Dynamics of chains with non-monotone stress-strain relations. 1. Model and Numerical Experiments. *

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Abstract

We discuss the dynamical processes in the materials with non-monotonic constitutive relation. We introduce a model of a chain of masses joined by springs with a non-monotone strain-stress relation. Numerical experiments are conducted to find the dynamics of that chain under slow external excitation. We find that the dynamics leads either to a vibrating steady state (*twinkling phase*) with radiation of energy, or (if a dissipation is introduced) to a hysteresis, rather than to an unique stress-strain dependence that would correspond to the energy minimization.

1 Introduction

The equilibrium of an elastic body with non-convex energy has been investigated by many authors, starting with [1, 2, 3]. The problem of finding the equilibrium arises in natural applications, such as phase separation or phase transition in solids. Recently, this problem was investigated in a number of papers. The mathematical formulation, and the martensite phase separation was discussed in [4, 5, 6, 7]. Examples are given by martensite phase transitions [4, 5], the shape memory alloys [7, 8], and similar phenomena of spontaneous self-organization of a complicated mosaic structure in materials [3, 9]. The computational aspects of the solution of these problems were highlighted in [10], see also references therein. The specific feature of the problem is the multiplicity of the equilibrium positions that correspond to meta-stable local minima of energy, see [12, 14]. The variational principle (Gibbs principle) is commonly used to select the equilibrium that minimizes the total energy of the system. The problem of the layout of the phases in the material under phase transition thus becomes a non-convex variational problem.

The problem of minimization of non-convex energy also occurs in structural optimization where the non-convexity of the minimizer follows from the optimality requirement, see e.g. [13]. These problems involving complicated materials are in many respects similar to structural optimization. In both cases, one deals with several materials or solid phases that are distributed in a domain in a specific way. The optimality requirement posed by a designer is similar to a natural variational principle of minimization of the total energy of the system. The transformation to another phase is similar to the use of another material in a design. Minimizing its energy, the system exhibits phase separation and forms a sort of natural composite that possesses an optimal microstructure.

The mentioned similarities suggest that similar mathematical methods could be applied to describe the natural mixtures with minimal energy. This concept was put forward and implemented to various systems in the works by Ball, Bhattacharia, Ericksen, Grinfield, James, Khachaturyan, Kinderlehrer, Kohn, Luskin, Rogers, Roitburd, Rosakis, Truskinovsky, and others. The methods of quasiconvexity are successively implemented for explanation of structures arising in some natural phase transitions; we refer to the works of the above mentioned authors.

However, the mentioned natural phenomena are much more complicated than the problems of structural optimization. Indeed, the best engineering system should reach the global minimum of the minimizing functional. On the contrary, a steady state of a natural system corresponds to any local minimum of the energy. Contrary to an optimal engineering construction, a realizable steady state of a natural system corresponds to a stable dynamical process that has led to it. The dynamics of phase transition allows the realizable minima to be identified. We mention several papers [14, 8, 15, 16] devoted to the dynamics of phase transition.

Finally, natural composites usually are a random mixture of the states that correspond to local minima. The search for a distribution of local minima requires different techniques than the techniques for the global minimum.

In dealing with an unstable material, it is not obvious what continuum model should be used. Dynamical parameters of the transition, such as the speed of the transition wave, are postulated in the continuum model. Here we develop another approach (see also [14, 17, 18]) that deals with a discrete model of unstable solids.

2 Chain of unstable elements

2.1 Gibbs principle.

It is generally accepted that the phase transition corresponds to a variational problem with non-convex energy in accordance with Gibbs variational principle (see e.g. [6, 4, 8, 7]).

Sometimes, the solid can exist in several different forms (as diamond and coal, martensite and austenite, etc). These forms are characterized by different forms of the elastic energy Π_i $(i = 1, 2, \ldots, n$, where n is the number of forms). The Gibbs principle states the mechanism for selection of these forms: the form with lowest energy is actually realized. The energy of the system becomes:

$$W_0(\nabla \boldsymbol{u}) = \min\left\{\Pi_1(\nabla \boldsymbol{u}),\ \Pi_2(\nabla \boldsymbol{u}), \ldots, \Pi_n(\nabla \boldsymbol{u})\right\}, \tag{2.1}$$

where $W_0(\nabla \boldsymbol{u})$ is the energy of the body, $\Pi_1(\nabla \boldsymbol{u})$, $\Pi_2(\nabla \boldsymbol{u})$,..., $\Pi_n(\nabla \boldsymbol{u})$ are the energies of the different forms. The energy $W_0(\nabla \boldsymbol{u})$ is non-convex; it is the minimum of several convex functions $\Pi_i(\nabla \boldsymbol{u})$ called wells; $W_0(\nabla \boldsymbol{u})$ is called the multiwell energy.

Phase separation. The non-convexity of $W_0(\nabla \boldsymbol{u})$ implies that the material can lose stability. When the homogeneous external strain $\nabla \boldsymbol{U} = \boldsymbol{V}_0$ =constant is applied to an unbounded region occupied by the material with the energy $W_0(\nabla \boldsymbol{u})$, it leads to a non-constant minimizer $\nabla \boldsymbol{u}(x)$. In other words, the solution with minimal energy turns out to be inhomogeneous. The minimizer (the strain) is discontinuous and it never takes values in a certain "forbidden" region of values. Here we deal with the one-dimensional case, and $\nabla \boldsymbol{u} = u_x$. In this case, the forbidden region is the region of nonconvexity of $W_0(\boldsymbol{u}_x)$. This phenomenon is called *phase separation.* When the energy tends to its minimum, the wells (phases) are fast alternated which results in the media with a microstructure.

The minimal energy of the effective medium system corresponds (in the one-dimensional case) to the convex envelope $\mathcal{C}W_0$ of W_0:

$$\mathcal{C}W_0(\boldsymbol{V}_0) = \min_{(c_i, \boldsymbol{v}_i)\in M} \sum_i c_i \Pi_i(\boldsymbol{v}_i) \le W_0(\boldsymbol{u}_0)$$

where

$$M = \left\{(c_i, \boldsymbol{v}_i): \quad c_i \ge 0, \sum c_i = 1, \sum c_i \boldsymbol{v}_i = \boldsymbol{V}_0\right\}$$

One replaces the multi-well energy with its convex envelope to describe the behavior of the unstable system.

Remark 2.1 *In the multidimensional problem, one needs less than convexity of multidimensional energy to ensure stability of minimizers. The needed property is quasiconvexity; it is discussed in detail in many papers, see e.g. [13] and the references therein. The reason to replace convexity with quasiconvexity is the required continuity of the deflection vector* $\boldsymbol{u}$*, which implies the vanishing of* $\nabla \times (\nabla \boldsymbol{u})$*. There exists an intensive literature on the quasiconvex envelope (see e.g. [13]). However, here we consider the system with one spatial coordinate, which is enough for our goal. In this case, quasiconvexity and classical convexity coincide.*

2.2 Model

Let us specify a simple one-dimensional model of a material with two-well energy (see Figure 1):

$$W_0 = \min\{\Pi_1,\ \Pi_2\}$$

where

$$\Pi_1 = \frac{D}{2}u_x^2, \quad \Pi_2 = \frac{D}{2}(u_x - 1)^2 + a, \quad W_0(u_x) = \{\Pi_1, \Pi_2\}.$$

This definition implies that

$$\begin{array}{lll} W_0 = \Pi_1 & \text{if} & u_x < v_c, \\ W_0 = \Pi_2 & \text{if} & u_x > v_c, \end{array}$$

the critical point, $v_c : (\Pi_1(v_c) = \Pi_2(v_c))$ is

$$v_c = \frac{a + D}{2D}.$$

We refer to these two branches with the energies Π_1 and Π_2 as "short" (u_x is smaller than v_c) and "long" (u_x is greater than v_c).

We choose the boundary conditions that describe a symmetric elongation of the sample:

$$u(-1) = -r, \quad u(+1) = r;$$

$L = 2r$ is the total elongation of the sample.

The calculation of the convex envelope $\mathcal{C}W_0$ of W_0 shows that it is equal to

$$\mathcal{C}W_0(v) = \begin{cases} \Pi_1(v) & \text{if} \quad v \le v_1, \\ \Pi_1(v_1) + v_c\,(v - v_1) & \text{if} \quad v_1 \le v \le v_2, \\ \Pi_2(v) & \text{if} \quad v \ge v_2, \end{cases} \tag{2.2}$$

where v_1 and v_2 are the boundaries of the interval of non-convexity of the energy W

$$v_1 = -\frac{a}{D}, \quad v_2 = 1 - \frac{a}{D}.$$

This system exhibits phase separation in the interval $u_x \in [v_1,\ v_2]$ of applied strains.

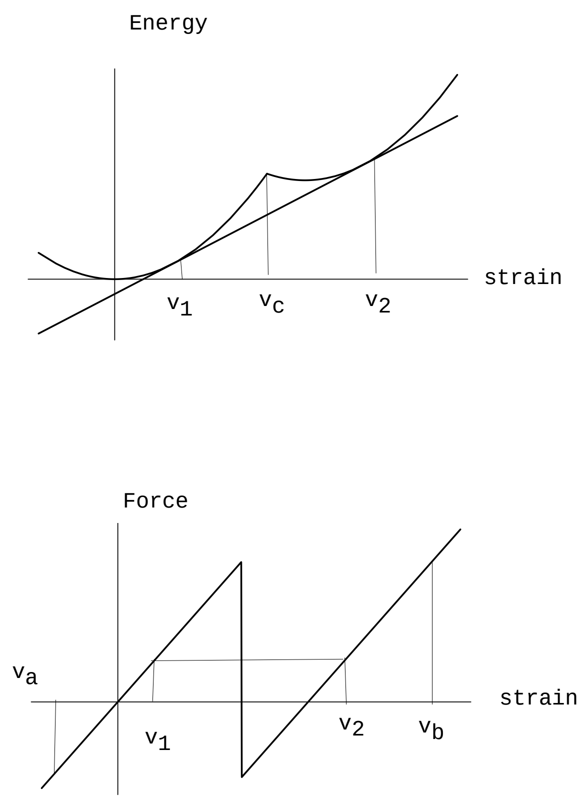

Figure 1: The two-well energy and the corresponding stress-strain relation. Here $[v_a, v_b]$ is the interval of non-uniqueness of strain for a given force and $[v_1, v_2]$ is the interval of non-convexity of the energy.

Stress-strain relation. Maxwell line. The stress $\sigma = \frac{\partial W_0}{\partial u_x}$ is computed as

$$\sigma(u_x) = \begin{cases} \sigma_1 = & D u_x & \text{if} & u_x < v_c \\ \sigma_2 = & D(u_x - 1) & \text{if} & u_x > v_c \end{cases}$$

and is a non-monotone function of u_x.

The equation $\sigma(u_x) = \gamma$ has two stable solutions

$$(u_x)_{short} = \frac{\sigma}{D} \quad \text{and} \quad (u_x)_{long} = \frac{\sigma - 1}{D}$$

in the interval $[v_a, v_b]$ (see Figure 1). Both solutions are locally stable: they correspond to local minima of the energy. The principle of minimal energy selects the solution that gives the global minimum of the energy; it selects the first solution in the interval $u_x \in [v_a, v_c]$ and the second solution in the interval $u_x \in [v_c, v_b]$ (see Figure 1).

The existence of two locally stable solutions explains the phase transition phenomenon. The chain can remain at rest even if some springs possess the strain v_{short} while the others — strain v_{long} (see Figure 1), provided these stresses correspond to the same elastic force

$$\sigma(v_{short}) = \sigma(v_{long}).$$

The system remains in locally stable equilibrium when the strain jumps from v_{short} to v_{long} and back in some elements of the chain. The principle of minimum energy determines transfers between branches and selects the best configuration. The resulting effective stress σ_{eff} in the convexified system is computed by using the effective energy and is equal to

$$\sigma_{eff} = \frac{\partial \mathcal{C}W}{\partial u_x} = \begin{cases} D\, u_x & \text{if} & u_x < v_1 & , \\ D\, v_1 & \text{if} & v_1 < u_x < v_2, \\ D\,(u_x - 1) & \text{if} & u_x > v_2 & . \end{cases}$$

The effective stress is monotone. The linear part of the convex envelope corresponds to the interval of constant stress $\sigma(v) = constant$. The horizontal line on the stress-strain graph corresponds to the Maxwell rule, which states that the areas below and above the line are equal to each other (see Figure 1). The Maxwell rule for the stress-strain relation leads to the convex envelope of the energy and vice versa.

The increase of an external elongation L, i.e. the increase of the mean strain u_x corresponds to the following variation of the state of minimal energy.

- In the interval of strains $u_x \in [-\infty,\ v_1]$, the local strain $v = v_{short}$ belongs to the "short" branch of the stress-strain curve (the second solution $v = v_{long}$ corresponds to a larger energy).

- In the interval of strains $u_x \in [v_2,\ \infty]$, the local strain $v = v_{long}$ belongs to the "long" branch of the stress-strain curve.

Figure 2: A mass-spring chain.

- In the interval of strains $u_x \in [v_1,\ v_2]$, the local strain alternates between two values v_1 and v_2 that belong to the "short" and to the "long" branches of the stress-strain curve. The total strain is equal to $cv_1 + (1-c)v_2$, where c is the fraction of the length where $v = v_1$.

  Variation of the external strain leads to the variation of the fractions c and $1-c$ of the phases that belong to the "short" and to the "long" branches, respectively, or to variation of the rate of the phase transition. The additional strain is achieved exclusively due to variation of the phase transition rate. Therefore the stress is constant.

The described quasistatic model of the transition looks consistent. However, questions about its applicability arise when the dynamics of the process is taken into account. Will the principle of minimal energy prevail if the inertia is taken into account, and the dynamics is considered? There is no doubt that the above configuration corresponds to a global minimum of the potential energy. However, there are many local minima of the energy, and it remains to be explained whether or not the system chooses the global minimum.

Remark 2.2 *Our model uses the piece-wise quadratic energy and piece-wise linear constitutive relation, which, however, contains a discontinuity. This model is clearly the simplest one that describes the non-monotonic stress-strain relation. The other commonly used model, the Ginsburg-Landau model, uses the polynomial energy of the type* $W(u_x) = (u_x^2 - 1)^2$.

The justification of the suggested model will be evident from the next sections: it is simple, it catches complicated dynamical behavior of the phase transition, and it is integrable.

2.3 Dynamics

We intend to check the principle of minimal energy by modeling the dynamics of the transition.

The model To be specific, let us consider a mass-spring chain, Figure 2, which is described by the following equations

$$\rho \ddot{u}_i = \sigma(u_i - u_{i-1}) - \sigma(u_i - u_{i+1}), \quad i = \ldots, -2, -1, 0, 1, 2, \ldots \tag{2.3}$$

Here x_i is the coordinate of the ith mass; ρ is the mass. The function $\sigma(v)$, defining the stress-strain relation of one spring, has the same form as in the continuous model (see Figure 1). There is an important advantage of this discrete model in comparison with the continuous model considered in the previous section. The discreteness allows us to consider vibrations

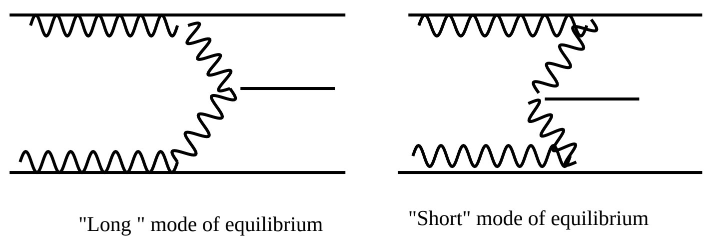

Figure 3: The first model of system with multiple equilibrium.

of individual masses. On the contrary, the continuous model assumes that the motions of neighboring masses are similar to each other.

Substituting the expression for $\sigma(v)$ into (2.3), we rewrite it in the form

$$\rho \ddot{u}_i = \mathcal{L}(u_i) + \mathcal{N}(u_i); \tag{2.4}$$

$$\mathcal{L}(u_i) = u_{i+1} - 2u_i + u_{i-1}, \tag{2.5}$$

$$\mathcal{N}(u_i) = \Theta(u_i - u_{i-1} + a) - \Theta(u_{i+1} - u_i - a). \tag{2.6}$$

Here, $\mathcal{L}$ is the linear part of the elastic force, and $\mathcal{N}$ is a nonlinear part, and it is assumed that $D = 1$ (compare with Section 2.2).

Notice that $\mathcal{N}(u_i) = 0$ if the magnitude of the relative oscillations is relatively small, $|u_i - u_{i-1}| < v_c, \ \forall i, \ \forall t$. In this case, the equation of motion of the chain becomes

$$\rho \ddot{u}_i = \mathcal{L}(u_i); \tag{2.7}$$

the homogenized equation (the limit $\rho \to 0, \ a \to 0$) obviously is

$$\ddot{u}_i = D u_{xx}.$$

If the magnitude is larger than the critical value v_c, the nonlinear force $\mathcal{N}$ emerges. In our model, the force $\mathcal{N}(t)$ at each moment t may take one out of only three values: $1, 0$, and -1. The process would be completely described by the specification of the instants of the application and release of the force.

2.4 The mechanical models

We suggest here two models of mechanical systems made of unstable elements.

Chain of springs with multiple equilibria. The first chain consists of the springs which can be in two equilibria as shown in the Figure 3. A similar construction can be made of shells that can flip-flop to either of two equilibria.

Notice that each element possesses two locally stable equilibrium positions, and the chain of N elements is characterized by 2^N equilibria.

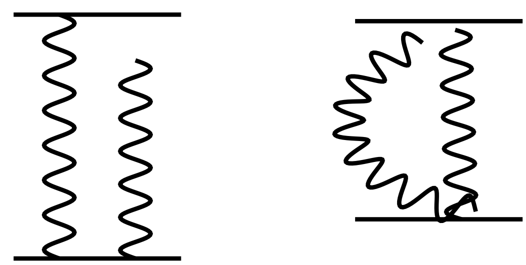

Figure 4: The second model of system with multiple equilibrium.

The chain of unstable "waiting elements". The second model is a chain that consists of elements that display Euler instability. Suppose that the adjacent masses in the chain are connected by a couple of elastic columns. Figure 4 shows one element of such chain. When the element is compressed by a sufficiently large force, greater than some critical value, the longer (left) column looses its stability and bends. After this, the shorter element takes the load, and the system stabilizes itself again. When the external load becomes smaller than the critical value, the lower columns becomes straight again and shows its original resistance. If we neglect the resistance that the "long" column shows after it has been bent, then the constitutive relation of one element has the form shown in the Figure 1.

3 Results of computer experiments

In this section, we describe the results of simulation of dynamics of the chain. We observe that the simple model (2.4) leads to a sophisticated dynamics, that include the wave of phase transition, radiation, randomization, and other phenomena typical for the phase transition.

3.1 Wave of phase transition

We consider a system of even number of masses $2N$. Initially all the masses are at rest: $\dot{u}_n(0) = 0$. All springs but the one in the middle are pre-stretched to a length slightly less than the critical value v_c: $u_i - u_{i-1} = v_c - \epsilon, i \neq N$. The middle spring is pre-stretched above this critical value $u_N - u_{N-1} = v_c + \epsilon$ (which we interpret as a fluctuation). This elongation initiates the dynamical process. The first and the last masses are kept at rest:

$$u_1(t) = 0, \quad u_{2N}(t) = (2N-1)v_c - (2N-3)\epsilon, \ \forall t > 0.$$

We numerically integrate the equations of the chain with these initial and boundary conditions using MATLAB.

We observe waves of phase transition (see Figure 5). They propagate with a constant speed and reflect from the boundaries of the chain. One can also see a shock wave that appear when a direct wave of phase transition collides with the reflected wave.

After the wave of phase transition, the chain enters an oscillatory regime which we call "the twinkling phase". In this regime, the springs oscillate between the "short" and the "long" states. The regular periodic motion lasts for about 10 periods of oscillations. After this, the motion of the chain becomes random.

Prior to the wave of phase transition, the sonic wave propagates (see Figure 5). Prior to the sonic wave, the masses are in rest and springs are in equilibrium. In the intermediate region (between the sonic wave and the wave of phase transition), the masses move with almost constant speed; the springs are contracted and are in the linear regime. This can be explained as follows. After the phase transition, the mean distance between neighboring masses becomes larger. Consequently, the distance between outer masses in the intermediate zone must become smaller. The appearance of this forerunning wave was predicted in [16] from conservation laws in the continuum model. Generally, if the constitutive relations are not piece-wise linear, the forerunning wave is a shock wave.

In the continuum limit, when the chain is infinitely long, it radiates the energy: a part of the energy is sent away from the transition zone in the form of sonic waves.

3.2 Radiation

To model the effect of radiation, the following computer experiment was performed (see Figure 6).

The chain initially rests at a position when the elongation $v_0 = u_i - u_{i-1}$ of each spring is smaller than the critical elongation v_c, but larger than the elongation v_1 of the Maxvellian transition (see Figure 1). In other words, the energy of the system is higher than the minimal energy W_* which corresponds to the convex envelope. The process is initiated by the large ($> v_c$) initial elongation of the middle spring, interpreted as "fluctuation". The ends of the chain are kept at rest.

The dynamics is governed by two competing processes. The transition from the metastable initial state to the stable state is accompanied by the energy release: Each mass releases energy after it jumps over the barrier. Due to this process the kinetic energy of oscillations would increase.

On the other hand, each jump of a mass initiates a sonic wave which carries the energy away from the zone of transition. This process can compensate (or even dominate) the release of energy. The example shown in Figure 6 demonstrates that the amount of energy sent away in the form of waves is larger than the released energy surplus. The transition process dries out and stops at a local minimum.

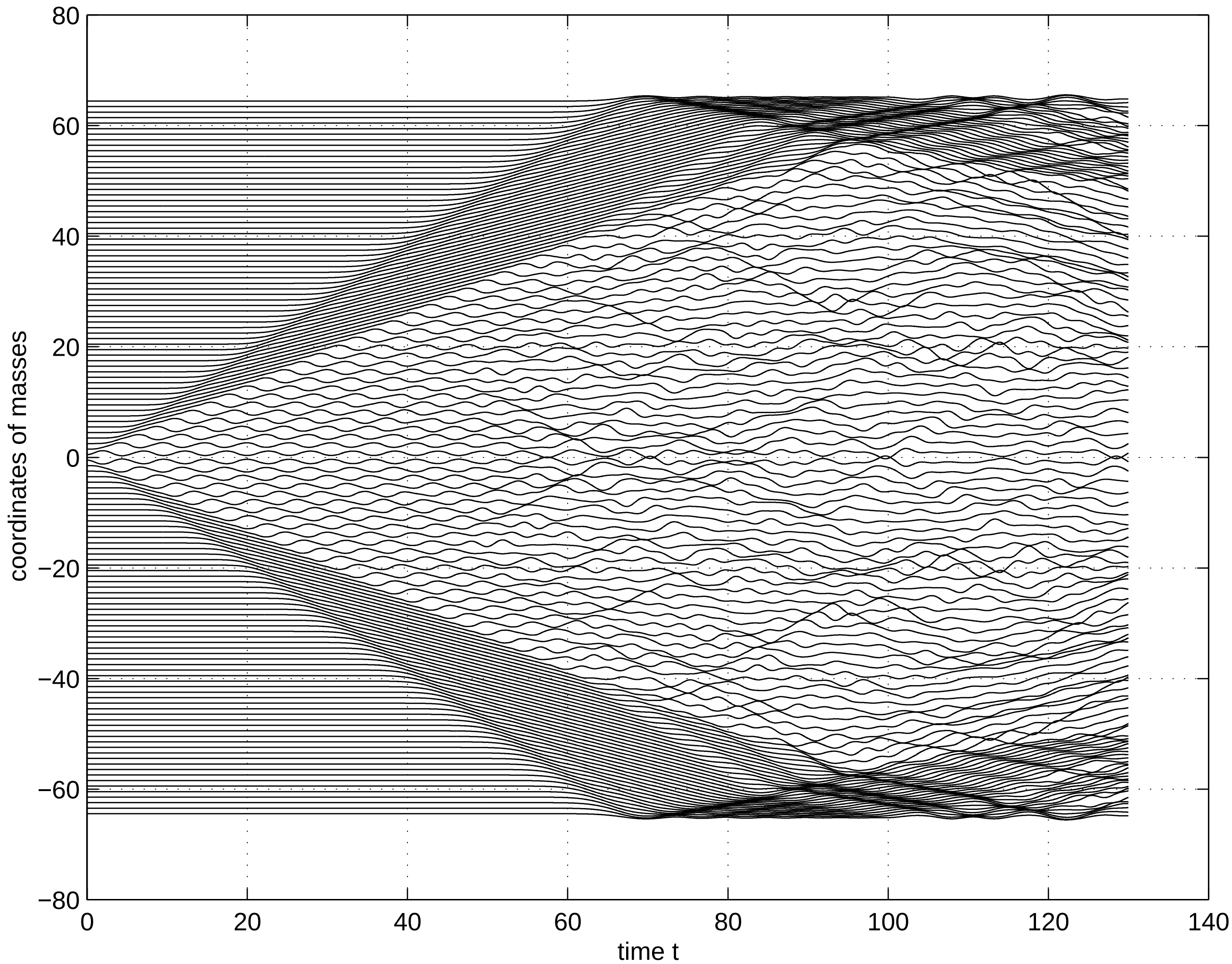

Figure 5: The motion of the chain with $N = 130$ masses.

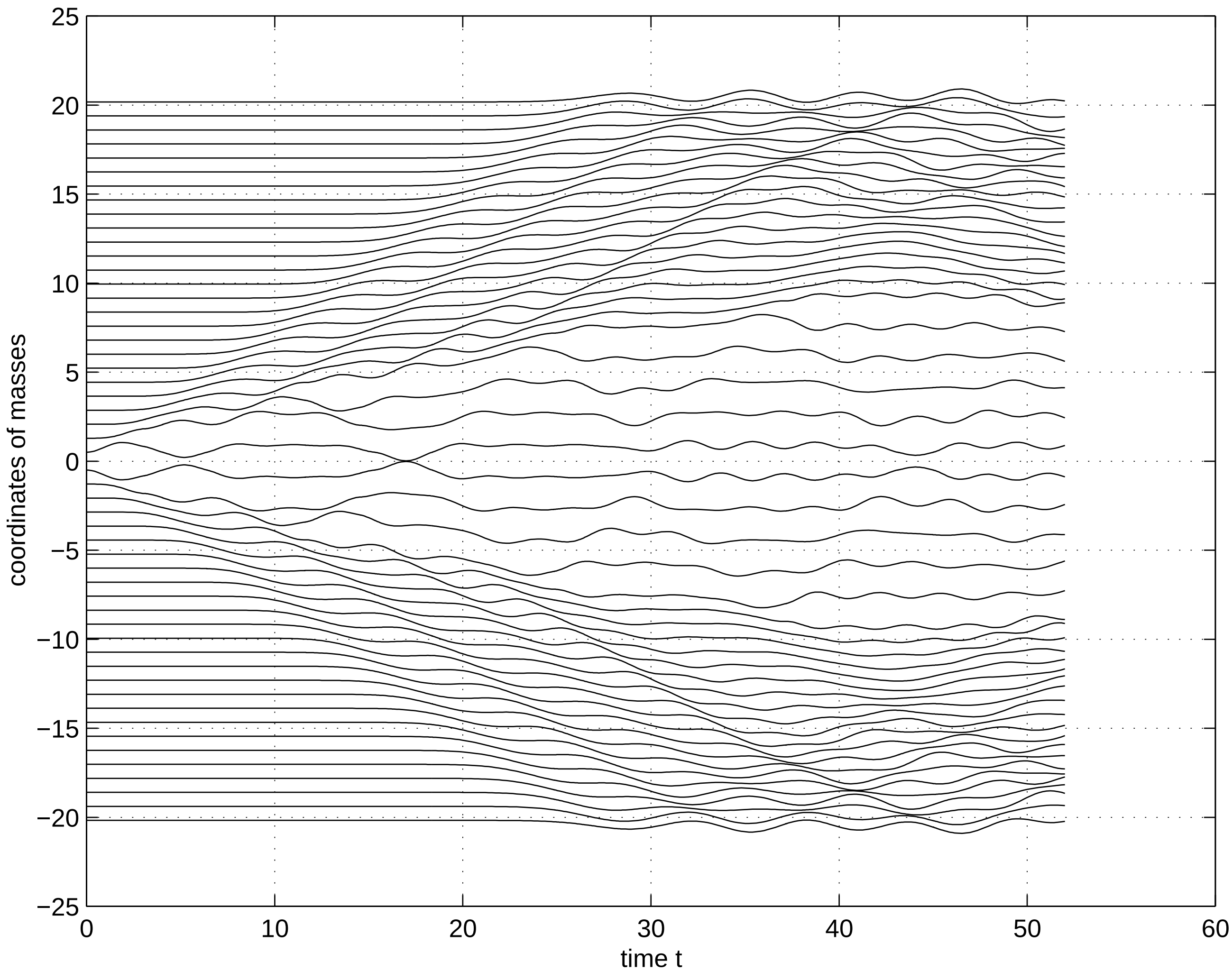

Figure 6: The motion of the chain with $N = 40$ masses. Initially all the masses are at rest. The elongation of each spring, besides the middle one is v_0 ($v_1 < v_0 < v_c$); and the elongation of the middle spring is slightly greater than v_c.

3.3 Chain under a slow external force

The next series of experiments studies reaction of the chain to the external elongation. To put the structure into motion it is slowly stretched. Namely, the left end of the chain is held fixed: $u_0 = 0$; the right end u_N is stretched by an external elongation: $u_N = L(t)$ (here $L(t)$ is the total elongation of the chain, a given "slow" function of time).

Specifically, we integrate the equations of the chain with the following initial and boundary conditions

$$\dot{u}_i(0) = 0, \quad u_i(0) = \beta\, i, \quad i = 0, 1, 2, \dots, N,$$
$$u_0(t) = 0, \quad u_N(t) = L(t) = N\beta - A\cos(t/T).$$

Here β is the initial elongation of the springs, $\beta < v_c$, the magnitude A is large enough so that all springs are forced to jump over the barrier. The period of external elongation T is much larger than the period of oscillation of a linear spring with a single mass so that the excitation is quasi-static.

Figure 7 shows the results of computer simulation of a mass-spring chain with $N = 4$. At first the masses move slowly, and the elongation of each spring is the same. However, when the elongation of one of the springs becomes greater than the critical elongation v_c, the springs start to oscillate with "high" frequencies, no matter how slowly we increase the total length $L(t)$. This is another reason to say that this mass-spring structure is *unstable*: high frequency oscillations appear in the structure while stress changes slowly.

The next experiment involves a chain with $N = 26$ masses. We take a chain with all masses being at rest and all springs being in the "short" branch ($v \ll v_a$, see Figure 1). We slowly stretch the chain up to the state when the elongation per one spring is sufficiently large so that all the springs are in the "long" branch, $v \gg v_b$, see Figure 1). Then we slowly release the chain to return it to the initial elongation. We repeat this process (stretching and releasing) two more times.

Figure 8 shows the dependence of the force F that we need to apply to the chain vs. the total elongation L. Figure 9 shows the corresponding total (potential plus kinetic) energy E of the chain vs. its total elongation L.

The modelling shows the following.

1. Once excited, the chain remains in the vibration mode (even if we return the total elongation $L(t)$ to its initial value). The motion is quickly randomized. Although the system in Hamiltonian, the motion of the system is *irreversible*: the part of the energy has been irreversibly transferred into the energy of the high frequency oscillations (it has been lost for the macroscopic motion).

2. The masses vibrate near equilibrium positions and all the springs spend some time in both "short" and "long" branches. We call this state of the system the *twinkling phase.*

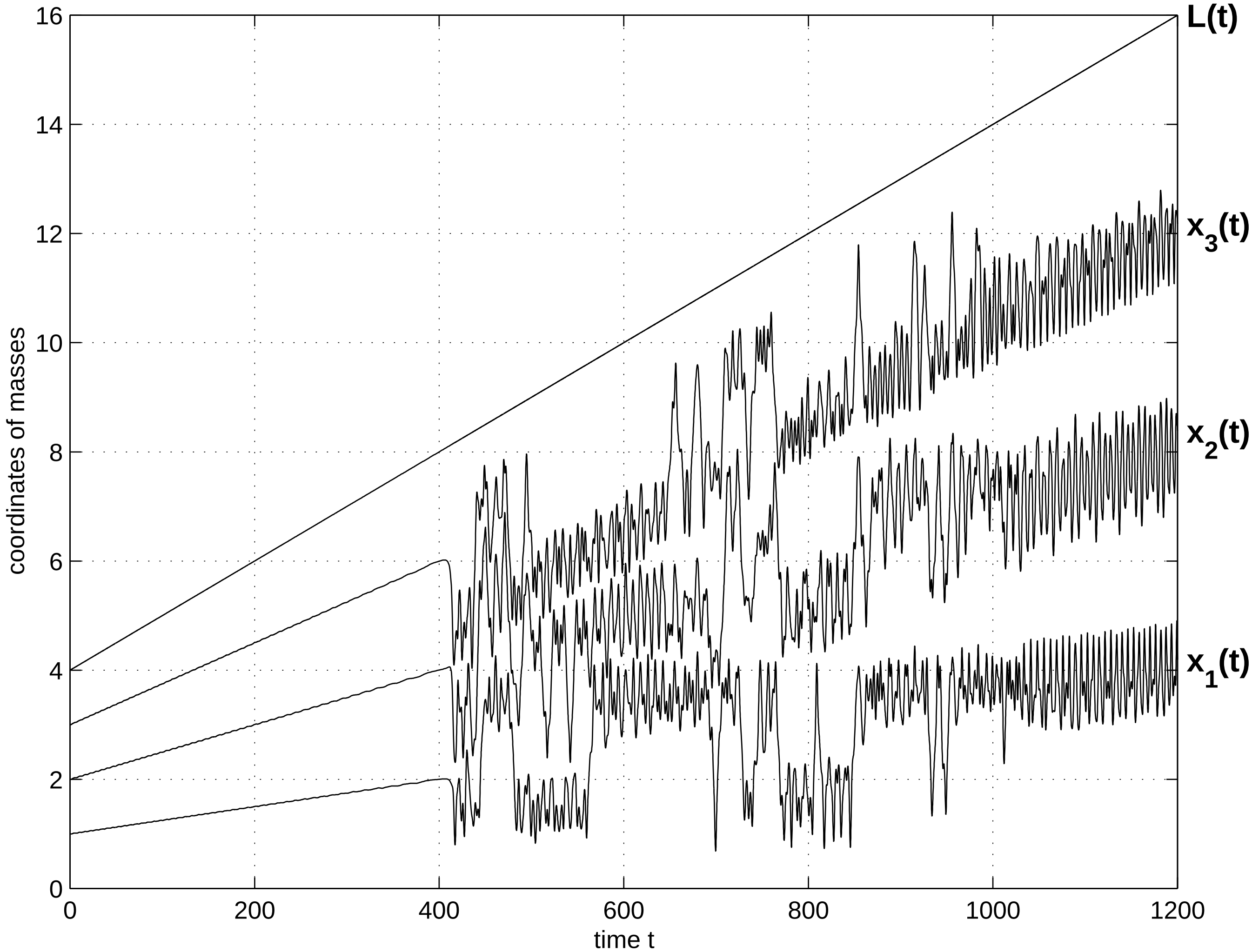

Figure 7: The elongation of each spring is the difference of the coordinates of the corresponding adjacent masses. Note that all masses switch to the twinkling face.

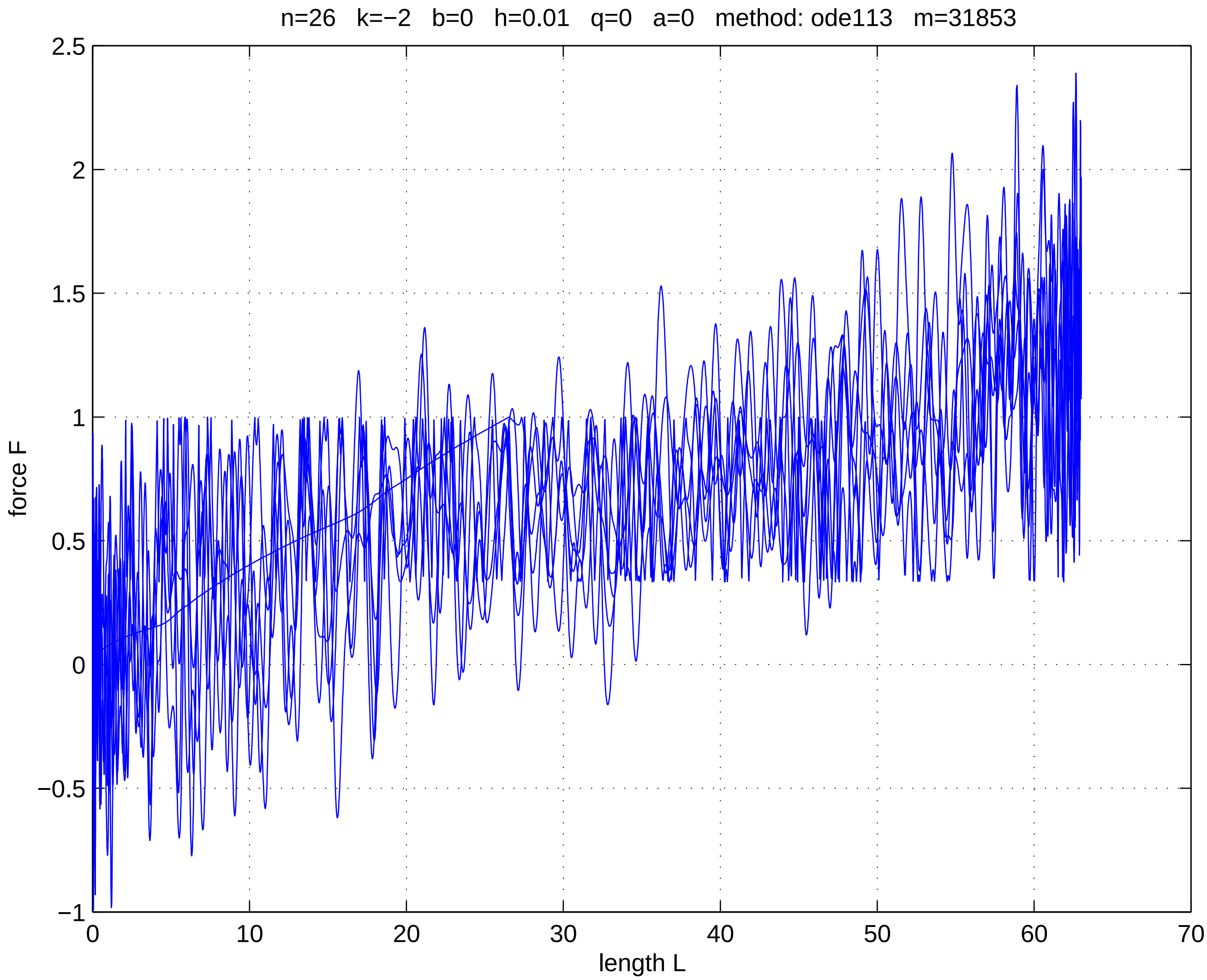

Figure 8: The dependence of the force F applied to chain vs. the total elongation L. The chain is slowly stretched and released three times. The smooth (almost linear) curve that goes from the origin to the value $F = 1$ corresponds to the initial stretching, when the masses do not oscillate.

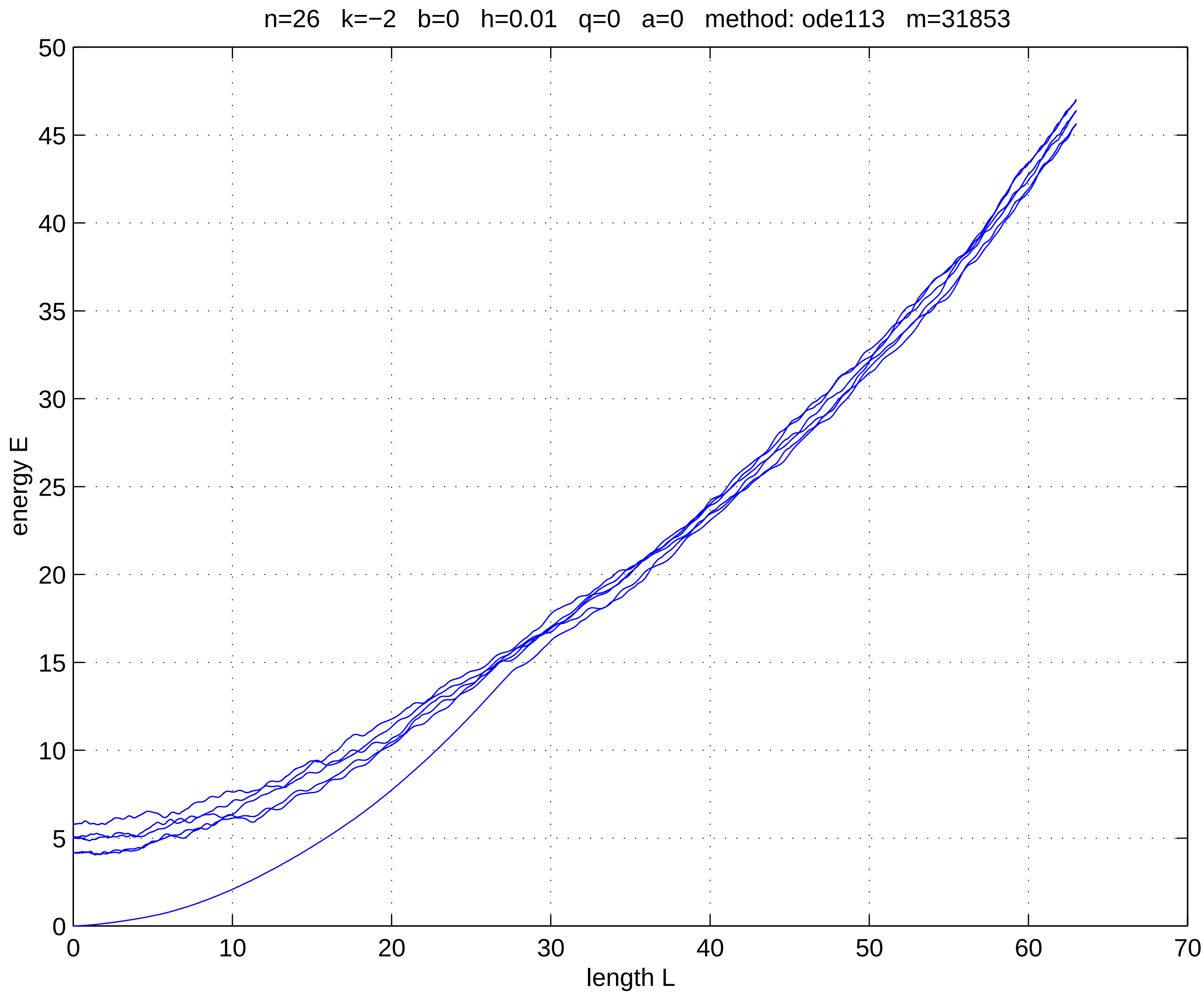

Figure 9: The total energy E of the chain vs. its total elongation L. This dependence corresponds to the dependence $F(L)$ shown in the previous figure. The smooth curve that goes from the origin corresponds to the initial stretching, when the masses do not oscillate.

3. In the beginning of the loading the structure is stable: it behaves as a linear system with the energy Π_1. When the vibrations are excited, their intensity does not increase as the cycles are repeated.

A part of the work of the external force irreversibly turns into the kinetic energy of high frequency oscillations. The steady state of the unstable chain is not quasistatic.

$$\lim_{\dot{L}\to 0} \frac{1}{t_f - t_0} \int_{t_0}^{t_f} T dt = T_0 \neq 0 \quad \text{in an unstable chain.}$$

Even for conservative chain, we have to treat T_0 as losses.

This shows that unstable chains with non-monotone stress-strain relations behave very differently than stable chains. When a stable chain is subject to a slow elongation ($\dot{L}/L << 1$), the kinetic energy T can be made arbitrarily small. In unstable chains, the inner instabilities excite an intensive dynamical process. This process is determined by the structure of the system. Its intensity is not small independently of the rate of the external elongation.

Remark 3.1 *There exists another source of energy losses: the trend of the energy to higher frequencies due to the nonlinearity of the stress-strain relation. This process is called the generation of higher harmonics, resulting in the energy cascade towards small scales. For instance, the hydrodynamic turbulence can dissipate energy even without any viscosity. Clearly this process occurs in unstable systems as well. However, it is much slower then the excitation of the high-frequency mode due to the above instability.*

3.4 Dissipation

The inner vibrations can be stabilized by a small dissipation. Here we check whether the introduction of a small dissipation leads to a stress-strain relation in a chain that corresponds to the minimum of the energy and to the Maxwell line. The modelling shows that this is not the case. We observe that introduced dissipation leads to a strong hysteresis instead of a steady state stress-strain dependence.

The model. In order to introduce dissipation, we add an extra term $\kappa \dot{u}_i$ to the left hand side of equation (2.3):

$$\rho_i \ddot{u}_i + \kappa \dot{u}_i = \sigma(u_i - u_{i-1}) + \sigma(u_i - u_{i+1}), \quad i = 1, 2, ...N$$

(κ is the dissipation rate).

Comparing a stable and an unstable chain, we notice that the speeds $\dot{u}_i \approx \frac{\dot{L}}{N}$ in a stable chain are as small as the rate $\dot{L}$ of an external elongation. Therefore, the losses due to dissipation can be made arbitrary small if the elongation is slow.

In an unstable chain, the speeds $\dot{u}_i$ are determined by its inner dynamics and are finite no matter what $\dot{L}$ is. The effective dissipation is determined by instabilities of the system, and it does not depend on the rate $\dot{L}$ of external excitation. The effective dissipation rate κ_{eff} is much larger than κ. The system exhibits strong hysteresis (see Figure 10).

Resume. The modelling shows that the slightly damped unstable structure is effectively a highly dissipative one, with strong hysteresis. Again, the convexification of the energy is not directly applicable to find the steady-state.

4 Discussion

Continuum limit. Consider a sequence of chains with different number N of masses, but of a constant total mass ρ_0. If N increases and each unit mass decreases: $\rho_N = \rho_0/N$, the structure becomes a continuum.

In this process, the stiffness of each spring should be proportional to N to keep the spring force constant. Therefore the characteristic frequency $\omega = \sqrt{C(N)/\rho(N)}$ is proportional to N:

$$\omega \sim N.$$

Thus, in the continuous limit some frequencies become infinite.

Notice the following properties of the limiting structure:

- A part of the energy stays in the form of the high frequency oscillations. This motion becomes "invisible" in the continuous limit. One can say that the corresponding kinetic energy is transferred into heat.
- Another part of the energy radiates in the form of sonic waves. In an infinite system this energy is lost. In a closed system with reflecting boundaries the radiation increases the "temperature" of the system.
- The twinkling phase is characterized by the following macro-parameters:
  1. The swelling distance;
  2. The kinetic energy that is in turn determined by the period of oscillations and the swelling distance;
  3. The appearance of this phase depends on the speed of the wave of the phase transition.

  In the next paper, we compute these parameters analytically.

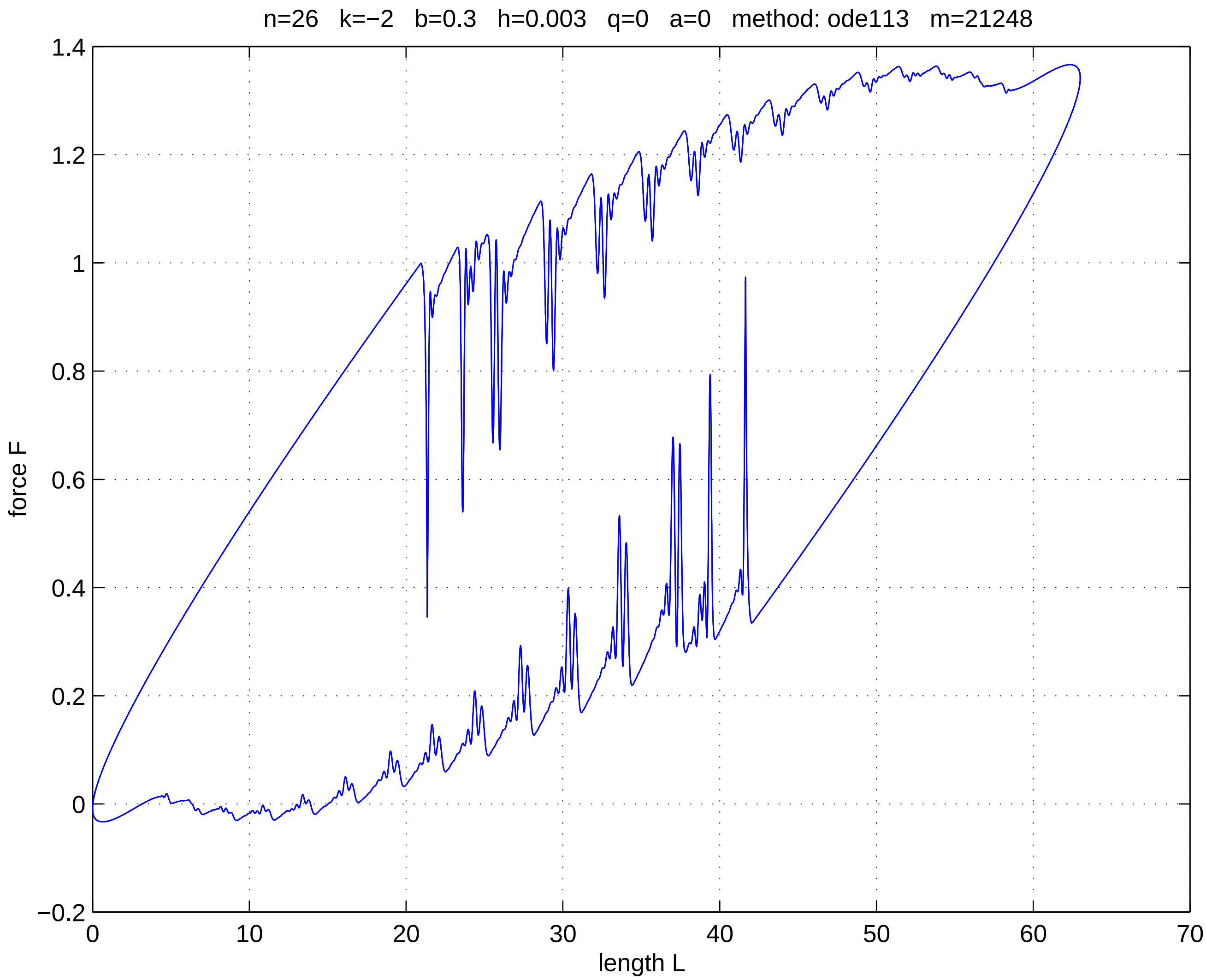

Figure 10: The hysteresis in the motion of the chain (here $N = 26$ and the dissipation coefficient $\kappa = 0.3$). The dependence of the force F applied to the chain vs. the total elongation L. The chain is slowly stretched and released three times; for each stretching and releasing the curves $F(L)$ are superimposed and give the same hysteresis curve.

- The consideration of the finite dimensional conservative model allows us to conclude that the convexification procedure does not describe the steady state of an unstable structure. This adiabatic system does not minimizes its potential energy, hence the Maxwell line in the constitutive relations is not achieved.

Use of unstable structures. Unstable structures described above possess abnormally high dissipation rate. When an unstable structure is subject to sufficiently strong slow perturbation, the high frequency oscillations are excited. They dissipate energy much faster than the long-wave quasi-static motions.

Thus these structures can be useful for building constructions that are able to withstand sufficiently strong repeated perturbations, e.g. for nuclear power plants in seismic areas. The construction is able to absorb energy of large perturbations (like those produced by seismic waves). When the external perturbation is gone, the construction returns to the original state.

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