the properties of the multicoated spheres introduced in [**29**].

The tensor $F(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$ depends linearly on the parameter γ: if $\gamma = 0$ or $\gamma = 1$ then the structure degenerates into the matrix layered composites of two different types the MCM(1,2) and the MCM(2,1) with the effective properties tensors $\mathcal{E}^*$ and $\mathcal{E}^{**}$ respectively.

Using the tensor $Y(\mathcal{E}_0)$ one can transform the relationship (3.43) into the form

$$(Y(\mathcal{E}_0)+\mathcal{E}_1)^{-1} = \gamma_1(Y(\mathcal{E}^*)+\mathcal{E}_1)^{-1} + \gamma_2\epsilon_1^{-1} n_{\beta 1}\otimes n_{\beta 1} + \gamma_3\epsilon_1^{-1} n_{\beta 2}\otimes n_{\beta 2}, \quad (3.47)$$

where

$$\begin{array}{c} \gamma_1 = \gamma, \quad \gamma_2 = \beta(1-\gamma), \quad \gamma_3 = (1-\beta)(1-\gamma) \\ \gamma_i \geq 0, \quad i = 1,2,3, \quad \gamma_1+\gamma_2+\gamma_3 = 1 \end{array}, \quad (3.48)$$

or into the form

$$(Y(\mathcal{E}_0) + \mathcal{E}_1)^{-1} = \gamma(Y(\mathcal{E}^*) + \mathcal{E}_1)^{-1} + (1-\gamma)(Y(\mathcal{E}^{**}) + \mathcal{E}_1)^{-1}. \quad (3.49)$$

This shows the tensor $(Y(\mathcal{E}_0) + \mathcal{E}_1)^{-1}$ of the MCM(1,2,1) to be a convex combination of the tensors of the MCM(1,2) and the MCM(2,1). The singular role of the material $\mathcal{E}_1$ corresponds to wrapping it around the MCM(1,2) on the last step.

Remark 3.6 *It is possible to construct a more complex MCM by enveloping the structure MCM(1,2,1) with the material $\mathcal{E}_2$. Such structure will be denoted by MCM(1,2,1,2); its properties tensor $\mathcal{E}_0$ can be found ¿From the relationship*

$$Y(\mathcal{E}_0)+\mathcal{E}_2)^{-1} = \delta_1(Y(\mathcal{E}^{***})+\mathcal{E}_2)^{-1} + \delta_2\epsilon_2^{-1} n_{\delta 1}\otimes n_{\delta 1} + \delta_3\epsilon_2^{-1} n_{\delta 2}\otimes n_{\delta 2} \quad (3.50)$$

where $\mathcal{E}^{***}$ is the MCM(1,2,1) properties tensor (this composite plays here the role of inclusions); the new parameters $\delta_1, \delta_2, \delta_3, n_{\delta 1}, n_{\delta 2}$ are similar to the $\gamma_1, \gamma_2, \gamma_3, n_{\beta 1}, n_{\beta 2}$.

3.3 THE CONSTRUCTION OF THE FUNCTIONALS

The known construction of the geometrically independent estimates of the $G_m U$ set is based on the studying of a minimization problem for some energy functional. The electrical and magnetic energy can be expressed as a quadratic form of fields of the intensities U and H as well as a quadratic form of the inductions D and B. In this section we discuss some heuristic arguments for choosing an appropriate functional.

Let us begin with the simplest problem of such kind.

To find the Reuss estimates (low bound for the electrical permeability) one can consider the minimization problem for the energy J_1 stored in the composite upon its structure (see [**41,5**]). The composite is supposed to be taken in the external electric field U_{10}

$$J_1 = \int_S U_1 \cdot \mathcal{E}(x) \cdot U_1 dS, \quad U_1 \in (2.2), \quad < U_1 >= U_{10}. \tag{3.51}$$

It is known [**5,41**] that the optimal structure storing the minimal energy is a layered composite; the layers are oriented perpendicular to the external field U_{10}.

To construct the upper bound (Voigt inequality) we should minimize the energy J_1^* stored in the composite taken in a fixed external induction field (see [**46,5**]) upon the microstructures of the mixture

$$J_1^* = \int_S D_1 \cdot \mathcal{E}^{-1}(x) \cdot D_1 dS, \quad D_i \in (2.3), \quad < D_i >= D_{i0}, \quad i = 1,2 \tag{3.52}$$

This functional is conjugate to the functional J_1. The solution of this problem also corresponds to the layered structure, but the layers are parallel to the applied field D_0.

The accurate estimates (3.33) of the G_m-closure for the electrical properties of a mixture were obtained [**23**] by minimization of the functional J_2 (see the equation (3.1)). This functional is equal to the sum of values of the energy stored in a composite medium exposed into two linearly independent external fields U_{10}, and U_{20}. For this problem the class of optimal structures must include both the layers orthogonal to the vector U_{10} and to the vector U_{20}. They are optimal in the cases when the field amplitude U_{20} or U_{10} vanishes. Indeed, the class of the optimal structures which represent the solution of the corresponding minimization problem is here the MLC (the MCM(2,1) when ϵ_1 is less then ϵ_2) which degenerates (see Fig.1) to both types of layers (see [**23**]).

Thus, the functional J_2 has been constructed as the sum of two simplest functionals of the J_1 type ; each of them corresponds to corner point of the G_mU set. The solution of the minimization problem for the functional (3.7) connects these points and forms one component of the G_mU boundary.

The other boundary component was founded by minimization of the functional J_2^* (see (3.12)) conjugated to the J_2; the optimal structures MCM(1,2) arising here also degenerate into two layered structures.

The construction of a functional for estimating the G_mU set for the electromagnetic problem (2.2)- (2.3) studied here is based on the same principles.

We mention first that the MLC structures of both types (i.e MCM(1,2) and MCM(2.1)) belong to the boundary of the five dimensional G_mU body because they possess the extremal electrical and magnetic properties. Therefore structures optimal for the electromagnetic problem must degenerate into these composites.

Consider the functional J_3

$$\begin{gathered} J_3 = \int_S [U_1 \cdot \mathcal{E}(x) \cdot U_1 + U_2 \cdot \mathcal{E}(x) \cdot U_2 + H_1 \cdot \mathcal{M}(x) \cdot H_1 + H_2 \cdot \mathcal{M}(x) \cdot H_2] dS \\ U_i \in (2.2), \quad < U_i >= U_{i0}, \quad i = 1, 2, \quad H_i \in (2.3), \quad < H_i >= H_{i0}, \quad i = 1, 2 \end{gathered} \tag{3.53}$$

which is the sum of the functionals of the J_2 type for the electrical and magnetic energies.

We will distinguish the case when the material constants ϵ_1, ϵ_2 and μ_1, μ_2 satisfy the relationship

$$(\epsilon_1 - \epsilon_2)(\mu_1 - \mu_2) \leq 0 \tag{3.54}$$

(case A) and the case when these constants satisfy the relationship

$$(\epsilon_1 - \epsilon_2)(\mu_1 - \mu_2) \geq 0 \tag{3.55}$$

(case B).

Let us study the case A. If the amplitude of the magnetic field vanishes ($H_{10} = H_{20} = 0$) the solution of the variational problem of estimating the functional (3.53) is attained on the structures MCM(2,1) (if ϵ_1 is less then ϵ_2). If the amplitude of the electrical field vanishes ($U_{10} = U_{20} = 0$) then it corresponds to the MCM(1,2) (because in the case A if ϵ_1 is less then ϵ_2, then μ_2 is less then μ_1). Indeed, in these both cases the problem of estimating the functional (3.53) degenerates into the solved problem for the functional of the (3.1) type. In general, the solution of the mentioned problem for the functional (3.53) corresponds to a four dimensional surface in the five dimensional space of invariants of the effective tensors. This surface connects the lines in 5-dimension space corresponding to the properties of the structures MCM(1,2) and the MCM(2,1). We will show later that the optimal structures for this problem are MCM(1,2,1) which obviously include the structures MCM(1,2) and MCM(2,1).

In case B, however, in both limits $U_{10} = U_{20} = 0$ and $H_{10} = H_{20} = 0$ the solution of the minimization problem (3.53) gives us only matrix composites MCM(2,1) (if ϵ_1 is less then ϵ_2, see [**23,24,41**]) because if the ϵ_1 is less then ϵ_2 then also μ_1 is less then μ_2 . We will see that also in the general case the solution of the minimization problem for this functional corresponds to the MCM(2,1) i.e. to the edge of the G_mU set.

Remark 3.7 *On the example of this functional we could see the differences between the so called well-ordered and badly-ordered cases (see Kohn, Milton [**15**]). The well-ordered case (case B in our example) is the case when one of the materials is "weaker" that the other one i.e. the difference between the matrices of properties is positive definite. In the badly-ordered case (case A in our example) there is no "weak" and "strong" material and the difference between the matrices of properties is an indefinite matrix.*

*The well-ordered case was examined in many papers. As was shown by Avellaneda in the paper [**1**] and by Milton in the paper [**36**] the solution of minimization problems for the well-ordered pairs of materials corresponds to the matrix layered composite materials. The badly-ordered case is in some sense richer. We could expect that more complex structures will arise in the minimization problem for such materials. For example, as we will show for the problem under study, it is necessary to introduce the multicoated matrix of the second rank.*

Consider now the case B.

We will show that the appropriate functional for obtaining the four-dimensional component of the G_mU-set boundary is J_4:

$$J_4 = \int_S [U_1 \cdot \mathcal{E}(x) \cdot U_1 + U_2 \cdot \mathcal{E}(x) \cdot U_2 + B_1 \cdot \mathcal{M}^{-1}(x) \cdot B_1 + B_2 \cdot \mathcal{M}^{-1}(x) \cdot B_2] dS$$
$$U_i \in (2.2), \quad < U_i >= U_{i0}, \quad i = 1,2, \quad B_i \in (2.3), \quad < B_i >= B_{i0}, \quad i = 1,2 \tag{3.56}$$

Indeed the solution of the minimization problem for this functional provides the appearance of the matrix composites of either type MCM(2,1) or MCM(1,2), in the particular cases when $U_1 = U_2 = 0$ or $B_1 = B_2 = 0$, respectively. To show this one can repeat the previous arguments keeping in mind that in the case B $\quad (\epsilon_1 - \epsilon_2)(1/\mu_1 - 1/1\mu_2) \geq 0$.

In the case A, the functional J_4 could give us the edge of the G_mU-set boundary, corresponding to the matrix composites MCM(1,2) (because for the case A , if ϵ_1 is less then ϵ_2 , then $1/\mu_1$ is less then $1/\mu_2$) and we are in the conditions of the well-ordered case.

By minimization of the functionals (3.53), (3.56) we determine one component of the G_mU set boundary. For obtaining the other component we consider the dual variational problems of minimization of the conjugate to the J_3 or J_4 functionals

$$\begin{gathered} J_3^* = \int_S [D_1 \cdot \mathcal{E}^{-1}(x) \cdot D_1 + D_2 \cdot \mathcal{E}^{-1}(x) \cdot D_2 \\ + B_1 \cdot \mathcal{M}^{-1}(x) \cdot B_1 + B_2 \cdot \mathcal{M}^{-1}(x) \cdot B_2] dS \\ \\ D_i \in (2.2), \quad < D_i >= D_{i0}, \quad i = 1,2, \\ B_i \in (2.3), \quad < B_i >= B_{i0}, \quad i = 1,2 \end{gathered} \tag{3.57}$$

in the case A and

$$\begin{gathered} J_4^* = \int_S [D_1 \cdot \mathcal{E}^{-1}(x) \cdot D_1 + D_2 \cdot \mathcal{E}^{-1}(x) \cdot D_2 \\ + H_1 \cdot \mathcal{M}(x) \cdot H_1 + H_2 \cdot \mathcal{M}(x) \cdot H_2] dS \\ \\ D_i \in (2.2), \quad < D_i >= D_{i0}, \quad i = 1,2, \\ H_i \in (2.3), \quad < H_i >= H_{i0}, \quad i = 1,2 \end{gathered} \tag{3.58}$$

in the case B.

We will also show that it is sufficient to study the minimization problems for the functionals (3.53), (3.56), (3.57), (3.58) to describe all the components of the boundaries of the G_mU set.

4 THE G_m-CLOSURE FOR THE OPERATOR OF THE ELECTROMAGNETOSTATIC PROBLEM.

Here we describe the G_mU set for the electromagnetic problem by using the methods and the notations introduced in the previous chapters. Without loss of generality we assume that the materials are numbered in such way that the inequality

$$\epsilon_1\mu_1 \le \epsilon_2\mu_2 \tag{4.1}$$

holds in the case (3.54) and the inequality

$$\epsilon_1/\mu_1 \le \epsilon_2/\mu_2 \tag{4.2}$$

holds in the case (3.55)

4.1 THE MAIN THEOREM

(i) The pair of the effective tensors $\mathcal{E}_0, \mathcal{M}_0$ belongs to the G_mU set for the electromagnetic problem (2.2) - (2.3) if and only if

1. They satisfy the inequalities (3.33) and the analogous inequalities for the magnetic properties tensors $\mathcal{M}_0$:

$$\begin{gathered} det[Y(\mathcal{E}_0)] \in [\epsilon_1^2,\ \epsilon_2^2], \\ det[Y(\mathcal{M}_0)] \in [\mu_1^2,\ \mu_2^2]. \end{gathered} \tag{4.3}$$

2. The following inequalities are valid:

2a. when the material constants are connected by the relationships (3.54) (case A) then

$$\frac{det(Y(\mathcal{E}_0)/\epsilon_i - Y(\mathcal{M}_0)/\mu_i)}{(1 - det(Y(\mathcal{E}_0)/\epsilon_i))(1 - det(Y(\mathcal{M}_0)/\mu_i)} \le c_1, \quad i = 1, 2, \tag{4.4}$$

$$\mathcal{E}_1 \cdot \mathcal{M}_1 \le Y(\mathcal{E}_0) \cdot R \cdot Y(\mathcal{M}_0) \cdot R^t \le \mathcal{E}_2 \cdot \mathcal{M}_2. \tag{4.5}$$

Here the constant c_1 is equal to

$$c_1 = \frac{(\epsilon_1\mu_2 - \epsilon_2\mu_1)^2}{(\epsilon_1^2 - \epsilon_2^2)(\mu_1^2 - \mu_2^2)}. \tag{4.6}$$

(Remember that

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.7}$$

is the tensor of the rotation on the right angle)

2b. when the material constants are connected by the relationship (3.55) (case B) then

$$\frac{det(Y(\mathcal{E}_0)/\epsilon_i - R \cdot Y^{-1}(\mathcal{M}_0) \cdot R^t \mu_i)}{(1 - det(Y(\mathcal{E}_0)/\epsilon_i))(1 - det^{-1}(Y(\mathcal{M}_0)/\mu_i))} \leq c_2, \qquad i = 1, 2 \tag{4.8}$$

$$\mathcal{E}_1 \cdot \mathcal{M}_1^{-1} \leq Y(\mathcal{E}_0) \cdot Y(\mathcal{M}_0)^{-1} \leq \mathcal{E}_2 \cdot \mathcal{M}_2^{-1}, \tag{4.9}$$

where the constant c_2 is equal to

$$c_2 = \frac{(\epsilon_1\mu_1 - \epsilon_2\mu_2)^2}{(\epsilon_1^2 - \epsilon_2^2)(\mu_2^2 - \mu_1^2)}. \tag{4.10}$$

(ii) The G_mU set boundary corresponds one to one (in each of cases (3.54) and (3.55)) to a set of pairs of the effective properties tensors ($\mathcal{E}_0$, $\mathcal{M}_0$) of the MCM(1,2,1) or MCM(2,1,2) structures and all the internal points of the G_mU set correspond to pairs of the effective tensors of their mixtures.

Two simple corollaries follow this theorem:

Corollary 1. If the constants of the initial materials are bounded by the relationship

$$\epsilon_1\mu_1 = \epsilon_2\mu_2 = v^2 \tag{4.11}$$

then the effective tensors $\mathcal{E}_0$ and $\mathcal{M}_0$ satisfy the equality

$$\mathcal{E}_0 \cdot R \cdot \mathcal{M}_0 \cdot R^t = v^2 I \tag{4.12}$$

and the inequalities (3.33).

The equality (4.12) is an extension to anisotropic case of the known relationship obtained by Dykhne [**7**] and Keller [**19**]. It was also proved by Mendelson [**39**].

Corollary. 2. If the constants of the initial materials are bounded by rthe elationship

$$\epsilon_1/\mu_1 = \epsilon_2/\mu_2 = k \tag{4.13}$$

then the effective tensors $\mathcal{E}_0$ and $\mathcal{M}_0$ satisfy the equality

$$\mathcal{E}_0 = k\mathcal{M}_0 \tag{4.14}$$

and the inequalities (3.33).

This statement is physically clear because the relationships (4.13) mean that the electrical and magnetic properties of components differ only by a scale coefficient.

Remark 4.1 *It is easy to check that the inequalities (4.4) for i=1, (4.8) for i=1 (and (4.4) for i=2, (4.8) for i=2) define the same surface in the space of invariants of the tensors* $(\mathcal{E}_0, \mathcal{M}_0)$ *pairs and correspond to the same structures MCM(1,2,1) (and the MCM(2,1,2))*

4.2 THE PROOF OF THE OPTIMAL ESTIMATES.

At first let us consider the case A when the constants of the initial materials satisfy the relationship (3.54)

Let us introduce the notation: Z is the eight-dimensional vector combined of the coordinates of two dimensional vectors (U_1, H_1, U_2, H_2) in the following way:

$$\begin{array}{ll} Z_1 = U_{1x_1}, & Z_2 = U_{1x_2}, \\ Z_3 = H_{1x_1}, & Z_4 = H_{1x_2}, \\ Z_5 = U_{2x_1}, & Z_6 = U_{2x_2}, \\ Z_7 = H_{2x_1}, & Z_8 = H_{2x_2}. \end{array} \tag{4.15}$$

Here $U_1, U_2 (H_1, H_2)$ are the pairs of the linearly independent vectors of the electrical (magnetic) field density, and A is the block matrix

$$A(\mathcal{E}(x), \mathcal{M}(x)) = \begin{pmatrix} \mathcal{E}(x) & 0 & 0 & 0 \\ 0 & \mathcal{M}(x) & 0 & 0 \\ 0 & 0 & \mathcal{E}(x) & 0 \\ 0 & 0 & 0 & \mathcal{M}(x) \end{pmatrix} \tag{4.16}$$

combined of (2×2) blocks; the tensors $\mathcal{E}(x), \mathcal{M}(x)$ are defined by the equations (2.4), (2.5).

Let Z_0 be the average value of the vector Z:

$$Z_0 = (U_{10}, H_{10}, U_{20}, H_{20}) =< (U_1, H_1, U_2, H_2) >=< Z > . \tag{4.17}$$

According to the well known variational principle (see for example [**24**]) the functional J_3 (see (3.53)) is equal to

$$J_3 = \inf_{Z \in \Sigma(Z_0)} \int_S Z \cdot A \cdot Z dS \tag{4.18}$$

where the set $\Sigma(Z_0)$ of the admissible fields of the vector Z is defined as

$$\Sigma(Z_0) = \{Z :< Z >= Z_0, \begin{array}{c} curl\ (Z_1, Z_2) = curl\ U_1 = \nabla \times U_1 = 0 \\ curl\ (Z_3, Z_4) = curl\ H_1 = \nabla \times H_1 = 0 \\ curl\ (Z_5, Z_6) = curl\ U_2 = \nabla \times U_2 = 0 \\ curl\ (Z_7, Z_8) = curl\ H_2 = \nabla \times H_2 = 0 \end{array} \}. \tag{4.19}$$

Remember that the functional J_3 being the sum of energies stored by composite in the uniform external field Z_0 is equal to the energy stored by equivalent homogeneous media possessing the effective properties tensors $\mathcal{E}_0, \mathcal{M}_0$ [**5,42**]. We construct the lower estimate of the infimum of the functional J_3 over all electromagnetic fields $Z(x)$ satisfying the differential restrictions (4.19) and among the all possible microstructures of the composites (i.e. among the all possible functions χ_1, χ_2). For this propose we use the construction described in the previous chapter.

First let us find the matrix T of the vector's Z quasiaffine quadratic form. We use the result of the compensated compactness theory (see Dacorogna [**6**], Murat, Tartar [**38**], Tartar [**44,45**], Strang, Kohn [**18**]):

The bilinear form $\phi(a, b) = a \cdot R \cdot b$ of two-dimensional vectors a and b such that $curl\ a = 0, curl\ b = 0$ is a quasiaffine function :

$$< \phi(a, b) >=< a \cdot R \cdot b >=< a > \cdot R \cdot < b >= \phi(< a > \cdot < b >). \tag{4.20}$$

Consider the quadratic form

$$\begin{array}{c} \Phi(Z, t_i) = Z \cdot T(t_i) \cdot Z = \\ = t_1(U_1 \cdot R \cdot U_2 + U_2 \cdot R^t \cdot U_1) + t_2(H_1 \cdot R \cdot H_2 + H_2 \cdot R^t \cdot H_1) + \\ +t_3(U_1 \cdot R \cdot H_2 + H_2 \cdot R^t \cdot U_1 + H_1 \cdot R \cdot U_2 + U_2 \cdot R^t \cdot U_1) \end{array} \tag{4.21}$$

where the matrix $T(t_i)$ is

$$T(t_i) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & t_1 & 0 & t_3 \\ 0 & 0 & 0 & 0 & -t_1 & 0 & -t_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & t_3 & 0 & t_2 \\ 0 & 0 & 0 & 0 & -t_3 & 0 & -t_2 & 0 \\ 0 & -t_1 & 0 & -t_3 & 0 & 0 & 0 & 0 \\ t_1 & 0 & t_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -t_3 & 0 & -t_2 & 0 & 0 & 0 & 0 \\ t_3 & 0 & t_2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.22}$$

or in the block form

$$T(t_i) = \begin{pmatrix} 0 & 0 & t_1 R & t_3 R \\ 0 & 0 & t_3 R & t_2 R \\ t_1 R^t & t_3 R^t & 0 & 0 \\ t_3 R^t & t_2 R^t & 0 & 0 \end{pmatrix}. \tag{4.23}$$

It is clear from (4.19), (4.20) that $\Phi(Z, t_i)$ is a quasiaffine function of the vector Z.

Now we are able to use the results of [**21,22**], see (3.35), (3.37).

$$Y(A_0) + T(t_i) \geq 0 \tag{4.24}$$

if

$$t_i \in \mathcal{T} = \{t_i : A_1 - T(t_i) \geq 0, \quad A_2 - T(t_i) \geq 0\} \tag{4.25}$$

(here we use the notations $A_i = A(\mathcal{E}_i, \mathcal{M}_i), \quad i = 1, 2, \quad A_0 = A(\mathcal{E}_0, \mathcal{M}_0)$.) Remember that restrictions (4.25) can be rewritten in the equivalent form

$$Y(A_1) + T(t_i) \leq 0, \quad Y(A_2) + T(t_i) \leq 0, \tag{4.26}$$

where

$$\begin{aligned} Y(A_0) &= (A_0 - A_h) \cdot (A_0 - A_a)^{-1} \cdot A_1 \cdot A_2 \cdot (A_h)^{-1}, \\ A_h &= (m_1 A_1^{-1} + m_2 A_2^{-1})^{-1}, \\ A_a &= m_1 A_1 + m_2 A_2. \end{aligned} \quad . \tag{4.27}$$

The estimates (4.4) for i=1 of the $G_m U$ set are obtained ¿From the corollary of the inequalities (4.24)

$$det\ (Y(A_0) + T(t_i)) \geq 0, \quad t_i \in \mathcal{T} \tag{4.28}$$

if the parameters t_i are chosen in the optimal way. Similar to [**24,25**] we choose these parameters in such way that the sum of the ranks of the matrices $(A_1 - T(t_i))$ and $(A_2 - T(t_i))$ are minimized:

$$t_i = \arg \min_{t_i} \{rank(A_1 - T(t_i)) + rank(A_2 - T(t_i))\} \tag{4.29}$$

(compare with the condition (3.28)).

It can be verified that these parameters are equal to:

$$\begin{aligned} t_1 &= -\theta\epsilon_1, \\ t_2 &= \theta\mu_1, \\ t_3 &= -\sqrt{1-\theta^2}\sqrt{\epsilon_1\mu_1}, \end{aligned} \tag{4.30}$$

where

$$\theta = \frac{(\epsilon_1\mu_1 - \epsilon_2\mu_2)^2}{(\epsilon_1\mu_2 - \epsilon_2\mu_1)^2} \tag{4.31}$$

and the corresponding ranks of the matrices $(A_1 - T(ti))$ and $(A_2 - T(t_i))$ are:

$$rank(A_1 - T(t_i)) = 4, \quad rank(A_2 - T(t_i)) = 6. \tag{4.32}$$

Substituting these values of the parameters into the relationship (4.28) after some algebraic transformations we come to the inequality (4.4) for i=1 (see Appendix).

The first inequality (4.5) follows from (4.28), if $\theta = 0$ (see Appendix).

To prove the estimate (4.28) for i=2 we consider the functional J_3^* conjugated to the functional J_3 (see (3.57)) :

$$J_3^* = \inf_{Z^* \in \Sigma(Z_0^*)} \int_S Z^* \cdot A^{-1} \cdot Z^* dS \tag{4.33}$$

where (see (2.3), (2.4))

$$Z^* = (D_1, B_1, D_2, B_2), \tag{4.34}$$

$$\Sigma(Z_0^*) = \{Z^* :< Z^* >= Z_0, \begin{cases} div(Z_1^*, Z_2^*) = divD_1 = \nabla \cdot D_1 = 0 \\ div(Z_3^*, Z_4^*) = divB_1 = \nabla \cdot B_1 = 0 \\ div(Z_5^*, Z_6^*) = divD_2 = \nabla \cdot D_2 = 0 \\ div(Z_7^*, Z_8^*) = divB_2 = \nabla \cdot B_2 = 0 \end{cases} \} \tag{4.35}$$

For any pair of the vectors a, b such that $div\ a = 0$ and $div\ b = 0$, the bilinear form $\phi(a, b)$ is a quasiaffine function [**6**]. Therefore the quadratic form $\Phi(Z^*, t_i)$ associated with the matrix $T(t_i)$ is also a quasiaffine function. The proof of the estimate (4.4) for i=2 is similar to the one given above. The only difference is the interchanging of the symbols Z and Z^*, A and A^{-1} and the indices 1 and 2. The inequalities obtained in such way can be easily transformed into the inequalities (4.4) for i = 2 of the main theorem.

This completes the proof of the main theorem estimates in case A .

Case B (see (3.55)) is treated in a similar way by using the estimates of the functional J_4 (see (3.56)). This functional is a quadratic form of two potential fields U_1 and U_2 and two divergence free fields B_1 and B_2. To reduce the problem of estimating this functional to the solved problem for the functional J_3, we transform the integrand of this functional into the quadratic form of four potential fields as the integrand of the J_3. To do this transformation it is enough to remark that in the two dimensional space each divergence free field B can be represented in the form $B = RH'$ where H' is a potential field. In this notation the integrand of the J_4 is the form $Z' \cdot A' \cdot Z'$ where

$$Z' = (U_1, H_1', U_2, H_2'), \tag{4.36}$$

and

$$A' = \begin{pmatrix} \mathcal{E} & 0 & 0 & 0 \\ 0 & R \cdot \mathcal{M}^{-1} \cdot R^t & 0 & 0 \\ 0 & 0 & \mathcal{E} & 0 \\ 0 & 0 & 0 & R \cdot \mathcal{M}^{-1} \cdot R^t \end{pmatrix}. \tag{4.37}$$

Comparing this expression with the integrand of the functional J_3 and keeping in mind that the estimates are independent of the fields Z' we find that the results for the case (3.55) could be obtained simply by interchanging the tensors $\mathcal{M}_i$ and $R \cdot \mathcal{M}_i^{-1} \cdot R^t$, $\;i = 0, 1, 2$ in the corresponding expressions for the case A.

The equality (4.12) is the corollary of (4.5), (4.11). Indeed, the relationships (4.5), (4.11) imply the equality:

$$Y(\mathcal{E}_0) = \epsilon_1 \mu_1 \; R^t \cdot Y^{-1}(\mathcal{M}_0) \cdot R \tag{4.38}$$

which can easily be transformed into the form (4.12) by using the conditions (4.11) and the definition of the tensor Y (see Appendix).

The equality (4.14) is proved in the same way by using (4.9), (4.13).

4.3 THE OPTIMAL STRUCTURES.

We show here that there is a correspondence between the set of the properties tensors of the described structures MCM(1,2,1) and MCM(2,1,2) and the set of all the boundary points of the G_mU. (We will consider only the case A, as the other cases are considered in the same way). Indeed, if substituting the expression (3.43) (and the same expression for the tensor of the magnetic properties) into the relationship (4.4) for i=1, one can observe that the dependence on the parameters of the structures $\alpha, \beta, \gamma, \omega$ vanishes and this inequality becomes an equality. The same is true for MCM(2,1,2): any pair $(\mathcal{E}_0, \mathcal{M}_0)$ of the effective properties tensors of these structures transforms the inequality (4.4) for i=2 into the equality.

Two four-dimensional components of the G_mU boundary, namely those described by (4.4) for i=1 and i=2 are intersected on the three-dimensional set which corresponds to the structures called [**20**] "the layers of the second rank" (see Fig.3). The both MCM(1,2,1) and MCM(2,1,2) structures degenerate into these ones. To obtain the layers of the second rank it is sufficient to assign to the parameter β of the external envelope the value 0.

Remark 4.2 *Precisely speaking, the geometrical picture of "the layers of the second rank" (see Fig.3) does not agree with our assertion about orthogonality of the normals $n_{\alpha 1}$ and $n_{\alpha 2}$. However, this assertion doesn't change the invariant characteristics of the effective properties tensor set of the structures.*

The mentioned three-dimensional set is in turn bounded by the inequalities (4.3) with the inequalities (4.5) or (4.9). These boundaries correspond to the next degree of degeneration of microstructures into MCM(1,2) and MCM(2,1). Both of the last types degenerate into the layered structure corresponding to the equalities in the Reuss-Voigt inequalities.

To complete the proof of the main theorem we have to demonstrate one-to-one correspondence between the sets (3.44) of the parameters $(\alpha, \beta, \gamma, \omega)$ of the MCM(1,2,1) structures and the boundary points of the G_mU given by the equation (4.4) for i=1 which is considered as the equality. Any pair of the tensors $(\mathcal{E}_0, \mathcal{M}_0)$ can be defined by the following matrix:

$$\begin{pmatrix} F(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2) & 0 \\ 0 & F(\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2) \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} & 0 & 0 \\ k_{12} & k_{22} & 0 & 0 \\ 0 & 0 & k_{33} & k_{34} \\ 0 & 0 & k_{34} & k_{44} \end{pmatrix}. \tag{4.39}$$

Here $k_{11}, k_{12}, k_{22} (k_{33}, k_{34}, k_{44})$ are the elements of the tensors $F(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$, $(F(\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2))$ in some fixed basis. It is convenient to choose the basis so that

$$k_{12} = k_{34} \geq 0 \tag{4.40}$$

and to introduce the parameters

$$\begin{array}{ccc} u_1 = k_{12}, & u_2 = k_{11} + k_{22}, & u_3 = k_{11} - k_{22}, \\ u_4 = k_{33} + k_{44}, & u_5 = k_{33} - k_{44}, & u_6 = k_{34}. \end{array} \tag{4.41}$$

The relationship (4.4) for i=1 considered as an equality allows us to find the parameter u_5 as a function of the others.

If $u_1 \neq 0$ then the parameters (u_1, u_2, u_3, u_4) define in one-to-one way the pair of the tensors ($\mathcal{E}_0, \mathcal{M}_0$) which belongs to the $G_m U$ set boundary. If $u_1 = 0$ there are two pairs of the tensors $(\mathcal{E}_0, \mathcal{M}_0)$ corresponding to each collection $(0, u_2, u_3, u_4)$. These pairs differ from each other by rotation on the right angle and therefore by the sign of the corresponding parameters u_3 and u_5. To distinguish one of these pairs we may adopt the additional agreement

$$(u_3 - u_5)(\epsilon_1\mu_2 - \epsilon_2\mu_1)(\epsilon_1\mu_1 - \epsilon_2\mu_2) \geq 0. \tag{4.42}$$

We now study the following mapping

$$\begin{gathered} (\alpha, \beta, \gamma, \omega) - -- > (u_1, u_2, u_3, u_4) : \\ \cdot \\ u_1 = (1-\gamma)(2\beta - 1) sin 2\omega, \\ \cdot \\ u_2 = 1 - \gamma + \gamma\left(\tfrac{\alpha\epsilon_1}{\alpha\epsilon_1 + (1-\alpha)\epsilon_2} + \tfrac{(1-\alpha)\epsilon_1}{\alpha\epsilon_2 + (1-\alpha)\epsilon_1}\right), \\ \cdot \\ u_3 = (1-\gamma)(2\beta - 1)\cos 2\omega + \gamma\left(\tfrac{\alpha\epsilon_1}{\alpha\epsilon_1 + (1-\alpha)\epsilon_2} + \tfrac{(1-\alpha)\epsilon_1}{(1-\alpha)\epsilon_2 + \alpha\epsilon_2 + (1-\alpha)\epsilon_1}\right), \\ \cdot \\ u_4 = 1 - \gamma + \gamma\left(\tfrac{\alpha\mu_1}{\alpha\mu_2 + (1-\alpha)\mu_1} + \tfrac{(1-\alpha)\mu_1}{\alpha\mu_2 + (1-\alpha)\mu_1}\right) \end{gathered} \tag{4.43}$$

which connects the parameters $u_1,\ u_2, u_3,\ u_4$ and the parameters of MCM(1,2,1) structures.

That this mapping is one to one is provided by nondegeneration of its Jacobian. The Jacobian I_4 of the mapping (4.43) is equal to

$$I_4 = det\ \frac{d(u_1, u_2, u_3, u_4)}{d(\alpha, \beta, \gamma, \omega)}. \tag{4.44}$$

After some calculations we can find that

$$I_4 = c_3\gamma(1-\gamma)^2(2\beta-1)(1-2\alpha)\alpha(1-\alpha)^2, \tag{4.45}$$

where

$$c_3 = \frac{4(\epsilon_1^2-\epsilon_2^2)(\mu_1^2-\mu_2^2)(\epsilon_1\mu_1-\epsilon_2\mu_2)(\epsilon_1\mu_2-\epsilon_2\mu_1)}{(\alpha\epsilon_1+(1-\alpha)\epsilon_2)^2(\alpha\epsilon_2+(1-\alpha)\epsilon_1)^2(\alpha\mu_1+(1-\alpha)\mu_2)^2(\alpha\mu_2+(1-\alpha)\mu_1)^2}. \tag{4.46}$$

The value of I_4 is never zero or infinity inside the admissible set (3.45) of the parameters $(\alpha, \beta, \gamma, \omega)$. Therefore, there is one-to-one correspondence between the interior of the set (3.45) of the MCM(1,2,1) structural parameters and the inner points of the examined component of the G_mU set boundary.

Now we explain the degenerations of the Jacobian I_4 at the boundary of the set (3.44) .

If the parameter β is zero then the structure degenerates to "the layers of the second rank" and the inequality (4.11) for i= 2 becomes an equality. The equalities $\alpha = 0, \gamma = 0$ or $\gamma = 1$ correspond to the degeneration of the structures into MCM(1,2) or MCM(2,1) (see Fig.2) and the inequality (4.5) or (3.33) is carried out as the equality. The values $\alpha = 1/2$ or $\beta = 1/2$ are connected with isotropy of the envelopes in the MCM structures which leads to the coincidence of the eigenvectors of the tensors $\mathcal{E}_0$ and $\mathcal{M}_0$.

The proof of one-to-one correspondence between the structures of MCM(2,1,2) and the points on the boundary (4.4) for i=2 of the G_mU set could be done in a similar fashion.

Note that the inner points of the G_mU set may be constructed by mixing of the MCM(1,2,1) and MCM(2,1,2) structures.

4.4 SOME PARTICULAR CASES. DISCUSSION.

Let us describe some sections of the G_mU set to illustrate their structure.

1. First we propose that the composite is isotropic in both electrical and magnetic properties. In this case the effective properties tensors can be represented in the form

$$\mathcal{E}_0 = \epsilon_0 I, \quad \mathcal{M}_0 = \mu_0 I \tag{4.47}$$

where ϵ_0, μ_0 are the material constants of an isotropic composite. For any isotropic mixture (i.e. for the isotropic section of the G_mU) these constants are bounded by the inequalities

$$\frac{(y(\epsilon_0)/\epsilon_i - y(\mu_0)/\mu_i)^2}{(1-y^2(\epsilon_0)/\epsilon_i^2)(1-y^2(\mu_0)/\mu_i^2)} \le c_1, \quad i=1,2 \tag{4.48}$$

$$\epsilon_1\mu_1 \le y(\epsilon_0)y(\mu_0) \le \epsilon_2\mu_2 \tag{4.49}$$

which follow from (4.4). Here

$$y(\epsilon_0) = -(\epsilon_0-\epsilon_h)(\epsilon_0-\epsilon_a)^{-1}\epsilon_1\epsilon_2\epsilon_h^{-1}, \tag{4.50}$$

$$y(\mu_0) = -(\mu_0-\mu_h)(\mu_0-\mu_a)^{-1}\mu_1\mu_2\mu_h^{-1}. \tag{4.51}$$

are the isotropy constants of the tensors $Y(\epsilon_0)$ and $Y(\mu_0)$, ϵ_a (μ_a) and ϵ_h (μ_h) are the arithmetic and harmonic means of the electric (magnetic) properties of the components (see (3.14)). Boundaries of the isotropic section of the G_mU set correspond to the isotropic MCM(1,2,1) or MCM(2,1,2) structures in which the parameters α, β are equal 1/2. The parameter γ varies from 0 to 1. The corners of this section (see Fig. 4) correspond to the values $\gamma = 0$ and $\gamma = 1$ i.e. to the matrix composites MCM(1,2) and MCM(2.1).

Remark 4.3 *One can check that the estimates (4.49) are exactly equivalent to inequalities obtained by Milton [**29**]. The isotropic structures of the double coated matrices MCM(1,2,1) and MCM(2,1,2) are also equivalent as regards to their effective properties to the double coated spheres introduced in [**29**].*

Fig. 4 illustrates the admissible set of the pairs (ϵ_0, μ_0) bounded by the inequalities (4.48). Here the parameters have the following values:

$$\begin{array}{lll} \epsilon_1 = 1, & \mu_1 = 1, & m_1 = 0.5, \\ \epsilon_2 = 10, & \mu_2 = 0.2, & m_2 = 0.5. \end{array} \tag{4.52}$$

The figure shows in particular that by fixing μ_0 for a mixture we can considerably decrease the admissible interval of possible values of ϵ_0.

2. Now let us propose that the composite is isotropic only in respect to the magnetic properties: $\mathcal{M}_0 = \mu_0 I$. In this case the inequalities (4.4) take the form

$$\frac{y^2(\mu_0)/\mu_i^2 - y(\mu_0)TrY(\mathcal{E}_0)/\epsilon_i\mu_i + detY(\mathcal{E}_0)/\epsilon_i^2}{(1-y^2(\epsilon_0)/\epsilon_i^2)(1-det(Y(\mu_0)/\mu_i))} \le c_1 \quad i=1,2 \tag{4.53}$$

They bound the set of the admissible values of the tensor $Y(\mathcal{E}_0)$ if the value of μ_0 is fixed. Fig.5 shows this set (values of the parameters are those of (4.52) and $\mu_0 = (\mu_a + \mu_h)/2 = 7/15$). Here the coordinates are the determinant and the trace of the tensor $Y(\mathcal{E}_0)$. The side (AB) of the curve triangle ABC bounded by the estimates (4.53) corresponds to the restriction

$$Tr^2 Y(\mathcal{E}_0) - 4\ det Y(\mathcal{E}_0) \geq 0, \tag{4.54}$$

valid for any tensor $Y(\mathcal{E}_0)$ with real eigenvalues. The sides (AC) and (BC) correspond to the restrictions (4.53) for i= 2, i = 1 respectively. The straight lines (DE) and (FG) correspond to the restrictions (3.33).

The same lines are shown on the Fig.6 in the coordinates $(\lambda_1^\epsilon, \lambda_2^\epsilon)$ where $\lambda_1^\epsilon, \lambda_2^\epsilon$ are the eigenvalues of the tensor $\mathcal{E}_0$.

We see that measuring the magnetic properties of a composite material allows us to determine its electric properties almost exactly (for reasonable choice of the initial material parameters)

3. The obtained result may be applied to the estimation of the effective properties of the three-dimensional fibrous composites whose properties are constant along some axis x_3. The characteristic functions $\chi_i \quad i = 1, 2$ for such composites depend only on variables x_1, x_2

$$\chi_i = \chi_i(x_1, x_2). \tag{4.55}$$

In this case the properties of the composite along axis x_3 are easily determined [**2**]:

$$\begin{array}{lll} \overline{\epsilon_{13}} = 0, & \overline{\epsilon_{23}} = 0, & \overline{\epsilon_{33}} = \epsilon_a, \\ \overline{\mu_{13}} = 0, & \overline{\mu_{23}} = 0, & \overline{\mu_{33}} = \mu_a \end{array} \tag{4.56}$$

Here $\overline{\epsilon_{ij}}$ ($\overline{\mu_{ij}}$) are the elements of the tensor $\overline{\mathcal{E}_0}$ ($\overline{\mathcal{M}_0}$) in the coordinate system x_1, x_2, x_3. The admissible pairs of (2×2) tensors

$$\mathcal{E}_0 = \begin{pmatrix} \overline{\epsilon_{11}} & \overline{\epsilon_{12}} \\ \overline{\epsilon_{12}} & \overline{\epsilon_{22}} \end{pmatrix}, \quad \mathcal{M}_0 = \begin{pmatrix} \overline{\mu_{11}} & \overline{\mu_{12}} \\ \overline{\mu_{12}} & \overline{\mu_{22}} \end{pmatrix} \tag{4.57}$$

which correspond to plane parts of the tensors $\overline{\mathcal{E}_0}$, $\overline{\mathcal{M}_0}$ are bounded by the main theorem.

The result obtained in the corollary (4.12) of the main theorem can be interpreted here as following. If the velocity of the light v has an equal value in both initial materials it has the same value (in x_3 direction) in any mixture of them.

5 G_m-CLOSURE PROBLEM FOR THE OPERATOR OF AN ELLIPTIC SYSTEM OF TWO EQUATIONS

The problem of description of the G_mU set for a more general operator can be solved using the obtained results. We consider the two-dimensional problem described by the following elliptic system:

$$\begin{aligned} \nabla \cdot (A(x) \cdot \nabla w_1 + B(x) \cdot \nabla w_2) = f_1, \\ \nabla \cdot (B(x) \cdot \nabla w_1 + C(x) \cdot \nabla w_2) = f_2, \end{aligned} \tag{5.1}$$

where w_1 and w_2 are the potentials, f_1, f_2 are the densities of external sources, and $A(x), B(x), C(x)$ are the properties of a nonhomogeneous material:

$$A(x) = A_1\chi_1(x) + A_2\chi_2(x) \tag{5.2}$$

and the same for $B(x)$ and $C(x)$. Here A_1, B_1, C_1 and A_2, B_2, C_2 are the isotropic tensors of the initial materials properties; corresponding small letters denote thir the scalar properties: $A_1 = a_1 I$ and so on. Ellipticity of the system (5.1) is provided by the relationships

$$(a_1c_1 - b_1^2) \geq 0, \quad (a_2c_2 - b_2^2) \geq 0. \tag{5.3}$$

Let A_0, B_0, C_0 be the (2×2) tensors of the effective properties of the mixture which is anisotropic in general. The G_m-closure problem for the system (5.1) is the problem of determination of invariants of possible triples of tensors A_0, B_0, C_0. We will show, however, that for any mixture the tensors A_0, B_0, C_0 are connected by a linear relationship and therefore only two of them are independent.

For description of the G_mU set we consider the energy of the process, i. e. the quadratic form

$$J = \int_S (\nabla w_1 \cdot A(x) \cdot \nabla w_1 + 2\ \nabla w_1 \cdot B(x) \cdot \nabla w_2 + \nabla w_2 \cdot C(x) \cdot \nabla w_2) dS. \tag{5.4}$$

We will show that the problem can be reduced to one already solved. To see this we, at any point x, transform the quadratic form

$$(\nabla w_1 \quad \nabla w_2) \cdot \begin{pmatrix} A(x) & B(x) \\ B(x) & C(x) \end{pmatrix} \cdot \begin{pmatrix} \nabla w_1 \\ \nabla w_2 \end{pmatrix} \tag{5.5}$$

to the sum of the squares by the appropriate linear transformation (independent of x) of the potentials w_1, w_2:

$$[w_1', w_2']^t = G[w_1, w_2]^t, G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{5.6}$$

Because the matrix

$$\begin{pmatrix} A(x) & B(x) \\ B(x) & C(x) \end{pmatrix} \tag{5.7}$$

is piece-wise constant, this transformation has to diagonalize only the two constant matrices

$$\begin{pmatrix} A_1 & B_1 \\ B_1 & C_1 \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ B_2 & C_2 \end{pmatrix} \tag{5.8}$$

This is always possible because both matrices are positive definite [**8**]. The parameters g_{ij} satisfy the equations

$$\begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix} \cdot \begin{pmatrix} a_i & b_i \\ b_i & c_i \end{pmatrix} \cdot \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \mathcal{E}_i & 0 \\ 0 & \mathcal{M}_i \end{pmatrix}, \quad i = 1, 2 \tag{5.9}$$

which introduce the fictitious "electrical" $\mathcal{E}_1, \mathcal{E}_2$ and "magnetic " $\mathcal{M}_1, \mathcal{M}_2$ permeabilities and give two constraints on the parameters g_{ij}:

$$(g_{11}g_{22} + g_{12}g_{21})\, b_i + g_{11}g_{12}\, a_i + g_{21}g_{22}\, c_i = 0,\ i = 1, 2. \tag{5.10}$$

To fix g_{ij} we may impose the additional conditions

$$det\ G = g_{11}g_{22} - g_{12}g_{21} = 1, \quad g_{11} = 1. \tag{5.11}$$

The effective properties tensors can be expressed in the new notations as

$$\begin{pmatrix} g_{11}I & g_{21}I \\ g_{12}I & g_{22}I \end{pmatrix} \cdot \begin{pmatrix} A_0 & B_0 \\ B_0^t & C_0 \end{pmatrix} \cdot \begin{pmatrix} g_{11}I & g_{12}I \\ g_{21}I & g_{22}I \end{pmatrix} = \begin{pmatrix} \mathcal{E}_0 & 0 \\ 0 & \mathcal{M}_0 \end{pmatrix} \tag{5.12}$$

where $\mathcal{E}_0, \mathcal{M}_0$ are the tensors of the effective fictitious properties. The set of the pairs $\mathcal{E}_0, \mathcal{M}_0$ have been treated in previous sections. See the main theorem in the section 4.1.

One can obtain from the equation (5.12) the following properties of the effective tensors A_0, B_0, C_0 of an arbitrary mixture

(i) The matrix B_0 is symmetric:

$$B_0 = B_0^t. \tag{5.13}$$

(ii) The matrices $A_0,\ B_0,\ C_0$ are connected by the linear equation

$$(g_{11}g_{22} + g_{12}g_{21})B_0 + g_{11}g_{12}A_0 + g_{21}g_{22}C_0 = 0 \tag{5.14}$$

(iii) and are expressed by the formulas

$$\begin{aligned} A_0 &= g_{22}^2\mathcal{E}_0 + g_{21}^2\mathcal{M}_0, \\ C_0 &= g_{12}^2\mathcal{E}_0 + g_{11}^2\mathcal{M}_0, \\ B_0 &= -g_{12}g_{22}\mathcal{E}_0 - g_{11}g_{21}\mathcal{M}_0. \end{aligned} \tag{5.15}$$

The formulas show that the G_mU set can be represented as a five-dimensional hypersurface in the eight-dimensional space of invariants of the tensor triples A_0, B_0, C_0.

Remark 5.1 *Milgrom and Shtrikman, in a recent paper [**28**], have studied a transformation similar to (5.9) for the more general case of the system of n equations in m dimensional space.*

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A APPENDIX

1. Proof of the formula (3.35).

¿From (3.26) we deduce that

$$A_a - A_0 \leq P(t_i), t_i \in \mathcal{T} \tag{A.1}$$

where

$$\begin{array}{c} P(t_i) = m_1 m_2 [m_1 (A_1 - T(t_i))^{-1} + m_2 (A_2 - T(t_i))^{-1}]^{-1} \cdot \\ \cdot (A_1 - T(t_i))^{-1} \cdot (A_1 - A_2) \cdot (A_2 - T(t_i))^{-1} \cdot (A_1 - A_2) \end{array} \tag{A.2}$$

As

$$A_h - A_a = -m_1 m_2 (A_1 - A_2)^2 \cdot (m_1 A_2 + m_2 A_1) \leq 0 \tag{A.3}$$

so

$$\begin{aligned} Y(A_0) = A_1 \cdot A_2 \cdot ((A_a - A_h) \cdot (A_a - A_0)^{-1} - I) \cdot A_h^{-1} \geq \\ A_1 \cdot A_2 \cdot ((A_a - A_h) \cdot P^{-1} - I) \cdot A_h^{-1} = \\ = -(A_1 - A_2) \cdot T(t_i) \cdot (A_1 - A_2)^{-1}, \end{aligned} \tag{A.4}$$

$$t_i \in \mathcal{T}. \tag{A.5}$$

Due to commutability of the matrices $Y(A_0)$ and A_1, A_2 we get

$$(A_1 - A_2) \cdot (Y(A_0) + T(t_i)) \cdot (A_1 - A_2)^{-1} \geq 0. \tag{A.6}$$

¿From here the statement (3.35) follows immediately.

2. Transformation of the estimates (4.28), (4.30)and (4.31) to the form (4.4) for i=1.

We obtain the estimate (4.4) for i=1 after appropriate choices of the coefficients t_i in the inequality (4.28), and after carrying out the series of algebraic transformations shown below.

We choose the coefficients t_i so that the ranks of the matrices $(A_1 - T(t_i))$ and $(A_2 - T(t_i))$ would be minimal. Let the relationships (4.30) hold. Then the matrix $(A_1 - T(t_i))$ has four positive and four zero eigenvalues and the inequality (3.38) may be written in the form

$$det \begin{pmatrix} Y(\mathcal{E}_0)/\epsilon_1 & 0 & -\theta R & -\sqrt{1-\theta^2}R \\ 0 & Y(\mathcal{M}_0)/\mu_1 & \sqrt{1-\theta^2}R^t & \theta R \\ -\theta R^t & \sqrt{1-\theta^2}R & Y(\mathcal{E}_0)/\epsilon_1 & 0 \\ -\sqrt{1-\theta^2}R^t & \theta R^t & 0 & Y(\mathcal{M}_0)/\mu_1 \end{pmatrix} \geq 0. \tag{A.7}$$

The parameter θ must be chosen in such way that the conditions

$$\begin{aligned} A_2 - T(t_i(\theta)) \geq 0, \\ 1 - \theta^2 \geq 0 \end{aligned} \tag{A.8}$$

are satisfied.

To simplify the expression (A.7) we will use the following properties of the block matrices [**9**]:

Let $K = K^t$, $L = L^t$ and M be the square matrices and

$$\begin{pmatrix} K & M \\ M^t & L \end{pmatrix} \geq 0. \tag{A.9}$$

Then

$$[K - M^t \cdot L^{-1} \cdot M] \geq 0 \tag{A.10}$$

and

$$[K \cdot M^{-1} \cdot L \cdot (M^{-1})^t - I] \geq 0. \tag{A.11}$$

We should note that the relationship (3.35) has the same form as (A.9) if

$$K = L = \begin{pmatrix} Y(\mathcal{E}_0)/\epsilon_1 & 0 \\ 0 & Y(\mathcal{M}_0)/\mu_1 \end{pmatrix} \tag{A.12}$$

$$M = \begin{pmatrix} -\theta R^t & -\sqrt{1-\theta^2}R \\ -\sqrt{1-\theta^2}R^t & \theta R \end{pmatrix} \tag{A.13}$$

Due to (A.10), we can deduce from (3.35) the inequality

$$\begin{pmatrix} K' & M' \\ M'^t & L' \end{pmatrix} \geq 0. \tag{A.14}$$

where

$$K' = Y(\mathcal{E}_0)/\epsilon_1 - \frac{\theta^2 Y(\mathcal{E}_0)/\epsilon_1}{det(Y(\mathcal{E}_0)/\epsilon_1)} - \frac{(1-\theta^2)Y(\mathcal{M}_0)/\mu_1}{det(Y(\mathcal{M}_0)/1\mu_1)}, \tag{A.15}$$

$$M' = -\theta\sqrt{1-\theta^2} - \frac{Y(\mathcal{E}_0)/\epsilon_1}{det(Y(\mathcal{E}_0)/\epsilon_1)} - \frac{Y(\mathcal{M}_0)/\mu_1}{det(Y(\mathcal{M}_0)/\mu_1)} \tag{A.16}$$

$$L' = Y(\mathcal{M}_0)/\mu_1 - \frac{\theta^2 Y(\mathcal{M}_0)/\mu_1}{det(Y(\mathcal{M}_0)/\mu_1)} - \frac{(1-\theta^2)Y(\mathcal{E}_0)/\epsilon_1}{det(Y(\mathcal{E}_0)/1\epsilon_1)} \tag{A.17}$$

Here the equalities

$$Y^{-1}(\mathcal{E}_0) = R \cdot Y(\mathcal{E}_0) \cdot R^t det Y(\mathcal{E}_0), \tag{A.18}$$

$$Y^{-1}(\mathcal{M}_0) = R \cdot Y(\mathcal{M}_0) \cdot R^t det Y(\mathcal{M}_0) \tag{A.19}$$

valid for any (2×2) matrix Y are used.

Again, using (A.13) we can obtain from formula (A.14) the inequality

$$(1 - det(Y(\mathcal{E}_0)/\epsilon_1))(1 - det(Y(\mathcal{M}_0)/\mu_1))I + (1-\theta^2)G \geq 0 \qquad \text{(A.20)}$$

where

$$\begin{array}{c} G = (det(Y(\mathcal{E}_0)/\epsilon_1) + det(Y(\mathcal{M}_0)/\mu_1))I - \\ -Y^{-1}(\mathcal{M}_0) \cdot Y(\mathcal{E}_0) det(Y(\mathcal{M}_0))/(\epsilon_1\mu_1) \\ -Y^{-1}(\mathcal{E}_0) \cdot Y(\mathcal{M}_0) det(Y(\mathcal{E}_0))/(\epsilon_1\mu_1). \end{array} \qquad \text{(A.21)}$$

It is easy to check that

$$G = det(Y(\mathcal{E}_0)/\epsilon_1 - Y(\mathcal{M}_0)/\mu_1)I \qquad \text{(A.22)}$$

and

$$(1 - det(Y(\mathcal{E}_0)/\epsilon_1))(1 - det(Y(\mathcal{M}_0)/\mu_1) \leq 0. \qquad \text{(A.23)}$$

The last inequality follows from the equations (3.54), (3.33) and the corresponding ones for the tensor $\mathcal{M}_0$.

Substituting (A.22) into (A.20), and taking into account the inequality (A.23), we obtain

$$\frac{det(Y(\mathcal{E}_0)/\epsilon_1 - Y(\mathcal{M}_0)/\mu_1)}{(1 - det(Y(\mathcal{E}_0)/\epsilon_1))(1 - det(Y(\mathcal{M}_0)/\mu_1)} \leq \frac{-1}{1-\theta^2} \qquad \text{(A.24)}$$

This estimate is valid for all values of the parameter θ satisfying the conditions (A.8). We choose θ so that

$$det(A_2 - T(t_i(\theta))) = 0. \qquad \text{(A.25)}$$

We then have (4.31). In this case the estimates (A.24) and (4.4) for i=1 coincide.

To prove the first inequality (4.5), we assign zero to the parameter θ in (3.35). Then we obtain from (4.28) the relationship

$$\begin{pmatrix} Y(\mathcal{E}_0)/\epsilon_1 & R \\ R^t & Y(\mathcal{M}_0)/\mu_1 \end{pmatrix} \geq 0. \qquad \text{(A.26)}$$

Due to the block matrices property (A.11), the inequality we need follows

$$Y(\mathcal{E}_0) \cdot R \cdot Y(\mathcal{M}_0) \cdot R^t - \epsilon_1\mu_1 I \geq 0. \qquad \text{(A.27)}$$

3. Transformation of the formula (4.38).

Let $\epsilon_1\mu_1 = \epsilon_2\mu_2$. It is then easy to obtain

$$\begin{array}{c} \epsilon_h = \epsilon_1\mu_1/\epsilon_a, \\ \epsilon_h = \epsilon_1\mu_1/\mu_h \end{array}. \tag{A.28}$$

By the definition of the tensor $Y(\mathcal{E}_0)$, we have

$$\begin{array}{c} \mathcal{E}_0 = (\epsilon_a\epsilon_h Y(\mathcal{E}_0) + \epsilon_1\epsilon_2\epsilon_h I)(\epsilon_h Y(\mathcal{E}_0) + \epsilon_1\epsilon_2 I)^{-1} \\ = \epsilon_1\mu_1(\mu_a^{-1}\mu_h^{-1}R\cdot Y^{-1}(\mathcal{M}_0)\cdot R^t\mu_1\mu_2 + \mu_a^{-1}I)(\mu_a^{-1}R\cdot Y^{-1}(\mathcal{M}_0)\cdot R^t\mu_1\mu_2 + I)^{-1} \\ \epsilon_1\mu_1(I + \mu_h R\cdot Y(\mathcal{M}_0)\cdot R^t\mu_1^{-1}\mu_2^{-1})(\mu_h + \mu_h\mu_a R\cdot Y(\mathcal{M}_0)\cdot R^t\mu_1^{-1}\mu_2^{-1})^{-1} \\ = \epsilon_1\mu_1(R\cdot\mathcal{M}_0\cdot R^t)^{-1}. \end{array} \tag{A.29}$$

The second equality in (A.29) is a result of the relationships (A.28) and (4.38) and the last one follows from the definition of the tensor $Y(\mathcal{M}_0)$. From (A.29) we immediately obtain the statement (4.12).

4.The independent proof of the equality (4.12) in corollary 1.

The statement (4.12) can be proved independently too. If the parameters t_i are chosen according to (4.30), and the parameter θ is equal to zero, all eigenvalues of the matrix $(m_1(A_1-T)^{-1}+m_2(A_2-T)^{-1})^{-1}$ are equal to zero. Then we obtain from (3.26)

$$A_0 - T(t_i(0)) \geq 0. \tag{A.30}$$

Hence it follows that

$$\begin{pmatrix} \mathcal{E}_0 & vR \\ vR^t & \mathcal{M}_0 \end{pmatrix} \geq 0 \tag{A.31}$$

$$(\epsilon_1\mu_1 = \epsilon_2\mu_2 = v^2). \tag{A.32}$$

In a similar way we can get

$$A_0^{-1} - T(t_i'(0)) \geq 0. \tag{A.33}$$

If

$$t_1' = 0, \quad t_2' = 0, \quad t_3' = 1/v \tag{A.34}$$

we deduce

$$\begin{pmatrix} \mathcal{E}_0^{-1} & v^{-1}R \\ v^{-1}R^t & \mathcal{M}_0^{-1} \end{pmatrix} \geq 0 \tag{A.35}$$

¿From (A.31), (A.35) we obtain the inequality

$$\mathcal{E}_0 \cdot R \cdot \mathcal{M}_0 \cdot R^t - v^2 I \geq 0 \tag{A.36}$$

and the reverse one

$$\begin{gathered} \mathcal{E}_0^{-1} \cdot R \cdot \mathcal{M}_0^{-1} \cdot R^t - v^{-2} I = \\ = v^{-2} \mathcal{E}_0^{-1} \cdot R \cdot \mathcal{M}_0^{-1} \cdot R^t \cdot (v^2 I - R \cdot \mathcal{M}_0 \cdot R^t \cdot \mathcal{E}_0) \geq 0 \,^{\cdot} \end{gathered} \tag{A.37}$$

The statement we prove follows.