Dynamics of solids with non-monotone stress-strain relations. 2. Nonlinear Waves and Waves of Phase Transition

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November 12, 1999

[1]

Abstract

We investigate the dynamics of one dimensional mass-spring chain with non-monotone dependence of the spring force vs. spring elongation. For this strongly nonlinear system we find a family of exact solutions that represent the nonlinear waves. We have found numerically that this system displays a dynamical phase transition from the stationary phase (when all masses are at rest) to the *twinkling* phase (when the masses oscillate in a wave motion). This transition has two fronts, which propagate with different speeds. We study this phase transition analytically and derive the relations between its quantitative characteristics.

1 Introduction

The system. We consider the dynamics of a one-dimensional chain of masses m connected by identical springs (see Figure 1). This system is described by the

[1]Submitted to J. Mech. Phys. Solids

Figure 1: A mass-spring chain.

following equations

$$\rho \ddot{x}_n = F(x_{n+1} - x_n) - F(x_n - x_{n-1}) \tag{1.1}$$

where $n = 0, \pm 1, \pm 2, ...,$ and x_n is the coordinate of the mass number n; the function $F(u)$ characterizes the dependence of the spring force vs. spring elongation. We are interested in the situation when this function is *non-monotone*; namely we consider the following basic model of a spring force

$$F(u) = \begin{cases} ku & \text{if } u < u_c, \\ ku - f & \text{if } u > u_c \end{cases} \tag{1.2}$$

where u_c is some critical elongation. In the range $u < u_c$, as well as in the range $u > u_c$ the force F depends linearly on the elongation u with the same slope k, but at the critical value u_c, the spring force $F(u)$ drops by f units, from the value ku_c to the value $ku_c - f$ (see Figure 2a).

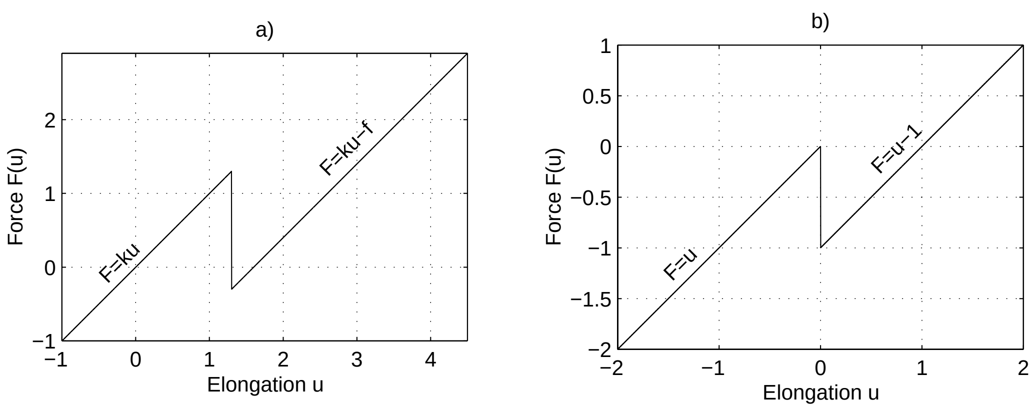

Figure 2: A model non-monotone constitutive relation (a) characterized by some values of k, u_c, and f. (b) normalized to $k = 1$, $u_c = 0$, and $f = 1$.

Motivation. One motivation of this paper arises from our previous work [1] on the analysis of the widely accepted *static* approach to the description of the phenomenon of phase transition in systems with non-convex elastic energy (see discussion in [1]). In numerical experiments we found quite regular pattern of phase transition (with two fronts, which propagate with different speeds), and our present goal is to describe this transition *analytically*.

Another motivation of the work is more general; the Hamiltonian system (1.1)-(1.2) is interesting from the view point of the general theory of nonlinear waves (see e.g. [2, 3]). We have found new kind of dynamical behavior, which we have not met in other systems (see the Figure 3). The example (1.1)-(1.2) appears to enrich one's intuition of nonlinear dynamics (of what can happen). Note that the system (1.1)-(1.2) is *strongly nonlinear*; for instance, it does not make sense to consider this system with small force drop f, since it can be always normalized to $f = 1$ (see below). The possibility of analytical description of this strongly nonlinear system is interesting as well.

Normalization. The system (1.1)-(1.2) can be rescaled to one and the same system, which is characterized by the unit mass $\rho = 1$, the unit spring constant $k = 1$, zero critical elongation $u_c = 0$, and the unit force drop $f = 1$. In order to see this, first of all, let us note that the dependence $F(u)$ can be shifted by an arbitrary vector (a, b) in the plane (u, F). Indeed, if we make the following change of variables

$$x_n = \tilde{x}_n + na \quad (n = 0, \pm 1, \pm 2, \ldots),$$

then the equations (1.1) can be written in the same form, but with a different force

$$\tilde{F}(u) = F(u - a) + b.$$

Therefore, we can shift the dependence $F(u)$ so that the discontinuity occurs at $u_c = 0$ and $F(0-) = 0$. Further, we can choose the units of mass so that $\rho = 1$, the units of time so that $k/\rho = 1$, and the units of length so that $\frac{1}{k}[F(0-) - F(0+)] = 1$ (see Figure 2b). Thus we can write our system in the form

$$\ddot{x}_n = F(x_{n+1} - x_n) - F(x_n - x_{n-1}) \quad \text{where} \quad F(u) = u - \Theta(u), \tag{1.3}$$

Θ is the Heaviside function.

The phenomenon observed in computer experiments. In computer experiments we have observed the following waves associated with a phase transition. We have stretched the chain so that the distance between any two adjacent masses is $u_c - \epsilon$ (ϵ is a "very small" number), and all masses are at rest; all spring forces have the same value $F(u_c - \epsilon)$. Because of some "small" fluctuations the elongation of one spring can become greater than u_c, say $u_c + \epsilon_0$ (ϵ_0 is also a "very small" number), and then the masses start moving. Here we describe the numerical results.

In computer experiments we considered the chain of large number N of masses with initial conditions

$$\begin{aligned} x_n(0) &= \begin{cases} (u_c - \epsilon)n & \text{if } n \le N/2 \\ (u_c + \epsilon_0) + (u_c - \epsilon)(n-1) & \text{if } n > N/2 \end{cases}, \\ \dot{x}_n(0) &= 0 \quad (n = 1, 2, \ldots, N; \quad \text{e.g. } \epsilon = \epsilon_0 = 10^{-6}) \end{aligned} \tag{1.4}$$

The result of our computer simulation is shown in the Figure 3. This computation reveals a *phase transition*: the system goes over from one steady state, when the elongation of each spring is constant, to another steady state, when the masses oscillate with some period p. The later phase can be called the *twinkling phase*.

The phase transition propagates symmetrically in the two opposite directions (corresponding to the mirror symmetry of the initial conditions(1.4)).

The phase transition has two fronts. The first of them propagates with the speed 1 (it takes time 1 to propagate from one mass to the next). This is the largest speed of propagation of linear waves (see below). Between the first front and the second front the masses move (in average) with "almost" constant speed. After the second front the masses start to oscillate in a wave motion. Several quantitative characteristics are clearly seen from the picture of this phase transition, in particular:

1. *the time period p of the oscillations in the second phase.* After the second front the masses start to oscillate in a wave motion with some period p in time.

2. *the "swelling" distance α.* Before the phase transition the distance between the adjacent masses is (almost) u_c; in the result of the transition the average distance between the adjacent masses becomes greater by some distance α. In other words, as a result of the phase transition, the chain is "swelling" by a distance α, times the number of masses transfered to the twinkling phase.

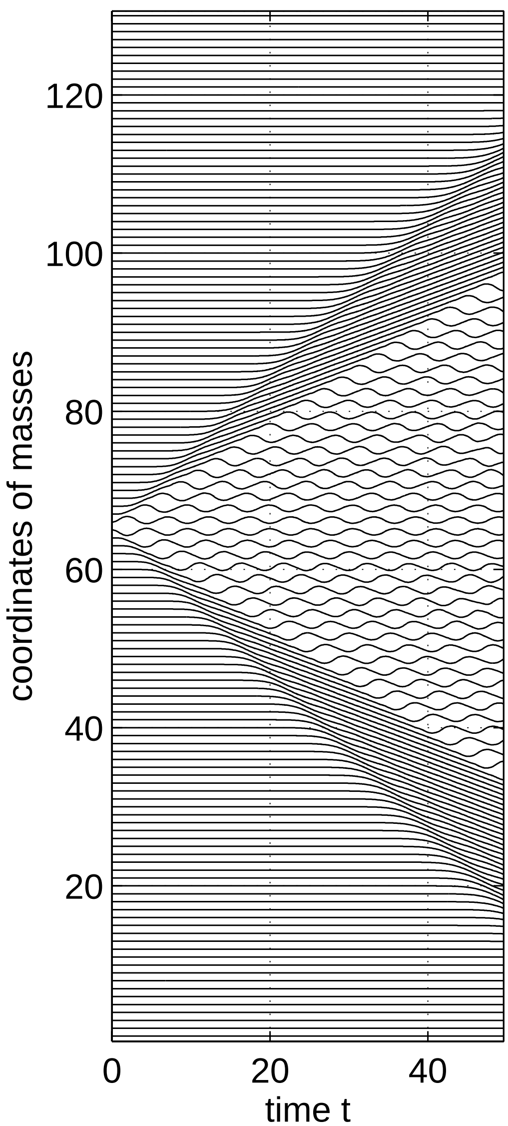

Figure 3: The result of our computer simulation of the chain with $N = 130$ masses. Initially, the chain is in rest, the distance between any pair of adjacent masses, besides the middle pair, is $u_c - \epsilon$, and the distance between the two middle masses is $u_c + \epsilon_0$ ($\epsilon = \epsilon_0 = 10^{-6}$), so that the chain is "almost" in equilibrium, and its instability "just starts to develop".

3. *the speed $1/\tau$ of the second front (after which the masses start to oscillate).* It takes a certain time τ for the second front to propagate from one mass to the next.

4. *the intermediate speed v.* Between the first and the second fronts the masses move with a certain "almost" constant speed v.

Our goal is to obtain these characteristics analytically. We note that similar phenomenon (when a wave of phase transition follows a forerunning sonic or shock wave that has a larger speed) was investigated by L. Truskinovsky [4] in a continous model. Continuous models, however, do not describe the oscillations of twinkling phase, just because they lack the internal degrees of freedom (responsible for the oscillations).

2 Nonlinear waves

The formulation. As a first step in the investigation of the system (1.3), we would like to find nonlinear waves that can propagate in this system. It is a strongly nonlinear system, and perturbation techniques utilized for finding waves in weakly nonlinear media do not work here.

We would like also to note the difference between the system considered in the present paper and the mass-spring chain with partially failing bonds [5]. The later situation takes place for instance when adjacent masses are joined by two linear springs, and at a certain critical stress one of the springs is torn. In this situation each bond displays nonlinearity only once. On the contrary, each spring in the present model is reversible and switches from one linear regime to another infinitely many times.

We look for the solutions $x_n(t)$ of (1.3) that are periodic in time

$$x_n(t+p) = x_n(t), \quad (n = 0, \pm 1, \pm 2, \ldots; \quad -\infty < t < \infty) \tag{2.1}$$

and satisfy the following self-similarity relation

$$x_{n+1}(t) = \alpha + x_n(t-\tau), \quad (n = 0, \pm 1, \pm 2, \ldots; \quad -\infty < t < \infty) \tag{2.2}$$

where p, α and τ are some a priori unknown constants. It is clear that the choice of parameter τ is not unique: the relation (2.2) will be satisfied if instead of τ we take $\tau + jp$ where j is an arbitrary integer; so we will assume that $|\tau| \leq p/2$. Then $|\tau|$ shows how much time it takes for the wave to propagate from one mass to the next, and sign(τ) shows the direction of the wave propagation; in other words $1/\tau$ is the Lagrangian velocity of the wave. By virtue of the self-similarity property (2.2), the coordinates $x_n(t)$ of all the masses can be expressed through the coordinate of only one mass with $n = 0$:

$$x_n(t) = n\alpha + x_0(t - n\tau), \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{2.3}$$

For the function $x_0(t)$, the equation (1.3) takes the following form

$$\ddot{x}_0(t) = F(\alpha + x_0(t - \tau) - x_0(t)) - F(\alpha + x_0(t) - x_0(t + \tau)). \tag{2.4}$$

We assume below that the origin of the x-axis is choosen such that the average value of x_0 is zero:

$$\int_0^p x_0(t)dt = 0. \tag{2.5}$$

Classes of regimes. The spring force (1.3) is piece-wise linear, and the switching from one linear dependence to the other occurs at the instants when the distance between the adjacent masses passes through the critical value $u_c = 0$. For instance, the switching of linear dependence of the first force in (2.3) occurs when the function

$$z(t) = x_1(t) - x_0(t) = \alpha + x_0(t - \tau) - x_0(t); \tag{2.6}$$

passes through zero. The solutions $x_0(t)$ of the equation (2.4) can be portioned into several classes depending on how many times the function (2.6) changes its sign during the period p.

Linear waves. If the function (2.6) does not change sign at all (it is either everywhere positive or everywhere negative), then the equation (2.4) takes the form

$$\ddot{x}_0(t) = x_0(t + \tau) + x_0(t - \tau) - 2x_0(t), \tag{2.7}$$

so that the mass-spring system is in linear regime. This equation has solutions

$$x_0(t) = \alpha + Ae^{i\Omega t}, \tag{2.8}$$

that are harmonic oscillations. Here the constant A is an arbitrary complex amplitude, and the frequency $\Omega = \frac{2\pi}{p}$ is connected with τ by the dispersion relation

$$-\Omega^2 = e^{i\Omega\tau} + e^{-i\Omega\tau} - 2,$$

which is equivalent to

$$\Omega = \pm 2 \sin \frac{\Omega\tau}{2}. \tag{2.9}$$

Note, that the slowest oscillations, with $\Omega = 0$, have the largest propagation speed $1/\tau = 1$. For the linear regime to take place, the amplitude A should be sufficiently small, so that the function (2.6) indeed does not change its sign, e.g. $|A| < \alpha/2$.

A nonlinear regime. Now consider the situation when the function (2.6) does change its sign. In the twinkling phase (Figure 3) the function (2.6) periodically changes its sign from "$-$" to "$+$" and then from "$+$" to "$-$" once per period p. In other words, the distance between any two adjacent masses once per period becomes greater than the critical distance $u_c = 0$, and once per period it becomes smaller than u_c. We choose zero of the time-axis so that

$$z(t) > 0 \quad \text{if} \quad 0 < t < q \qquad \text{and} \qquad z(t) < 0 \quad \text{if} \quad q < t < p \tag{2.10}$$

where q is some instant q between 0 and p. This means that the distance between mass #0 and mass #1 is greater than u_c during time intervals $(0, q), (\pm p, \pm p + q), (\pm 2p, \pm 2p + q), \ldots$ and smaller than u_c during time intervals $(q, p), (\pm p + q, \pm p + p), (\pm 2p + q, \pm 2p + p), \ldots$. The parameters p and q (a priory unknown) determine the instants of switching between the linear regimes of the spring force (see Figure 2) Now the equation (2.4) can be written in the form

$$\ddot{x}_0(t) = [\alpha + x_0(t-\tau) - x_0(t) - g(t)] - [\alpha + x_0(t) - x_0(t+\tau) - g(t+\tau)]$$

where

$$g(t) = \sum_{j=-\infty}^{\infty} [\Theta(t - jp) - \Theta(t - jp - q)]. \tag{2.11}$$

Thus the consideration of periodic waves enables us to reduce the nonlinear equation (2.4) to a linear equation

$$\ddot{x}_0(t) = x_0(t+\tau) + x_0(t-\tau) - 2x_0(t) + g(t+\tau) - g(t), \tag{2.12}$$

but with external forcing $g(t+\tau) - g(t)$, which contains two unknown parameters p and q. We will find the solution of this linear equation in terms of parameters τ, p, q. Then we will require that the function (2.6) satisfies the condition (2.10), so that the solution is self-consistent: the switching between the linear branches of the spring force occurs at the "right" instants, i.e. at $t = 0, q, \pm p, \pm p + q, \pm 2p, \pm 2p + q$, and so on periodically with period p. This will give us a nonlinear algebraic equation for the four parameters τ, p, q, and α.

We wish to emphasize that we were able to reduce the strongly nonlinear equation (2.4) to the linear equation (2.12) because we have chosen the dependence $F(u)$ consisting of two linear parts with THE SAME slope (see Figure 2). Then the nonlinearity is reduced to turning on constant external force at certain instants. If the motion is periodic (and the switching on and off occurs once per period), then the instants of switching (and therefore the external force) are completely characterized by two numbers p and q (see (2.11)). In this situation the nonlinear difference-differential equation (2.4) is reduced to the linear difference-differential equation (2.12) and a system of four nonlinear algebraic equations for the parameters τ, p, q, and α.

The solution of the linear equation. We solve the linear difference-differential equation (2.12) by means of Fourier transform. The Fourier series expansion

$$x_0(t) = \sum_{k=-\infty}^{+\infty} X_k e^{ik\nu t}, \qquad \nu = 2\pi/p, \tag{2.13}$$

reduces the equation (2.12) to the form

$$\left(-k^2\nu^2 - e^{ik\nu\tau} - e^{-ik\nu\tau} + 2\right) X_k = \left(e^{ik\nu\tau} - 1\right) G_k \tag{2.14}$$

where G_k is the Fourier coefficient of the function (2.11):

$$G_k = \frac{1}{p}\int_0^p g(t) e^{-ik\nu t} dt = \frac{1}{2\pi i k}\left(1 - e^{-ik\nu q}\right).$$

Hence

$$X_k = \frac{1 - e^{-ik\nu q}}{2\pi i k} \; \frac{e^{ik\nu\tau} - 1}{\left(2\sin\frac{k\nu\tau}{2}\right)^2 - (k\nu)^2}, \qquad \nu = 2\pi/p, \tag{2.15}$$

($k = \pm 1, \pm 2, \ldots$; $X_0 = 0$ according to (2.5)), and the solution $x_0(t)$ is given by (2.13).

The nonlinear algebraic equation for the four parameters τ, p, q, and α. Now we need to ensure that the solution (2.13)-(2.15) is consistent with the condition (2.10), i.e. the function (2.6) indeed vanish at $t = 0$ and $t = q$. This means that

$$x_0(t) - x_0(t-\tau) = \sum_{k=-\infty}^{+\infty} X_k \left(1 - e^{-ik\nu\tau}\right) e^{ik\nu t} \tag{2.16}$$

equals α when $t = 0$ and $t = q$. Using the solution (2.15), we find that condition (2.16) gives

$$\begin{aligned} \text{at } t &= 0: \qquad \sum_{k=-\infty}^{+\infty} \frac{\sin^2 \frac{k\nu\tau}{2}}{\left(\frac{k\nu}{2}\right)^2 - \left(\sin \frac{k\nu\tau}{2}\right)^2} \frac{1 - e^{-ik\nu q}}{2\pi i k} = \alpha, \\ \text{at } t &= q: \qquad \sum_{k=-\infty}^{+\infty} \frac{\sin^2 \frac{k\nu\tau}{2}}{\left(\frac{k\nu}{2}\right)^2 - \left(\sin \frac{k\nu\tau}{2}\right)^2} \frac{e^{ik\nu q} - 1}{2\pi i k} = \alpha \end{aligned}$$

(in these sums $k \neq 0$). The sum of these equations gives us the expression for α in terms of the parameters τ, p, q:

$$\alpha = \sum_{k=1}^{\infty} \frac{\sin^2 \frac{k\nu\tau}{2}}{\left(\frac{k\nu}{2}\right)^2 - \left(\sin \frac{k\nu\tau}{2}\right)^2} \frac{\sin k\nu q}{\pi k} \qquad (\nu = 2\pi/p), \tag{2.17}$$

while the difference of these equations is satisfied identically (since the resulting summand is odd with respect to the index k).

Thus we have a three-parameter family[2] of nonlinear waves: given τ, p, q, we can find α by formula (2.17) and the corresponding nonlinear wave — by formulas (2.15),(2.13). Note that in the linear regime we also have a three-parameter family of waves, but in this case τ and p are connected by the dispersion relation (2.9), and the independent parameters are p, A, α.

It is also instructive to compare the waves of the present model with water waves, which are characterized by the wave length λ and wave amplitude A (see e.g. ([2])). The time period p is similar to the wave length λ. The parameter q plays the part of the wave amplitude (it characterizes the nonlinearity). In the present model we have additionally the third parameter α (the average distance between masses), which has no analog for water waves. This parameter arises because we can pre-strech the chain (before exciting the oscillations). We call the relation (2.17) the *nonlinear dispersion relation* for the nonlinear waves in our model.

[2]The phase of the wave is the fourth parameter (related to the choice of an initial instant t_0) that we do not count.

3 Transition from the stationary to the twinkling phase

The wave motion in our computer experiment appears to be a nonlinear wave defined by the formulas (2.13), (2.15), (2.17), (2.3). However, we have a three-parameter family of nonlinear waves, while our computer experiment is well reproduced and gives rise to a unique nonlinear wave of the twinkling phase. It is not clear which wave from the three-parameter family describes the wave motion in the computer experiment.

In this section we will show that the values of the parameters p, q, τ are uniquely determined by the condition that the corresponding nonlinear wave originates from the phase transition. In other words, in this section we describe the entire transition pattern, shown in the Figure 3; the parameters p, q, τ are uniquely determined from the condition that the twinkling phase can be matched with the stationary phase.

Reduction to the linear difference-differential equation with undetermined parameters. The equation (1.3) of the mass spring chain can be written in the form

$$\ddot{x}_n = (x_{n+1} - x_n - g_n) - (x_n - x_{n-1} - g_{n-1}) \tag{3.1}$$

where

$$g_n = \Theta(x_{n+1} - x_n) \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{3.2}$$

In accordance with our computer experiment, we consider zero initial conditions:

$$x_n(0) = \dot{x}_n(0) = 0, \quad n = 0, \pm 1, \pm 2, \ldots$$

and assume that the phase transition propagates symmetrically so that

$$x_{1-n} = x_n \quad \Rightarrow \quad g_{-n} = g_n. \tag{3.3}$$

Let us choose zero of the time-axis t such that the quantity $x_2(t) - x_1(t)$ is less than zero when $t < \tau$, and it becomes positive for the first time when $t = \tau+$. The Figure 3 suggests that after the second front, the masses move periodically with some period p

$$x_n(t + p) = x_n(t) \qquad \text{for} \quad t > \tau n \tag{3.4}$$

in a wave motion with some wave speed $1/\tau$

$$x_{n+1}(t) = \alpha + x_n(t-\tau) \quad \text{for} \quad t > \tau n \tag{3.5}$$

($n = 1, 2, \ldots$). In the previous consideration of nonlinear waves (Section 2) we required that these relations hold for all $t \in (-\infty, \infty)$. This time we take into account the instant of excitation of the wave and consider the equations (3.4)-(3.5) only for $t > \tau n$.

The difference $x_2(t) - x_1(t)$ remains negative up to the instant $t = \tau$; then it becomes positive and stays positive for some time q ($0 < q < p$), i.e. during time interval $(\tau, \tau+q)$; then, at instant $\tau+q$ it becomes negative again and stays negative during time interval $(\tau+q, \tau+p)$. After that it repeats periodically with period p. By virtue of the self-similarity condition (3.5) we have that

$$x_{n+1}(t) - x_n(t) \begin{cases} < 0 & \text{if } t < \tau|n|, \\ > 0 & \text{if } \tau|n| + pj < t < \tau|n| + pj + q, \\ < 0 & \text{if } \tau|n| + pj + q < t < \tau|n| + p(j+1) \end{cases} \tag{3.6}$$

($n = 1, 2, 3, \ldots; \quad j = 0, 1, 2, \ldots$). This implies that we know the form of the functions $g_n(t)$ (see 3.2)) up-to three undetermined parameters p, q, τ:

$$g_n(t) = \sum_{j=0}^{\infty} \left[\Theta(t - n\tau - jp) - \Theta(t - n\tau - jp - q)\right]. \tag{3.7}$$

This expression is valid for positive integers $n = 1, 2, 3, \ldots$. By virtue of the symmetry (3.3) we have a similar expression for $g_n(t)$ with negative n ($n = -1, -2, -3, \ldots$). The case $n = 0$ is special since 0-th and 1-st masses move symmetrically; the sign of $x_1(t) - x_0(t)$ is described by two additional parameters τ_0 and q_0:

$$x_1(t) - x_0(t) \begin{cases} < 0 & \text{if } t < \tau_0, \\ > 0 & \text{if } \tau_0 + pj < t < \tau_0 + pj + q_0, \\ < 0 & \text{if } \tau_0 + pj + q_0 < t < \tau_0 + p(j+1), \end{cases} \tag{3.8}$$

and therefore

$$g_0(t) = \sum_{j=0}^{\infty} \left[\Theta(t - \tau_0 - jp) - \Theta(t - \tau_0 - jp - q_0)\right]. \tag{3.9}$$

The close look at the Figure 2 reveals that the wave motion is approximate: $x_{n+1}(t)$ is not exactly $\alpha + x_n(t-\tau)$ for $n = 1, 2, \ldots$. The relation (3.5) seems to hold only when $n \to \infty$. Therefore, we should have introduced different τ_n and q_n for all $n = 0, 1, 2, \ldots$ (not only the case $n = 0$ is special). Moreover, we will see that the periodicity (3.4) is also approximate and seems to hold when $t \to \infty$. However, the numerical experiment shows that the accuracy of the above approximations (3.4),(3.5) is "good", and we assume these approximations. Thus, similar to the case of nonlinear waves considered in the previous section, we reduce the nonlinear system (3.1) to a linear form

$$\ddot{x}_n = x_{n+1} + x_{n-1} - 2x_n + g_{n-1}(t) - g_n(t). \tag{3.10}$$

with the forcing depending on undetermined parameters. This time we need five parameters τ, p, q, τ_0, q_0 instead of three parameters τ, p, q in the case of nonlinear waves, extended from $x = -\infty$ to $x = +\infty$ (see Section 2). We find the solution of the linear system (3.10) in terms of the parameters τ, p, q, τ_0, q_0. Then we require that the conditions (3.6) and (3.8) are satisfied, so that the solution is self-consistent. To do this, we have the five undetermined parameters. However, it turns out that it is impossible to choose the values of the parameters to satisfy these conditions exactly. We satisfy these conditions approximately. This approximation leads to a unique choice of the values of the parameters τ, p, q, τ_0, q_0, and therefore to a unique nonlinear wave of the twinkling phase. The approximation is based on the postulated form, rather than on a small parameter. Hence our approximation is similar to a variational approach. The comparison with the computer experiment justifies the approximation.

Solution of the linear equation. To solve the linear equation (3.10), we use the Laplace transform with respect to the time t and the Fourier transform, or z-transform, with respect to the index n.

The Laplace transform

$$X_n(s) = \int_0^\infty x_n(t)\, e^{-st}\, dt, \qquad G_n(s) = \int_0^\infty g_n(t)\, e^{-st}\, dt$$

leads to the equation

$$s^2 X_n = X_{n+1} + X_{n-1} - 2X_n + G_{n-1}(s) - G_n(s) \quad (n = 0, \pm 1, \pm 2, \ldots) \tag{3.11}$$

where

$$G_n(s) = \frac{1 - e^{-qs}}{s(1 - e^{-ps})} e^{-\tau n s}, \quad G_{-n}(s) = G_n(s), \quad (n = 1, 2, 3, \ldots),$$

$$G_0(s) = \frac{1 - e^{-q_0 s}}{s(1 - e^{-ps})} e^{-\tau_0 s}.$$

The Fourier transform with respect to the index n (z-transform)

$$X(\phi,s)=\sum_{n=-\infty}^{+\infty}X_n(s)\,e^{in\phi},\quad G(\phi,s)=\sum_{n=-\infty}^{+\infty}G_n(s)\,e^{in\phi}$$

reduces the equation (3.11) to the form

$$(s^2-e^{-i\phi}-e^{i\phi}+2)X(\phi,s)=(e^{i\phi}-1)G(\phi,s) \tag{3.12}$$

where

$$\begin{gathered}G(\phi,s)=\frac{1}{s(1-e^{-ps})}\times \\ \left\{(1-e^{-qs})\left[\frac{e^{-\tau s+i\phi}}{1-e^{-\tau s+i\phi}}+\frac{e^{-\tau s-i\phi}}{1-e^{-\tau s-i\phi}}\right]+(1-e^{-q_0 s})e^{-\tau_0 s}\right\}.\end{gathered} \tag{3.13}$$

Inverse z-transform. From (3.12)-(3.13) we can find $X(\phi,s)$ and then making the inverse Fourier transform find $X_n(s)$. Let's consider $n\geq 1$ (the masses with $n\leq 0$ move symmetrically, see (3.3)) and make the substitution $z=e^{-i\phi}$; then

$$\begin{gathered}X_n(s)=\frac{-1}{s(1-e^{-ps})}\times \\ \frac{1}{2\pi i}\oint\left\{(1-e^{-qs})\left[\frac{1}{z-e^{-\tau s}}+\frac{z}{1-ze^{-\tau s}}\right]+(1-e^{-q_0 s})e^{-\tau_0 s}\right\}\frac{(z-1)z^{n-1}\,dz}{z^2-(s^2+2)+1}\end{gathered} \tag{3.14}$$

where the integration is carried out along the unit circle clockwise. In order to calculate this integral, we find the singularities of the integrand. The quadratic equation

$$z^2-(s^2+2)z+1=0 \tag{3.15}$$

has two roots whose product is 1. This equation has a root with absolute value 1 only when $\Re(s)=0$ and $-2<\Im(s)<2$. Let $\zeta(s)$ denote the root of the equation (3.15) whose absolute value is less than 1 when $\Re(s)>0$:

$$\zeta(s)=\frac{s^2+2-s\sqrt{s^2+4}}{2} \tag{3.16}$$

(here we consider the principal value of the square root, with argument between $-\pi/2$ and $\pi/2$); $1/\zeta(s)$ is the other root, whose absolute value is greater than 1 when s is in the right half-plane. Thus, when $\Re(s) > 0$, the integrand in (3.14) has two poles inside the unit circle: $z = \zeta(s)$ and $z = e^{-\tau s}$, and we find

$$X_n(s) = \frac{1 - e^{-qs}}{s(1 - e^{-ps})} \frac{\zeta(s)}{[\zeta(s) - e^{-\tau s}]\,[\zeta(s) - e^{\tau s}]} \tag{3.17}$$
$$\left\{ (e^{\tau s} - 1)e^{-n\tau s} + \frac{2e^{-\tau s} - \zeta(s) - 1/\zeta(s)}{\zeta(s) + 1}[\zeta(s)]^n \right\} + \frac{1 - e^{-q_0 s}}{s(1 - e^{-ps})} e^{-\tau_0 s} \frac{[\zeta(s)]^n}{\zeta(s) + 1}$$

($n = 1, 2, \ldots$). These functions are analytic in the right half-plane $\Re(s) > 0$. Indeed, when $\Re(s) > 0$, we have $|\zeta(s)| < 1$, $|e^{\tau s}| > 1$, so that the denominator in (3.17) vanishes only when $\zeta(s) = e^{-\tau s}$, but then the expression in parentheses also vanishes. This analyticity implies that $x_n(t) = 0$ when $t < 0$ ($n = 0, 1, 2, \ldots$).

In the right half-plane, $\Re(s) > 0$, we have $|\zeta(s)| < 1$ and $|e^{-\tau s}| < 1$; therefore, $X_n(s) \to 0$ and $x_n(t) \to 0$ when $n \to \infty$. This corresponds to the motion of distant masses that at the instant t are not yet reached even by the first front.

Remark about analytical continuation. The functions (3.17) can be analytically continued to the entire complex plane, besides cuts and poles on the imaginary axis. Indeed, the function $\zeta(s)$ can be analytically continued to the entire complex plane with cuts

$$\{s : \Re(s) = 0, \Im(s) > 2\} \quad \text{and} \quad \{s : \Re(s) = 0, \Im(s) < -2\}. \tag{3.18}$$

Then the functions $X_n(s)$ can be meromorphically continued to the complex plane with cuts (3.18). The formula (3.16) with principal value of the square root actually gives this analytical continuation.

The asymptotic periodicity of $x_n(t)$ as $t \to \infty$. The choice of parameter q. The asymptotics of $x_n(t)$ when $t \to \infty$ are determined by the singularities of $X_n(s)$ on the imaginary axis. These are the poles at the points

$$s = i\nu \;\text{ where }\; \nu = \frac{2\pi}{p}k \quad (k = 0, \pm 1, \pm 2, \ldots), \tag{3.19}$$

as well as at the points of the imaginary axis where

$$\zeta(s) = e^{\tau s}. \tag{3.20}$$

The last equation has solutions $s = i\Omega$ where $\Omega = \Omega(\tau)$ is a real non-zero roots of the equation (2.9). Note that at $s = 0$ the expression (3.17) has a simple pole included in the set (3.19). Thus,

$$x_n(t) \sim \sum_{k=-\infty}^{\infty} \mathcal{C}_k e^{i\frac{2\pi}{p}kt} \; + \; \left[\mathcal{D}e^{i\Omega t} + \mathcal{D}^* e^{-i\Omega t}\right] \qquad (t \to \infty). \tag{3.21}$$

where $\mathcal{C}_k$, $(k = 0, \pm 1, \pm 2, \ldots; \quad \mathcal{C}_k = \mathcal{C}^*_{-k})$ are the residues corresponding to the poles (3.19), and $\mathcal{D}$, $\mathcal{D}^*$ are the residues corresponding to the non-zero solutions $s = \pm i\Omega(\tau)$ of the equation (2.9).[3] We should also note that the analytic functions (3.17) have cuts (3.18) on the imaginary axis; also at the points $s = \pm 2i$ these functions have "one over square-root singularities", since $\zeta(s)+1$ vanishes at $s = \pm 2$ as $\sqrt{s \mp 2}$. These integrable singularities, as well as the cuts, contribute some terms to the asymptotics of $x_n(t)$ at $t \to \infty$. However, these terms approach zero as $t \to \infty$. Hence the asymptotics of $x_n(t)$ at $t \to \infty$ are determined only by the poles of $X_n(s)$ on the imaginary axis, and we indeed have (3.21). Asymptics (3.21) shows that $x_n(t)$ becomes p-periodic as $t \to \infty$ only if $\mathcal{D} = 0$. In turn this is realized if q takes some special value so that the factor $1 - e^{-qs}$ in the numerator of (3.17) vanishes at all zeros of the factor $[\zeta(s) - e^{\tau s}]$ in the denominator. This condition, which could be called the *pole cancellation condition*, gives a relation between the parameter q and τ:

$$q\Omega(\tau) = 2\pi \quad \text{where } \Omega(\tau) \text{ is the positive solution of the equation (2.9).} \tag{3.22}$$

In our computer experiment we find $\tau = 2.2 \;\Rightarrow\; \Omega = 1.8$ and $q = 3.4$, which are in agreement with (3.22).

The "swelling" distance α. The average value of $x_n(t)$ as $t \to \infty$ is determined by the residue of $X_n(s)$ at $s = 0$. Thus

$$\text{as } t \to \infty, \;\; x_n(t) \text{ oscillates around } \; \alpha(n - \frac{1}{2}) + \beta \tag{3.23}$$

where

$$\alpha = \frac{q\tau}{p(1+\tau)}, \qquad \beta = \frac{q - q_0}{2p}. \tag{3.24}$$

According to (3.23), the constant α is the average distance between adjacent masses, i.e. α is "the swelling distance". In our computer experiment we find $q = 3.4$, $\tau = 2.2$, $p = 5.3$ and $\alpha = 0.44$, which fits the theoretical result (3.24).

[3] We have assumed implicitly that the equation (2.9) has exactly one pair of non-zero solutions $\pm\Omega(\tau)$, which is true only for some range of τ. The value of parameter τ in our computer experiment is clearly inside this range.

Solution near the front of the phase transition. In order to obtain the characteristics of the phase transition (in particular the intermediate velocity v between the first and the second fronts), we go over to the frame of reference moving with the front of the phase transition (the second front). In other words, that we describe the system using "local" time $\eta = t - n\tau$ (around the instant when the second front reaches mass number n). We introduce the function

$$y_n(\eta) = x_n(n\tau+\eta) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} X_n(s)e^{(n\tau+\eta)s}\,ds = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} Y_n(s)e^{\eta s}\,ds \quad (3.25)$$

(σ is an arbitrary positive number). According to (3.17) we have

$$Y_n(s) \;=\; Y(s) + Z(s)\left[\zeta(s)\,e^{\tau s}\right]^n, \quad (3.26)$$

where

$$\begin{aligned} Y(s) &= \frac{(1-e^{-qs})(1-e^{\tau s})}{s(1-e^{-ps})(e^{-\tau s}+e^{\tau s}-s^2-2))}, \\ Z(s) &= \frac{1}{s(1-e^{-ps})}\left[\frac{(1-e^{-qs})(s^2+2-2e^{-\tau s})}{e^{-\tau s}+e^{\tau s}-s^2-2} + (1-e^{-q_0 s})e^{-\tau_0 s}\right]\frac{1}{\zeta(s)+1}. \end{aligned} \quad (3.27)$$

Here we have taken into account that $\zeta(s)$ and $1/\zeta(s)$ are the roots of the equation (3.15), so that their sum is s^2+2, and therefore,

$$\frac{\zeta(s)}{\left[\zeta(s)-e^{-\tau s}\right]\left[\zeta(s)-e^{\tau s}\right]} = -\frac{1}{e^{-\tau s}+e^{\tau s}-(s^2+2)}.$$

In the new frame the self-similarity property (3.5) takes the form

$$y_{n+1}(t) = \alpha + x_n(t) \qquad \text{for} \quad t > \tau n \quad (3.28)$$

The transformation of the integration contour in the inverse Laplace transform. Our idea now is to transform an integration line in (3.25) to some path C on which $|\zeta(s)e^{\tau s}| < 1$, so that the second term in the right hand side of (3.26) goes away as $n \to \infty$. In the vicinity of the imaginary axis, when $s = \sigma + i\omega$ (σ being small), we have

$$|\zeta(s)e^{\tau s}| = \begin{cases} 1+\sigma\left(\tau - \frac{2}{\sqrt{4-\omega^2}}\right) + O(\sigma^2) & \text{if } |\omega| < 2 \quad (\sigma \to 0), \\ \frac{\omega^2}{2} - \frac{|\omega|}{2}\sqrt{\omega^2-4} + O(\sigma) < 1 & \text{if } |\omega| > 2 \quad (\sigma \to +0). \end{cases}$$

Therefore $|\zeta(s)e^{\tau s}| < 1$ on the path C shown in the Figure 4. This path goes along the imaginary axis from $-i\infty$ to $+i\infty$. Whether the path goes to the left or to the right of the imaginary axis is determined by the constant

$$h = 2\sqrt{1 - \frac{1}{\tau^2}}. \tag{3.29}$$

If $|\omega| < h$, the path C goes on the left of the axis; if $|\omega| > h$, it goes on the right of the axis. Denoted by crosses in the Figure 4 are the poles (3.19) of the functions $Y_n(s)$ (or $X_n(s)$). According to our computer experiment (Figure 3), $\tau = 2.2 \;\Rightarrow\; h = 1.8$ and $p = 5.3 \;\Rightarrow\; \nu = 2\pi/p = 1.2$; therefore three of the poles (3.19) lie to the right of the path C, and the rest lies to the left of C. The integral (3.25) along a straight line from $\sigma - i\infty$ to $\sigma + i\infty$ $(\sigma > 0)$ can be replaced by the same integral along the path C, plus integrals along the small circles (counterclockwise) around the three poles that lie to the right of C (see Figure 4); the latter are given by the corresponding residues:

$$y_n(\eta) = \frac{1}{2\pi i}\int_C Y_n(s)e^{s\eta}\,ds + \mathrm{res}_{s=0}[Y_n] + e^{i\nu\eta}\mathrm{res}_{s=i\nu}[Y_n] + e^{-i\nu\eta}\mathrm{res}_{s=-i\nu}[Y_n]. \tag{3.30}$$

When $n \to \infty$, the function $Y_n(s)$ on the path C approaches $Y(s)$ and becomes independent of the index n. The second term in (3.30) is

$$\mathrm{res}_{s=0}[Y_n(s)] = \alpha(n - \frac{1}{2}) + \beta \tag{3.31}$$

(cf. (3.23)). In order for the self-similarity property (3.28) to take place, the last two terms in (3.30) should be independent of the index n: It means that the factor $Z(s)$ in (3.26) should vanish at $s = \pm i\nu$:

$$\frac{(1 - e^{-i\nu q})(2 - \nu^2 - 2e^{-i\nu\tau})}{e^{-i\nu\tau} + e^{i\nu\tau} + \nu^2 - 2} = -(1 - e^{-i\nu q_0})e^{-i\nu\tau_0},. \tag{3.32}$$

To satisfy this complex equation, we choose the appropriate values of the two real parameters τ_0 and q_0. Thus, asymptotically as $n \to \infty$ the formula (3.30) takes the form

$$y_n(\eta) = \frac{1}{2\pi i}\int_C Y(s)e^{s\eta}\,ds + [\alpha(n - \frac{1}{2}) + \beta] + e^{i\nu\eta}\mathrm{res}_{s=i\nu}[Y] + e^{-i\nu\eta}\mathrm{res}_{s=-i\nu}[Y].$$

Combining together the integral term and the last two residual terms, we re-write this formula in the form

$$y_n(\eta) = \frac{1}{2\pi i}\int_\Gamma Y(s)e^{s\eta}\,ds + \alpha(n - \frac{1}{2}) + \beta \tag{3.33}$$

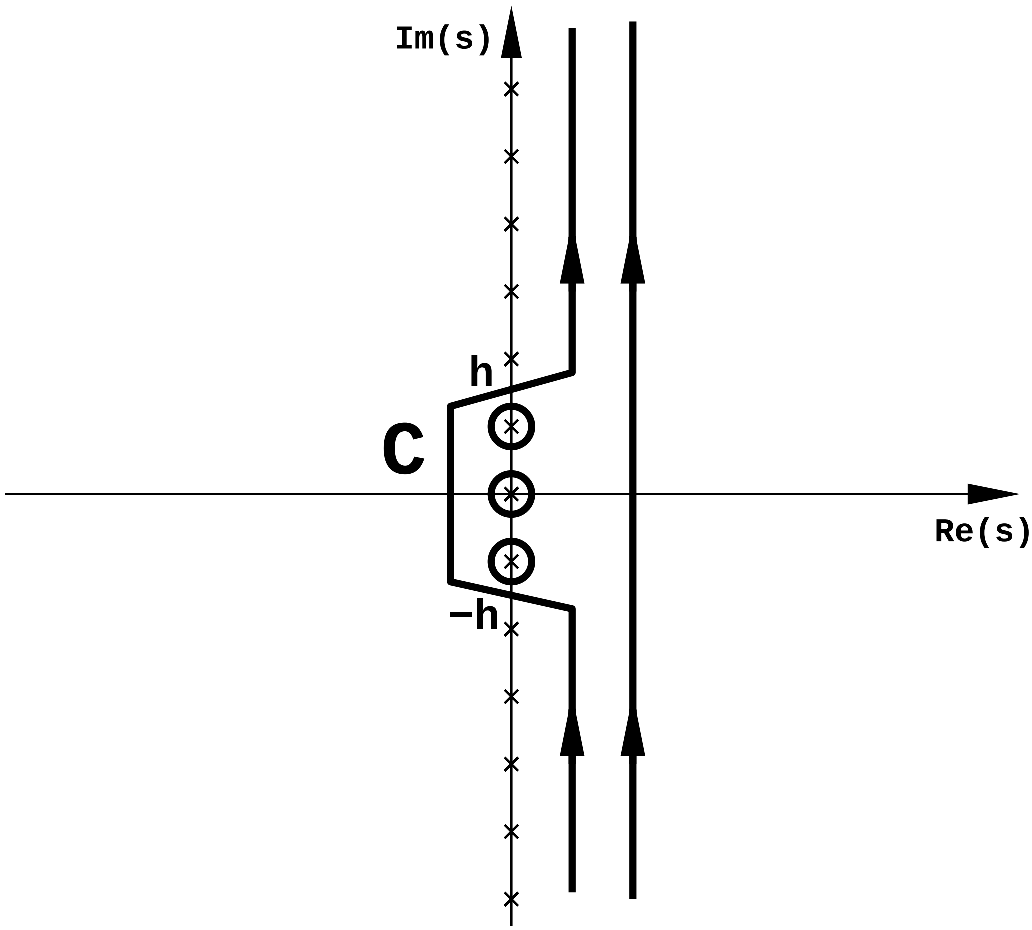

Figure 4: The line $\Im(s) = \sigma$ of integration in (3.25) is transformed to the path C. This path lies near the imaginary axis, in the right half-plane if $|\omega| > h$ and in the left half-plane if $|\omega| < h$. The poles (3.19) are denoted by crosses. Three of these poles, $s = 0$, $s = i\nu$, $s = -i\nu$, lie to the right of C, and the rest lies to the left of C.

where , is the path shown in the Figure 5; it goes from $-i\infty$ to $+i\infty$ near the imaginary axis passing all the poles (3.19) of the function (3.27) on the right, besides one pole at the origin, which is passed on the left.

An alternative approach to describe the self-similar solution near the front of the phase transition. We could find the solution near the front of the

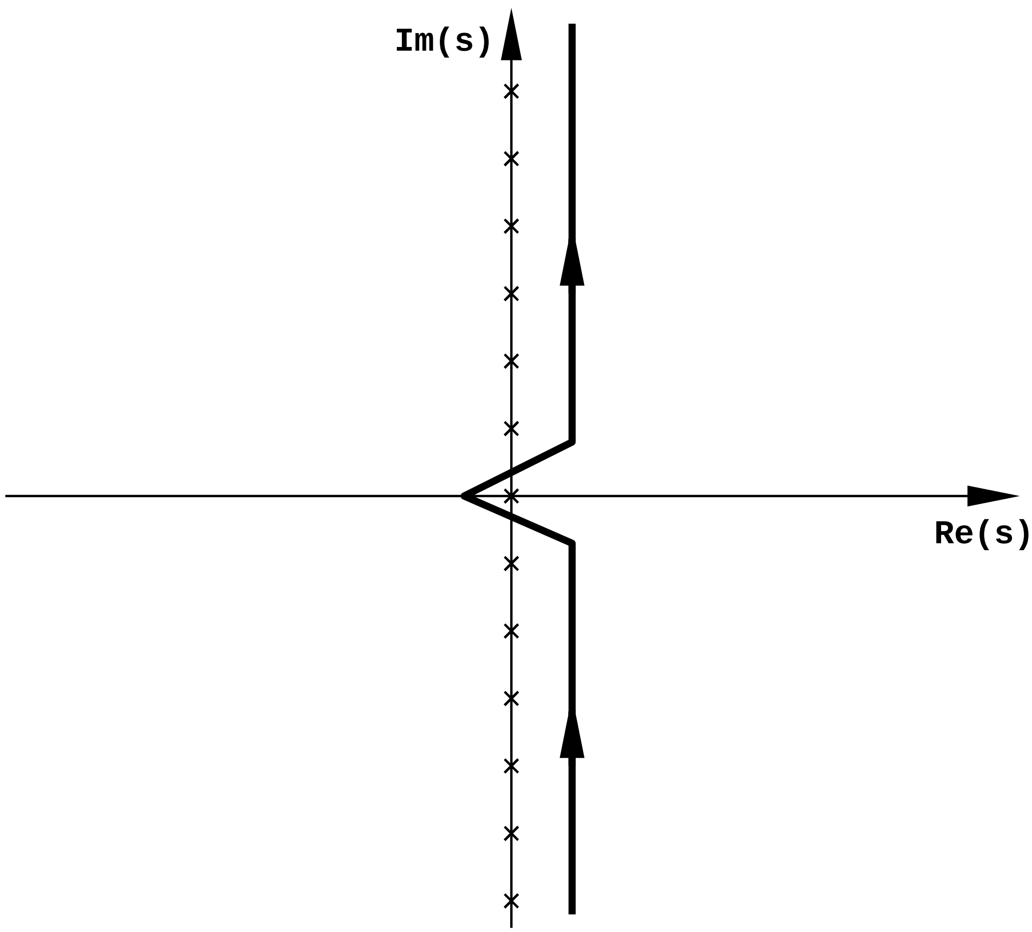

Figure 5: The path , . It lies near the imaginary axis; all the poles (3.19) lie to the left of , , besides one pole at $s = 0$ that lies to the right of , .

phase transition directly, assuming the self-similarity of this solution:

$$x_n(t) = y(t - n\tau),$$

where τ is an undetermined parameter, and $y(\eta)$ is a p-periodic function. For this function we would derive the following equation

$$\ddot{y}(\eta) = y(\eta + \tau) + y(\eta - \tau) - 2y(\eta) + [g(\eta + \tau) - g(\eta)]$$

where

$$g(\eta) = \sum_{j=0}^{\infty}[\Theta(\eta - jp) - \Theta(\eta - jp - q)]$$

(cf. (2.12),(2.11) and (3.10),(3.7)). For the Laplace transform of the function $y(\eta)$ we would find the expression (3.27). However, it would be unclear what contour we should take in the inverse Laplace transform. Our approach enabled us to derive the formula (3.33) with the integration over contour , , shown in Figure (5). Our approach also shows the necessity of two extra parameters τ_0 and q_0, which desribe the dynamics of the middle spring.

The intermediate speed v. In order to find the speed v between the first and second fronts (see Figure 3), we consider the asymptotics of the solution (3.33) as $\eta \to -\infty$. It is defined by the pole of $Y(s)$ at $s = 0$:

$$Y(s) = -\frac{1}{s^2}\frac{q\tau}{p(\tau^2 - 1)}[1 + s(p + \tau - q)] + O(1) \quad (s \to 0).$$

Therefore

$$y_n(\eta) \sim v\eta + n\alpha + \gamma \quad \text{as} \quad \eta \to -\infty$$

where the speed v is

$$v = \frac{q\tau}{p(\tau^2 - 1)}, \tag{3.34}$$

$\gamma = v(p + \tau - q) + \beta$, and the constants α and β are defined in (3.24). Comparing the expression (3.34) for the intermediate speed and the expression (3.24) for the "swelling" distance α, we obtain the *kinematic relation*

$$v(\tau - 1) = \alpha, \tag{3.35}$$

which has the following simple kinematic interpretation. It takes time 1 for the first front to propagate from mass number n to the mass number $(n+1)$; after this front the masses start moving with the speed v. It takes time $\tau > 1$ for the second front to propagate from from mass number n to the mass number $(n+1)$; after this front each mass oscillates around a fixed coordinate $n\alpha + \beta$. Thus the mass number $(n + 1)$ is moving with the speed v by time $\tau - 1$ longer than the mass number n, which is expressed by the relation (3.35).

It also follows from the formula (3.33) that the masses "almost" do not oscillate between the fronts. Indeed, the asymptotics $y(\eta)$ as $\eta \to -\infty$ is defined by the singularities of $Y(s)$ that lie on the imaginary axis and are located to the right of the path , ; there is only one such singularity — $s = 0$ (and there are no singularities with non-zero imaginary part, which would lead to oscillations).

Parameters p and τ. The nonlinear dispersion relation. Up to now we did not fix the values of the two parameters p and τ. We need to choose these values to make our solution self-consistent; we need to require that the distance between masses number n and number $(n+1)$ indeed vanishes (and switching between linear regimes of the spring force indeed occurs) at the "right" instants. Namely

$$x_{n+1}(t) - x_n(t) = 0 \;\text{ when }\; t = n\tau + pj \text{ and } t = n\tau + pj + q \tag{3.36}$$

(n=1,2,3,...; j=0,1,2,...; cf. (3.6)). The Laplace transform $X_n(s)$, given by the formula (3.14), is not a meromorphic function, and therefore, the solution $x_n(t)$ cannot be periodic. It is impossible to satisfy (3.36) exactly. However, the computer experiment suggests that $x_n(t)$ is periodic with sufficiently high accuracy. We satisfy the condition (3.36) approximately, and moreover consider this condition for large n, when we can use the asymptotic formula (3.33). Since $y_n(\eta) = x_n(n\tau + \eta)$, we have

$$x_{n+1}(n\tau + \eta) - x_n(n\tau + \eta) = \frac{1}{2\pi i}\int_\Gamma Y(s)(e^{-\tau s} - 1)e^{s\eta}\, ds + \alpha.$$

Therefore, the condition (3.36) takes the form

$$\frac{1}{2\pi i}\int_\Gamma Y(s)(e^{-\tau s} - 1)e^{s\eta}\, ds + \alpha = 0 \tag{3.37}$$

for instants

$$\eta = pj, \quad \eta = pj + q \quad (j = 0, 1, 2, \ldots). \tag{3.38}$$

We satisfy this condition asymptotically for instants (3.38) as $j \to \infty$. Asymptotics of $y_n(\eta)$ as $\eta \to +\infty$ are determined by the poles (3.19), and we find that as $\eta \to +\infty$, the chain performs the wave motion described by the formulas (2.13),(2.15). Hence we arrive at the nonlinear dispersion relation (2.17) (see Section 2). Since the parameter α is given now by the formula (3.24), the dispersion relation represents an equation connecting the wave period p, the wave speed $1/\tau$, and the parameter q (the latter plays the part of the wave amplitude).

Thus the asymptotic considerations of this section give us three equations for the four parameters p, q, τ, and α. We also need to require that the condition (3.36) holds at $t = \tau n$, i.e. the wave of phase transition indeed reaches the mass number n at the instant $t = n\tau$ (see (3.6)). This condition is the equation (3.37) with $\eta = 0$:

$$\frac{1}{2\pi i}\int_\Gamma \frac{(1-e^{-qs})(1-e^{\tau s})(e^{-\tau s}-1)}{s(1-e^{-ps})(e^{-\tau s}+e^{\tau s}-s^2-2))}\,ds + \alpha = 0. \tag{3.39}$$

It represents the fourth equation for the four parameters p, q, τ and α.

4 Conclusion

In computer experiments we found a new regular pattern of dynamical phase transition in the mass-spring chain (see Figure 3). The phase transition has two fronts propagating with different speeds. Due to this transition the mass-spring chain transfers from the stationary state, when all the masses are at rest, to the twinkling state, when the masses perform a wave motion. The pattern is well reproduced and has several quantitative characteristics, e.g. (1) the time period p of waves, (2) the "swelling" distance α, (3) the speed $1/\tau$ of the second wave front, (4) the intermediate speed v (see the end of Section 1).

In order to approach this strongly nonlinear system analytically, we have considered the special form of the force-vs.-elongation dependence (see Figure 2); the dependence consists of two linear parts with *the same* slope.

Analytically we have found a three parameter family of exact solutions, which represent the nonlinear waves (extending from $-\infty$ to $+\infty$). The wave motion of the twinkling phase, which we observed in the computer experiment, represents such a nonlinear wave. (One would observe this nonlinear wave if he is in "the middle" of the twinkling phase and knows nothing about the boundaries, as well as about the "pre-history".)

A natural question arises: "What particular wave out of that three-parameter family of nonlinear waves is realized in computer experiment?" In order to answer this question, we have analytically considered the entire transition dynamics. This has lead us to the system of algebraic equations for the parameters p, τ, α and a certain parameter q. The latter is similar to the wave amplitude in the general theory of nonlinear waves; it indicates the instants of switching between the linear parts of the force dependence.

We summarize here the main steps in the derivation of this nonlinear system.

1. We require that the solution that we construct is asymptotically time-periodic as $t \to +\infty$. This is possible only if a certain cancellation of poles occurs. This requirement (the pole cancellation condition, see Section 3) gives us the equation (3.22), which determines q in terms of parameters τ:

$$q\Omega(\tau) = 2\pi, \tag{4.1}$$

where $\Omega(\tau)$ is defined by the linear dispersion relation (2.9): $\Omega(\tau)$ is the positive root of the equation

$$\Omega = 2 \sin \frac{\Omega\tau}{2}.$$

2. We consider the asymptotics of our solution as $t \to +\infty$ and find "the swelling distance" α in terms of the parameters p, q, τ (formula (3.24)):

$$\alpha = \frac{q\tau}{p(1+\tau)}. \tag{4.2}$$

3. We consider our solution near the second front, propagating with the speed $1/\tau$. Moreover, we consider the asymptotic regime when $n \to +\infty$ and $t \to +\infty$, but $\eta = t - n\tau$ stays finite. The requirement that our solution asymptotically represents a self-similar wave defines the parameters τ_0 and q_0 (characterizing the dynamics of the "middle" spring, see (3.8) and (3.32)) and leads to the asymptotic formula (3.33):

$$x_n(n\tau + \eta) = \frac{1}{2\pi i} \int_\Gamma Y(s) e^{s\eta} \, ds + \alpha(n - \frac{1}{2}) + \beta. \tag{4.3}$$

Here the function $Y(s)$ is given by (3.27):

$$Y(s) = \frac{(1 - e^{-qs})(1 - e^{\tau s})}{s(1 - e^{-ps})(e^{-\tau s} + e^{\tau s} - s^2 - 2))}, \tag{4.4}$$

the path , is shown in the Figure 5, the "swelling distance" α and the constant β are defined in (3.24). The singularities of the function $Y(s)$ on the imaginary axis and the form of the path , (see Figure 5) define the asymptotical behavior of the integral (4.3) as $\eta \to \pm\infty$. The behavior at $\eta \to -\infty$ is defined by the only pole to the right of , — the double pole at $s = 0$. We find that the function (4.3) asymptotically as $\eta \to -\infty$ grows linearly with η (and there are no oscillations in the intermediate region, between the fronts). This gives us a simple kinematic expression for the intermediate velocity v

$$v = \frac{\alpha}{\tau - 1} \tag{4.5}$$

(see (3.35) and the physical interpretation after that equation).

The simple poles of $Y(s)$, located on the imaginary axis to the left of the path , , determine the asymptotic behavior of (4.3) at $\eta \to +\infty$: asymptotically solutions $x_n(n\tau + \eta)$ become p-periodic functions and take the form of a nonlinear wave found in Section 2. Thus we have the nonlinear dispersion relation (2.17):

$$\alpha = \sum_{k=1}^{\infty} \frac{\sin^2 \frac{k\nu\tau}{2}}{\left(\frac{k\nu}{2}\right)^2 - \left(\sin \frac{k\nu\tau}{2}\right)^2} \frac{\sin k\nu q}{\pi k} \qquad (\nu = 2\pi/p). \tag{4.6}$$

4. The last equation on the parameters comes from the requirement that the function $x_{n+1}(n\tau) - x_n(n\tau) = 0$, so that the masses indeed transfer to the twinkling state when the second front reaches them. Using the asymptotic formula (4.3), we arrive at the equation (3.39):

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{(1 - e^{-qs})(1 - e^{\tau s})(e^{-\tau s} - 1)}{s(1 - e^{-ps})(e^{-\tau s} + e^{\tau s} - s^2 - 2))} \, ds + \alpha = 0. \tag{4.7}$$

The four equations (4.1),(4.2),(4.6),(4.7) define the values of the parameters p, q, τ, and α. Once these parameters are determined, the formulas (4.3),(4.4) determine the self-similar wave of phase transition.

Comparisson with Numerics. Our analysis is based on the hypothesis that the solution represents wave motion when $t > n$ (i.e. after the front of phase transition has passed the mass). We would like to check this hypothesis not only qualitatively, but also quantitatively. In order to do this, we check that the values of the parameters p, q, τ, α, found in computer simulation, agree with the four equations (4.1),(4.2),(4.6),(4.7) In our computer experiments we found $p = 5.3, q = 3.4, \tau = 2.2, \alpha = 0.44$. These values indeed agree (within the measurement accuracy) with the four equations.

Turbulence. We should note that the regular pattern shown in the Figure 3 starts to disintegrate after about ten periods of oscillations. If we continue to compute further (beyond the time interval of $t \approx 50$ presented in the Figure 3), the oscillations become irregular and chaotic. At present we are not sure whether this disintegration is due to the numerics or is inherent to the real nonlinear dynamics. Probably, this disintegration takes place because of the instability of the nonlinear wave (in the oscillating phase). In the following papers we intend to describe the developed turbulence of random nonlinear waves in this system.

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