Variational principles for complex conductivity, viscoelasticity and similar problems in media with complex moduli.

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Abstract

We consider linear processes in media with dissipation arising in conductivity, optics, viscoelasticity, etc. Time-periodic fields in such media are described by linear differential equations for complex-valued potentials. The properties of the media are characterized by complex valued tensors, for example, by complex conductivity or complex elasticity tensors. We suggest variational formulations for such problems: we construct the functionals whose Euler equations coincide with the original ones. We get four equivalent variational principles: two minimax and two minimal ones. The functionals of the obtained minimal variational principles are proportional to the energy dissipation averaged over the period of oscillation. The last principles can be used in the homogenization theory to obtain the bounds on the effective properties of composite materials with complex valued properties tensors.

1 Introduction

The paper deals with the processes in media which can be described by the linear elliptic equations with complex coefficients and solutions. The examples under study are the conductivity in a medium that possesses inductance and capacity along with resistivity, and viscoelasticity. We formulate four equivalent systems of equations that describe the conductance of an oscillating field. We obtain two pairs of mutually conjugated variational principles whose Euler equations coincide with mentioned systems of equations. Two of these principles are minimax ones and two others are minimal ones; they are similar to Dirichlet's and Thompson's variational principles for the electrostatic problem. Then we discuss similar results for the viscoelasticity problem and the applications of the proposed variational principles to the homogenization theory. We are pleased to thanks G.W. Milton[1] for the extensive citation of our results (that were unpublished at that time). In the mentioned paper Milton studied the minimal variational principle that describes the behavior of the complex conducting media and generalized this approach to a non self-adjoin problems. The variational formulation of the problem was used[1] to find the bounds on effective complex conductivity of composite. Gibiansky and Milton[2] found the bounds on the complex bulk modulus of a composite by using the variational formulation of the viscoelasticity problem suggested in the present paper . They also used it[3] to find the bounds on the effective shear modulus of a two-phase two-dimensional viscoelastic composite. The variational formulation of a non self-ajoint problem similar to the conductivity problem was suggested independently by Fannjiang and Papanicolaou[4].

2 Complex conductivity problem

2.1 Physical preliminaries

Consider a body Ω occupied by a conducting material and suppose that this body is loaded on the boundary $S = \partial\Omega$ by the time-dependent external electrical field which causes the current with density $\boldsymbol{j}(\boldsymbol{x}, t)$ and the corresponding electrical field $\boldsymbol{e}(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \Omega$. We assume for simplicity that there are no free bulk sources of current and electrical fields inside Ω. In other words, we assume the divergence-free nature of the current field and the curl-free nature of the electrical field:

$$\nabla \cdot \boldsymbol{j} = 0, \quad \nabla \times \boldsymbol{e} = 0. \tag{2.1}$$

This allows us to introduce a vector potential $\boldsymbol{a}$ of the current field $\boldsymbol{j}$ and a scalar potential ϕ of the electrical field $\boldsymbol{e}$ through the relations

$$\boldsymbol{j} = \nabla \times \boldsymbol{a}, \quad \boldsymbol{e} = -\nabla\phi. \tag{2.2}$$

The boundary conditions can be described by the usual way as

$$\phi = \phi_0 \quad \text{on} \quad S_1, \qquad \boldsymbol{n} \cdot \boldsymbol{j} = j_0 \quad \text{on} \quad S_2, \quad S_1 \cup S_2 = S, \tag{2.3}$$

where $\boldsymbol{n}$ is a normal vector to S; we assume the necessary smoothness of the boundary S and of the data ϕ_0, j_0.

The material is supposed to be linear in a sense that a linear combination of the current and its time derivatives linearly depends on a linear combination of the field and its time derivatives:

$$\sum_l a_l \frac{\partial^l \boldsymbol{j}}{\partial t^l} = \sum_l b_l \frac{\partial^l \boldsymbol{e}}{\partial t^l}. \tag{2.4}$$

Here $a_l = a_l(\boldsymbol{x})$ and $b_l = b_l(\boldsymbol{x})$ are some time-independent coefficients which are scalars (for the isotropic conductors) or symmetric matrices (for the anisotropic ones); we assume that the properties of the material (i.e. the scalar or matrix parameters a_l, b_l,) do not depend on time. We also assume that the properties of the material are local in space and in time: the current field and its derivatives at a point $\boldsymbol{x} \in \Omega$ at the moment t depend only on electrical field and its derivatives at the same point at the same moment of time.

An example of the described material is provided by an conducting medium possessing resistivity, inductance and capacity[5]. The other example of similar state law is given by an optical transparent medium[6].

We begin the consideration of the problem with the assumption that the external load as well as electrical and current fields are monochromatic, i.e.

$$\phi_0^s(\boldsymbol{x}, t) = (\Phi_0(\boldsymbol{x}) e^{-i\omega t})' = \Phi_0'(\boldsymbol{x}) \cos \omega t + \Phi_0''(\boldsymbol{x}) \sin \omega t, \tag{2.5}$$

$$j_0^s(\boldsymbol{x}, t) = (J_0(\boldsymbol{x}) e^{-i\omega t})' = J_0'(\boldsymbol{x}) \cos \omega t + J_0''(\boldsymbol{x}) \sin \omega t, \tag{2.6}$$

$$\boldsymbol{j}^s(\boldsymbol{x}, t) = (\boldsymbol{J}(\boldsymbol{x}) e^{-i\omega t})' = \boldsymbol{J}'(\boldsymbol{x}) \cos \omega t + \boldsymbol{J}''(\boldsymbol{x}) \sin \omega t \;\;, \tag{2.7}$$

$$\boldsymbol{e}^s(\boldsymbol{x}, t) = (\boldsymbol{E}(\boldsymbol{x}) e^{-i\omega t})' = \boldsymbol{E}'(\boldsymbol{x}) \cos \omega t + \boldsymbol{E}''(\boldsymbol{x}) \sin \omega t \tag{2.8}$$

where $\Phi_0(\boldsymbol{x})$, $J_0(\boldsymbol{x})$, $\boldsymbol{J}(\boldsymbol{x})$, and $\boldsymbol{E}(\boldsymbol{x})$ are the complex valued Fourie coefficients of corresponding functions. It means that we restrict our attention to the steady-state oscillations in a dissipative media.

Remark 1. *We use the symbols* $'$ *and* $''$ *throughout the paper to denote the real and imaginary parts of variables and we use the symbol* * *to denote the complex conjugate variable, i.e.* $\boldsymbol{c} = \boldsymbol{c}' + i\boldsymbol{c}''$, and $\boldsymbol{c}^* = \boldsymbol{c}' - i\boldsymbol{c}''$ where i is the imaginary unit.

Remark 2. *To use the results for the periodic nonharmonic oscillations we need to use the Fourie serias representation of the fields. General case of nonperiodic fields can* *be treated by using the Fourie transformation of the equations. In these situations our analyses should be applied to each of the harmonics separately.*

The linearity of the state low (??) leads to a linear relationship between the vectors $\boldsymbol{J}(\boldsymbol{x})$ and $\boldsymbol{E}(\boldsymbol{x})$:

$$\boldsymbol{J} = \boldsymbol{\sigma}\boldsymbol{E} \tag{2.9}$$

where $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\omega) = \boldsymbol{\sigma}'(\omega) + i\boldsymbol{\sigma}''(\omega)$ is a complex conductivity tensor that depends on frequency of oscillations[5]. For an isotropic material with state law (??) the tensor $\boldsymbol{\sigma}$ is defined by

$$\boldsymbol{\sigma} = \frac{\sum_l (-i\omega)^l a_l}{\sum_l (-i\omega)^l b_l} \boldsymbol{I}, \tag{2.10}$$

where $\boldsymbol{I}$ is a unit matrix. It can be determined similarly for an anisotropic material.

The divergence-free nature of the current field and the curl-free nature of the electrical field means that the Fourier coefficients of these fields satisfy to the relations similar to (??)

$$\nabla \cdot \boldsymbol{J} = 0, \qquad \nabla \times \boldsymbol{E} = 0. \tag{2.11}$$

Therefore they allow the representation (see (??))

$$\boldsymbol{J} = \nabla \times \ \boldsymbol{A}, \qquad \boldsymbol{E} = -\nabla \Phi, \tag{2.12}$$

where $\boldsymbol{A}$ and Φ are the Fourier coefficients of the potentials $\boldsymbol{a}$ and ϕ.

The boundary conditions (??) leads to the relations

$$\Phi = \Phi_0 \ \text{ on } \ S_1, \quad \boldsymbol{n} \cdot \boldsymbol{J} = J_0 \ \text{on} \ S_2, \quad S_1 \cup S_2 = S, \tag{2.13}$$

where Φ_0 and J_0 are the Fourier coefficients of the functions ϕ_0 and j_0.

We have shown that harmonic oscillation in the conducting media can be described by the state law (??), differential equations (??), (??) in conjunction with the boundary conditions (??).

The equations (??), (??)-(??) describe the conductance of the medium. They look exactly like equations for the real conductivity; however one should remember that they describe more complicated processes and have more complicated structure. Endeed, the complex-valued differential equations (??)-(??) form a fourth-order system of differential equations for real variables:

$$\nabla \cdot \boldsymbol{J}' = 0 \ \ , \nabla \cdot \boldsymbol{J}'' = 0, \quad \nabla \times \ \boldsymbol{E}' = 0, \quad \nabla \times \ \boldsymbol{E}'' = 0, \tag{2.14}$$

These equations are satisfied if four potentials of fields were introduced:

$$\boldsymbol{J}' = \nabla \times \ \boldsymbol{A}', \qquad \boldsymbol{J}'' = \nabla \times \ \boldsymbol{A}'', \qquad \boldsymbol{E}' = -\nabla \Phi', \qquad \boldsymbol{E}'' = -\nabla \Phi''. \tag{2.15}$$

The currents and electrical fields are connected by the state low (??)

$$-\boldsymbol{J}' = -\boldsymbol{\sigma}' \cdot \boldsymbol{E}' + \boldsymbol{\sigma}'' \cdot \boldsymbol{E}'', \tag{2.16}$$

$$\boldsymbol{J}'' = \boldsymbol{\sigma}'' \cdot \boldsymbol{E}' + \boldsymbol{\sigma}' \cdot \boldsymbol{E}''. \tag{2.17}$$

The vector form of the last equations is:

$$\begin{pmatrix} -\boldsymbol{J}' \\ \boldsymbol{J}'' \end{pmatrix} = \boldsymbol{D}_{EE} \begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{E}'' \end{pmatrix}, \tag{2.18}$$

where

$$\boldsymbol{D}_{EE} = \begin{pmatrix} -\boldsymbol{\sigma}' & \boldsymbol{\sigma}'' \\ \boldsymbol{\sigma}'' & \boldsymbol{\sigma}' \end{pmatrix} \tag{2.19}$$

is the conductivity matrix of the medium. The boundary conditions (??) can be rewritten as

$$\Phi' = \Phi'_0 \quad \text{on} \quad S_1, \tag{2.20}$$

$$\Phi'' = \Phi''_0 \quad \text{on} \quad S_1, \tag{2.21}$$

$$\boldsymbol{n} \cdot \boldsymbol{J}' = J'_0 \quad \text{on} \quad S_2, \tag{2.22}$$

$$\boldsymbol{n} \cdot \boldsymbol{J}'' = J''_0 \quad \text{on} \quad S_2. \tag{2.23}$$

We have formulated the system of the differential equations and the boundary conditions that describes conductivity of the complex conducting medium. As it has been expected, it has a double dimensions comparing to the real conductivity problem.

Note that the real part $\boldsymbol{\sigma}'$ of the complex conductivity tensor $\boldsymbol{\sigma}$ defined by (??) must be nonnegative:

$$\boldsymbol{\sigma}' \geq 0, \tag{2.24}$$

which represents the positiveness to the energy dissipation rate. Indeed, the density of the dissipation energy averaged over the period of oscillations is equal[5] to:

$$\frac{\omega}{2\pi} \int_t^{t+\frac{2\pi}{\omega}} \boldsymbol{j}^s \cdot \boldsymbol{e}^s dt = \frac{1}{2}(\boldsymbol{J}' \cdot \boldsymbol{E}' + \boldsymbol{J}'' \cdot \boldsymbol{E}'') = \frac{1}{2}(\boldsymbol{E}' \cdot \boldsymbol{\sigma}' \boldsymbol{E}' + \boldsymbol{E}'' \cdot \boldsymbol{\sigma}' \boldsymbol{E}'') \geq 0. \tag{2.25}$$

The last equality in (??) can be obtained by using (??), (??).

2.2 Reformulating of the equations

Here we rewrite the system of four first order differential equations as a system of two second order equations by excluding a pair of potentials. We will do it in four different ways, and we will end up with four equivalent systems. Each of them turn out to be Euler equations for a related variational problem, and we are going to formulate these problems.

Let us show that the system of equations (??), (??), (??) is equivalent to two second order equations for the scalar potentials Φ' and Φ''. Indeed, we take the divergence $(\nabla\cdot)$ of the right and left hand side terms of the equations (??), (??). The left hand side terms vanish (see (??)) and we get:

$$\nabla \cdot [-\boldsymbol{\sigma}' \cdot \nabla\Phi' + \boldsymbol{\sigma}'' \cdot \nabla\Phi''] = 0, \tag{2.26}$$

$$\nabla \cdot [\boldsymbol{\sigma}'' \cdot \nabla\Phi' + \boldsymbol{\sigma}' \cdot \nabla\Phi''] = 0. \tag{2.27}$$

Thus we have excluded the vector potentials $\boldsymbol{A}'$ and $\boldsymbol{A}''$ from the equations (??), (??), (??).

We may also rewrite this system of equations leaving any other pair of four scalar and vector potentials (??) and excluding the other two of them. For example, let us exclude the fields $\boldsymbol{E}'$ and $\boldsymbol{E}''$. First, we solve the equations (??), (??) for $\boldsymbol{E}'$ and $\boldsymbol{E}''$:

$$\boldsymbol{E}' = (\boldsymbol{\sigma}' + \boldsymbol{\sigma}''\boldsymbol{\sigma}'^{-1}\boldsymbol{\sigma}'')^{-1}\boldsymbol{J}' + (\boldsymbol{\sigma}'' + \boldsymbol{\sigma}'\boldsymbol{\sigma}''^{-1}\boldsymbol{\sigma}')^{-1}\boldsymbol{J}'', \tag{2.28}$$

$$\boldsymbol{E}'' = -(\boldsymbol{\sigma}'' + \boldsymbol{\sigma}'\boldsymbol{\sigma}''^{-1}\boldsymbol{\sigma}')^{-1}\boldsymbol{J}' + (\boldsymbol{\sigma}' + \boldsymbol{\sigma}''\boldsymbol{\sigma}'^{-1}\boldsymbol{\sigma}'')^{-1}\boldsymbol{J}''. \tag{2.29}$$

The last relations can be presented in a vector form as

$$\begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{E}'' \end{pmatrix} = \boldsymbol{D}_{JJ} \begin{pmatrix} -\boldsymbol{J}' \\ \boldsymbol{J}'' \end{pmatrix}, \tag{2.30}$$

where

$$\boldsymbol{D}_{JJ} = \begin{pmatrix} -(\boldsymbol{\sigma}' + \boldsymbol{\sigma}''\boldsymbol{\sigma}'^{-1}\boldsymbol{\sigma}'')^{-1} & (\boldsymbol{\sigma}'' + \boldsymbol{\sigma}'\boldsymbol{\sigma}''^{-1}\boldsymbol{\sigma}')^{-1} \\ (\boldsymbol{\sigma}'' + \boldsymbol{\sigma}'\boldsymbol{\sigma}''^{-1}\boldsymbol{\sigma}')^{-1} & (\boldsymbol{\sigma}' + \boldsymbol{\sigma}''\boldsymbol{\sigma}'^{-1}\boldsymbol{\sigma}'')^{-1} \end{pmatrix}. \tag{2.31}$$

(Note that matrices $\boldsymbol{D}_{EE}$ and $\boldsymbol{D}_{JJ}$ are reciprocally inverse ones). By taking the curl $(\nabla\times)$ of the right and left hand sides of (??), (??) and by using the last two equations (??) and the representation (??) of current fields $\boldsymbol{J}'$ and $\boldsymbol{J}''$ through the vector potentials $\boldsymbol{A}'$ and $\boldsymbol{A}''$, we obtain

$$\nabla \times [(\boldsymbol{\sigma}' + \boldsymbol{\sigma}''\boldsymbol{\sigma}'^{-1}\boldsymbol{\sigma}'')^{-1}\nabla \times \boldsymbol{A}' + (\boldsymbol{\sigma}'' + \boldsymbol{\sigma}'\boldsymbol{\sigma}''^{-1}\boldsymbol{\sigma}')^{-1}\nabla \times \boldsymbol{A}''] = 0, \tag{2.32}$$

$$\nabla \times [-(\boldsymbol{\sigma}'' + \boldsymbol{\sigma}'\boldsymbol{\sigma}''^{-1}\boldsymbol{\sigma}')^{-1}\nabla \times \boldsymbol{A}' + (\boldsymbol{\sigma}' + \boldsymbol{\sigma}''\boldsymbol{\sigma}'^{-1}\boldsymbol{\sigma}'')^{-1}\nabla \times \boldsymbol{A}''] = 0. \tag{2.33}$$

We may as well solve (??)-(??) for the fields $\boldsymbol{E}'$ and $\boldsymbol{J}''$ as

$$\boldsymbol{E}' = (\boldsymbol{\sigma}')^{-1}\ \boldsymbol{J}' + (\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}''\ \boldsymbol{E}'', \tag{2.34}$$

$$\boldsymbol{J}'' = \boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1}\ \boldsymbol{J}' + (\boldsymbol{\sigma}' + \boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}'')\ \boldsymbol{E}''. \tag{2.35}$$

The vector form of the last equations is:

$$\begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{J}'' \end{pmatrix} = \boldsymbol{D}_{JE} \begin{pmatrix} \boldsymbol{J}' \\ \boldsymbol{E}'' \end{pmatrix}, \tag{2.36}$$

where

$$\boldsymbol{D}_{JE} = \begin{pmatrix} (\boldsymbol{\sigma}')^{-1} & (\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}'' \\ \boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1} & \boldsymbol{\sigma}' + \boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}'' \end{pmatrix}. \tag{2.37}$$

According to (??)-(??) the field $\boldsymbol{E}'$ is curl-free and the field $\boldsymbol{J}''$ is divergence-free. Therefore, by using (??) - (??) we arrive at the following system of the second order equations

$$\nabla \times [(\boldsymbol{\sigma}')^{-1}\nabla \times \boldsymbol{A}' - (\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}''\nabla\Phi''] = 0, \tag{2.38}$$

$$\nabla \cdot [\boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1}\nabla \times \boldsymbol{A}' - (\boldsymbol{\sigma}' + \boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}'')\nabla\Phi'] = 0. \tag{2.39}$$

Similarly we have

$$\boldsymbol{J}' = [\boldsymbol{\sigma}' + \boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}'']\boldsymbol{E}' - (\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}''\boldsymbol{J}'', \tag{2.40}$$

$$\boldsymbol{E}'' = -\boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1}\boldsymbol{E}' + (\boldsymbol{\sigma}')^{-1}\boldsymbol{J}'', \tag{2.41}$$

or in the vector form

$$\begin{pmatrix} \boldsymbol{J}' \\ \boldsymbol{E}'' \end{pmatrix} = \boldsymbol{D}_{EJ} \begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{J}'' \end{pmatrix}, \tag{2.42}$$

where

$$\boldsymbol{D}_{EJ} = \begin{pmatrix} \boldsymbol{\sigma}' + \boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}'' & -(\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}'' \\ -\boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1} & (\boldsymbol{\sigma}')^{-1} \end{pmatrix}. \tag{2.43}$$

(Note that the matrices $\boldsymbol{D}_{JE}$ and $\boldsymbol{D}_{EJ}$ are reciprocally inverse ones, $\boldsymbol{D}_{JE} = \boldsymbol{D}_{EJ}^{-1}$). Again, the operations $(\nabla\cdot)$ and $(\nabla\times)$ vanish the left hand side terms in the equations (??) and (??) respectively:

$$\nabla \cdot [(\boldsymbol{\sigma}' + \boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}'')\nabla\Phi' + (\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}''\nabla \times \boldsymbol{A}''] = 0, \tag{2.44}$$

$$\nabla \times [\boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1}\nabla\Phi' + (\boldsymbol{\sigma}')^{-1}\nabla \times \boldsymbol{A}''] = 0. \tag{2.45}$$

We have written down different interpretation of the same equations. All four systems of equations (??) - (??), (??) - (??), (??) - (??), and (??) - (??) are equivalent to each other and to the original system (??). Each of them in conjunction with the boundary conditions (??) - (??) allows to find the solution that describes the processes in the conducting medium. We will show now that each of them represents the Euler equations for a corresponding variational problem.

It is interesting to note that matrices $\boldsymbol{D}_{EJ}$ and $\boldsymbol{D}_{JE}$ are unimodal, i.e.

$$\det \boldsymbol{D}_{EJ} = \det \boldsymbol{D}_{JE} = 1. \tag{2.46}$$

Indeed, they are positive definite and satisfy the relations

$$\boldsymbol{D}_{EJ} = \boldsymbol{D}_{JE}^{-1}, \qquad \det \boldsymbol{D}_{EJ} = [\det \boldsymbol{D}_{JE}]^{-1}, \tag{2.47}$$

and

$$\boldsymbol{D}_{EJ} = \boldsymbol{R} \cdot \boldsymbol{D}_{JE} \boldsymbol{R}, \qquad \det \boldsymbol{D}_{EJ} = (\det \boldsymbol{R})^2 \det \boldsymbol{D}_{JE} \tag{2.48}$$

(compare (??) and (??)). Here $\boldsymbol{R}$ is an unimodal block matrix

$$\boldsymbol{R} = \begin{pmatrix} 0 & \boldsymbol{I} \\ -\boldsymbol{I} & 0 \end{pmatrix}, \qquad (\det \boldsymbol{R})^2 = 1 \tag{2.49}$$

and $\boldsymbol{I}$ is a unit matrix. The conditions (??) and (??) - (??) lead to (??).

3 Variational principles for the complex conducting medium

In this section we establish the variational principles for the problem of complex conductivity. There is no direct complex analogue to the variational principles for the real valued problem because the inequalities in the complex plane cannot be considered. However, the systems of real valued differential equations described above are the stationary conditions for some real-valued functionals. These functionals lead to variational principles that describe the complex conductivity processes.

First, we formulate two minimax variational principles. They follow naturally from the equations in the form (??) - (??) and (??)-(??). Then we obtain two minimal variational principles based on the equations of the problem in the form (??) - (??) and (??) - (??). Finally, we discuss the relation between these four principles referring on the procedure of Legandre transformation.

3.1 Minimax variational principles

3.1.1 The minimax variational principle in terms of the electrical fields

Let us consider the following variational min-max problem

$$\min_{\substack{\boldsymbol{E}'' \\ \boldsymbol{E}''=-\nabla\Phi'', \\ \Phi''=\Phi''_0 \ \text{on} \ S_1}} \quad \max_{\substack{\boldsymbol{E}' \\ \boldsymbol{E}'=-\nabla\Phi', \\ \Phi'=\Phi'_0 \ \text{on} \ S_1}} \quad U_{EE}, \tag{3.1}$$

where

$$U_{EE} = \int_{\Omega} W_{EE}(\boldsymbol{E}', \boldsymbol{E}'')d\Omega + \int_{S_2} [\Phi'' J''_0 - \Phi' J'_0]dS \tag{3.2}$$

and

$$W_{EE}(\boldsymbol{E}', \boldsymbol{E}'') = \frac{1}{2}\begin{pmatrix}\boldsymbol{E}' & \boldsymbol{E}''\end{pmatrix} \cdot \boldsymbol{D}_{EE}\begin{pmatrix}\boldsymbol{E}' \\ \boldsymbol{E}''\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\boldsymbol{E}' & \boldsymbol{E}''\end{pmatrix}\begin{pmatrix}-\boldsymbol{\sigma}' & \boldsymbol{\sigma}'' \\ \boldsymbol{\sigma}'' & \boldsymbol{\sigma}'\end{pmatrix}\begin{pmatrix}\boldsymbol{E}' \\ \boldsymbol{E}''\end{pmatrix}. \tag{3.3}$$

Remark 3. *We use the field variables $\boldsymbol{E}'$, $\boldsymbol{E}''$ instead of potentials Φ', Φ'' as minimizers in (??). Actually, this is a question of suitable notations; the problem remains the same no matter what variables are used. We find more convenient to deal with the field variables because of various transformations of the problem (??) which are discussed in next sections.*

The first variation of the functional U_{EE} (see (??)) must be vanished[7] which leads to the condition:

$$\delta U_{EE} = \int_{\Omega} [(\nabla\delta\Phi') \cdot \boldsymbol{J}' - (\nabla\delta\Phi'') \cdot \boldsymbol{J}'']d\Omega + \int_{S_2} [\delta\Phi'' J''_0 - \delta\Phi' J'_0]dS = 0, \tag{3.4}$$

where

$$-\boldsymbol{J}' = \frac{\partial W_{EE}(\boldsymbol{E}', \boldsymbol{E}'')}{\partial \boldsymbol{E}'} = -\boldsymbol{\sigma}'\boldsymbol{E}' + \boldsymbol{\sigma}''\boldsymbol{E}'', \tag{3.5}$$

$$\boldsymbol{J}'' = \frac{\partial W_{EE}(\boldsymbol{E}', \boldsymbol{E}'')}{\partial \boldsymbol{E}''} = \boldsymbol{\sigma}''\boldsymbol{E}' + \boldsymbol{\sigma}'\boldsymbol{E}''. \tag{3.6}$$

The last equations coincide with (??) - (??). It can be shown in a standard way, that the vectors $\boldsymbol{J}'$, $\boldsymbol{J}''$ are divergence-free. Indeed, we get by integration by parts:

$$\int_{\Omega} \nabla\delta\Phi \cdot \boldsymbol{J} d\Omega = -\int_{\Omega} \delta\Phi \nabla \cdot \boldsymbol{J} d\Omega + \int_{S} \boldsymbol{n} \cdot \boldsymbol{J} \delta\Phi dS. \tag{3.7}$$

Taking into account (??) and the boundary conditions (??), (??) (which require $\delta\Phi' = \delta\Phi'' = 0$ on S_1) we receive

$$\delta U_{EE} = -\int_\Omega [\delta\Phi'\nabla\cdot\boldsymbol{J}' - \delta\Phi''\nabla\cdot\boldsymbol{J}'']d\Omega + \int_{S_2}[\delta\Phi''(J_0'' - \boldsymbol{n}\cdot\boldsymbol{J}'') - \delta\Phi'(J_0' - \boldsymbol{n}\cdot\boldsymbol{J}')]dS = 0 \quad (3.8)$$

for any variations $\delta\Phi'$ and $\delta\Phi''$. The coefficients by these variations should vanish identically; this requirement leads to the Euler equations for the functional (??):

$$\nabla\cdot\boldsymbol{J}' = 0, \quad \nabla\cdot\boldsymbol{J}'' = 0; \quad \boldsymbol{n}\cdot\boldsymbol{J}' = J_0' \ \text{ on } \ S_2\ , \quad \boldsymbol{n}\cdot\boldsymbol{J}'' = J_0'' \ \text{ on } \ S_2. \qquad (3.9)$$

One can check that they coincide with the original system of equations in the form (??), (??) and with the boundary conditions (??), (??). The conditions (??), (??) have been taken into account explicitly.

Then, the second variation U_{EE} is equal

$$\begin{gathered}\delta^2 U_{EE} = \int_\Omega W_{EE}(\delta\boldsymbol{E}', \delta\boldsymbol{E}'')d\Omega = \\ \frac{1}{2}\int_\Omega [\delta\boldsymbol{E}''\cdot\boldsymbol{\sigma}'\delta\boldsymbol{E}'' - \delta\boldsymbol{E}'\cdot\boldsymbol{\sigma}'\delta\boldsymbol{E}' + 2\delta\boldsymbol{E}''\cdot\boldsymbol{\sigma}''\delta\boldsymbol{E}']d\Omega = \\ \frac{1}{2}\int_\Omega (\delta\boldsymbol{E}' \ \ \delta\boldsymbol{E}'')\cdot\boldsymbol{D}_{EE}\begin{pmatrix}\delta\boldsymbol{E}' \\ \delta\boldsymbol{E}''\end{pmatrix}d\Omega\end{gathered} \qquad (3.10)$$

Remember, that $\boldsymbol{D}_{EE}$ is determined by (??) and have both positive and negative eigenvalues. This implies the minimax nature of the variational problem and the saddle point nature of the stationary solution.

Remark 4. *We mention that (at least weak) solution of the quadratic minimax problem (??) exists*[7] *for any reasonable smooth boundary conditions (??) - (??). However, we do not want to discuss here the existence theorems for the variational problems, which are quite well-known for the quadratic integrand. This remark is made only to justify the use of the symbols* max *and* min *instead of* sup *and* inf *respectively in this section and below.*

3.1.2 The minimax variational principle in terms of current fields

Let us now consider the variational problem

$$\max_{\substack{\boldsymbol{J}' \\ \boldsymbol{J}' = \nabla\times\boldsymbol{A}', \\ \boldsymbol{n}\cdot\boldsymbol{J}' = J_0' \text{ on } S_2}} \quad \min_{\substack{\boldsymbol{J}'' \\ \boldsymbol{J}'' = \nabla\times\boldsymbol{A}, \\ \boldsymbol{n}\cdot\boldsymbol{J}'' = J_0'' \text{ on } S_2}} \quad U_{JJ}, \qquad (3.11)$$

where

$$U_{JJ} = \int_\Omega W_{JJ}(\boldsymbol{J}',\ \boldsymbol{J}'')d\Omega + \int_{S_1}[\Phi_0''\ \boldsymbol{n}\cdot\boldsymbol{J}'' - \Phi_0'\ \boldsymbol{n}\cdot\boldsymbol{J}']dS. \tag{3.12}$$

and

$$W_{JJ}(\boldsymbol{J}',\ \boldsymbol{J}'') = \frac{1}{2}(-\boldsymbol{J}'\quad \boldsymbol{J}'')\cdot\boldsymbol{D}_{JJ}\begin{pmatrix}-\boldsymbol{J}'\\ \boldsymbol{J}''\end{pmatrix}; \tag{3.13}$$

the matrix $\boldsymbol{D}_{JJ}$ is defined by (**??**).

Similarly to the previous section we check that the Euler equations for the functional (**??**) coincide with one of the original systems. Indeed, the stationary conditions for the functional U_{JJ} (see (**??**)) have a form

$$\delta U_{JJ} = \int_\Omega[(\nabla\times\delta\boldsymbol{A}'')\cdot\boldsymbol{E}'' - (\nabla\times\delta\boldsymbol{A}')\cdot\boldsymbol{E}']d\Omega + \int_{S_1}[\Phi_0''\boldsymbol{n}\cdot\nabla\times\delta\boldsymbol{A}'' - \Phi_0'\boldsymbol{n}\cdot\nabla\times\delta\boldsymbol{A}']dS = 0, \tag{3.14}$$

where

$$\boldsymbol{E}' = \frac{\partial W_{JJ}(\boldsymbol{J}',\boldsymbol{J}'')}{\partial\boldsymbol{J}''} = -(\boldsymbol{\sigma}'' + \boldsymbol{\sigma}'\boldsymbol{\sigma}''^{-1}\boldsymbol{\sigma}')^{-1}\boldsymbol{J}' + (\boldsymbol{\sigma}' + \boldsymbol{\sigma}''\boldsymbol{\sigma}'^{-1}\boldsymbol{\sigma}'')^{-1}\boldsymbol{J}'', \tag{3.15}$$

$$\boldsymbol{E}' = -\frac{\partial W_{JJ}(\boldsymbol{J}',\boldsymbol{J}'')}{\partial\boldsymbol{J}'} = (\boldsymbol{\sigma}' + \boldsymbol{\sigma}''\boldsymbol{\sigma}'^{-1}\boldsymbol{\sigma}'')^{-1}\boldsymbol{J}' + (\boldsymbol{\sigma}'' + \boldsymbol{\sigma}'\boldsymbol{\sigma}''^{-1}\boldsymbol{\sigma}')^{-1}\boldsymbol{J}''. \tag{3.16}$$

They coincide with the equations (**??**) - (**??**). Using the formula

$$\int_\Omega(\nabla\times\delta\boldsymbol{A})\cdot\boldsymbol{E}d\Omega = \int_\Omega[\nabla\cdot(\delta\boldsymbol{A}\times\boldsymbol{E}) + \delta\boldsymbol{A}\cdot(\nabla\times\boldsymbol{E})]d\Omega = \int_\Omega\delta\boldsymbol{A}\cdot(\nabla\times\boldsymbol{E})d\Omega + \int_S\boldsymbol{n}\cdot\delta\boldsymbol{A}\times\boldsymbol{E}dS \tag{3.17}$$

we rewrite the condition (**??**) as

$$\begin{aligned}\delta U_{JJ} = &\int_\Omega[\delta\boldsymbol{A}''\cdot\nabla\times\boldsymbol{E}'' - \delta\boldsymbol{A}'\cdot\nabla\times\boldsymbol{E}']d\Omega\\ &+ \int_S[\boldsymbol{n}\cdot\delta\boldsymbol{A}'\times\boldsymbol{E}' - \boldsymbol{n}\cdot\delta\boldsymbol{A}''\times\boldsymbol{E}'']dS\\ &+ \int_{S_1}[\Phi_0''\ \boldsymbol{n}\cdot\nabla\times\delta\boldsymbol{A}'' -\ \Phi_0'\boldsymbol{n}\cdot\nabla\times\delta\boldsymbol{A}']dS = 0.\end{aligned} \tag{3.18}$$

The variation (**??**) must be equal to zero for any variations $\delta\boldsymbol{A}'$ and $\delta\boldsymbol{A}''$. Hence, the fields $\boldsymbol{E}'$ and $\boldsymbol{E}''$ must be curl-free

$$\nabla\times\boldsymbol{E}' = 0, \qquad \nabla\times\boldsymbol{E}'' = 0. \tag{3.19}$$

By substitution (**??**), (**??**) into (**??**) we end up with the equations (**??**), (**??**). The boundary terms in (**??**) must be equal to zero which leads to the boundary conditions (**??**), (**??**). To show it let us introduce a potentials of the fields $\boldsymbol{E}'$ and $\boldsymbol{E}''$

$$\boldsymbol{E}' = -\nabla\Phi', \quad \boldsymbol{E}'' = -\nabla\Phi''. \tag{3.20}$$

We get by integration by parts

$$\int_\Omega \nabla\Phi\cdot\boldsymbol{J}\; d\Omega = \int_\Omega \nabla\cdot(\boldsymbol{J}\Phi)d\Omega - \int_\Omega \Phi\nabla\cdot\boldsymbol{J}\; d\Omega = \int_S \boldsymbol{n}\cdot\boldsymbol{J}\Phi\; dS \tag{3.21}$$

for any current field $\boldsymbol{J}$. Then we transform the equation (??) to the form

$$\delta U_{JJ} = \int_{S_1} [\boldsymbol{n}\cdot\delta\boldsymbol{J}''(\Phi_0'' - \Phi'') - \boldsymbol{n}\cdot\delta\boldsymbol{J}'(\Phi_0' - \Phi')]dS = 0. \tag{3.22}$$

by using the relation (??), the boundary conditions (??) - (??) (that require $\boldsymbol{n}\cdot\partial\boldsymbol{J}' = \boldsymbol{n}\cdot\partial\boldsymbol{J}'' = 0$ on S_2), and the stationary conditions (??). We arrive immediately at the boundary conditions (??), (??), whereas the conditions (??), (??) have been taken into account explicitly. We see that the equations (??) - (??) coincide with the Euler equations for the functional (??).

The second variation of the functional U_{JJ} is equal

$$\delta^2 U_{JJ} = \int_\Omega W_{EE}(\delta\boldsymbol{J}', \delta\boldsymbol{J}'')d\Omega = \frac{1}{2}\int_\Omega (-\delta\boldsymbol{J}' \quad \delta\boldsymbol{J}'')\cdot\boldsymbol{D}_{JJ}\begin{pmatrix}-\delta\boldsymbol{J}'\\ \delta\boldsymbol{J}''\end{pmatrix} d\Omega. \tag{3.23}$$

The matrix $\boldsymbol{D}_{JJ}$ of the quadratic form in (??) is also not positive definite, which implies the minimax nature of the stationary solution of the variational problem (??).

Remark 5 *The minimax nature of both principles (??) and (??) do not allow to use many well-developed variational methods that essentially based on the minimal nature of the variational problems. They use the fact that the energy (i.e. the value of the functional) on any trial field should exceed the actual energy, stored in the material. Therefore the energy on any trial field provides the upper bound on the actual energy. For the minimax principles (??), (??), however, the situation is different. Consider, for example, the problem (??) and let us calculate the energy on a trial fields of two potentials Φ' and Φ''. The actual energy is increased if the trial field $\nabla\Phi''$ differs from the optimal one and is decreased if the other trial field $\nabla\Phi'$ is not optimal. Therefore, the value of the functional (??) on the trial fields can be lower or higher then the actual energy and cannot bound the functional (??) from either side.*

3.2 The minimal variational principles for complex conductivity

Now we are going to establish two minimal variational principles for the problem under consideration.

3.2.1 The first minimal variational principle

Let us consider the variational problem

$$\min_{\substack{\boldsymbol{J}' \\ \boldsymbol{J}' = \nabla\times\boldsymbol{A}' \\ \boldsymbol{n}\cdot\boldsymbol{J}' = J_0' \text{ on } S_2}} \quad \min_{\substack{\boldsymbol{E}'' \\ \boldsymbol{E}'' = -\nabla\Phi'' \\ \Phi'' = \Phi_0'' \text{ on } S_1}} \quad U_{JE}, \tag{3.24}$$

where

$$U_{JE} = \int_\Omega W_{JE}(\boldsymbol{J}', \boldsymbol{E}'')d\Omega - \int_{S_1} \boldsymbol{n}\cdot\boldsymbol{J}'\Phi_0' dS + \int_{S_2} \Phi'' J_0'' dS, \tag{3.25}$$

and

$$W_{JE}(\boldsymbol{J}', \boldsymbol{E}'') = \frac{1}{2}\begin{pmatrix}\boldsymbol{J}' & \boldsymbol{E}''\end{pmatrix}\cdot\begin{pmatrix}(\boldsymbol{\sigma}')^{-1} & (\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}'' \\ \boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1} & \boldsymbol{\sigma}' + \boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}''\end{pmatrix}\begin{pmatrix}\boldsymbol{J}' \\ \boldsymbol{E}''\end{pmatrix}. \tag{3.26}$$

The first variation of the functional (??) has a form

$$\delta U_{JE} = \int_\Omega [\delta\boldsymbol{J}'\cdot\boldsymbol{E}' + \delta\boldsymbol{E}''\cdot\boldsymbol{J}''] - \int_{S_1} \boldsymbol{n}\cdot\delta\boldsymbol{J}'\Phi_0' dS + \int_{S_2} \delta\Phi'' J_0'' dS = 0, \tag{3.27}$$

where

$$\boldsymbol{E}' = \frac{\partial W_{JE}(\boldsymbol{J}', \boldsymbol{E}'')}{\partial \boldsymbol{J}'} = (\boldsymbol{\sigma}')^{-1}\boldsymbol{J}' + (\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}''\boldsymbol{E}'', \tag{3.28}$$

$$\boldsymbol{J}'' = \frac{\delta W_{JE}(\boldsymbol{J}', \boldsymbol{E}'')}{\delta \boldsymbol{E}''} = \boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1}\boldsymbol{J}' + (\boldsymbol{\sigma}' + \boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}'')\boldsymbol{E}'', \tag{3.29}$$

(compare with (??) - (??)). Further we directly follow the arguments of the paragraph 2.1.1 dealing with the fields $\boldsymbol{J}'$ and $\boldsymbol{E}'$ and the arguments of the paragraph 2.1.2 dealing with the fields $\boldsymbol{J}''$ and $\boldsymbol{E}''$. We conclude in a similar way that the Euler equations for the functional (??) coincide with the system of original equations in the form (??) - (??) and the boundary conditions (??), (??). The positiveness of the quadratic form (??) follows from the physically clear condition (??). Indeed, as follows from (??) and (??) - (??)

$$W_{JE}(\boldsymbol{J}', \boldsymbol{E}'') = \frac{1}{2}(\boldsymbol{J}'\cdot\boldsymbol{E}' + \boldsymbol{J}''\cdot\boldsymbol{E}'') = \frac{1}{2}(\boldsymbol{E}'\cdot\boldsymbol{\sigma}'\boldsymbol{E}' + \boldsymbol{E}''\cdot\boldsymbol{\sigma}'\boldsymbol{E}'') \geq 0. \tag{3.30}$$

As we see, this functional is equal to the whole energy dissipated in the body Ω during one period of oscillation (see (??)).

The second variation $\delta^2 U_{JE}$ of the functional (??)

$$\delta^2 U_{JE} = \int_\Omega W_{JE}(\delta\boldsymbol{J}', \delta\boldsymbol{E}'')d\Omega = \frac{1}{2}\int_\Omega \begin{pmatrix}\delta\boldsymbol{J}' & \delta\boldsymbol{E}''\end{pmatrix}\cdot\boldsymbol{D}_{JE}\begin{pmatrix}\delta\boldsymbol{J}' \\ \delta\boldsymbol{E}''\end{pmatrix} d\Omega. \tag{3.31}$$

is positive due to the positiveness of the matrix $\boldsymbol{D}_{JE}$, see (??). It gives a sufficient condition for the quadratic functional (??) to provide its minimum at the stationary point[7]; the existence of the minimum justifies the notation " min " instead of "inf" in the variational principle (??).

3.2.2 The second minimal variational principle

Similarly, we consider the variational problem

$$\begin{array}{ccc} \min\limits_{\boldsymbol{J}''} & \min\limits_{\boldsymbol{E}'} & U_{EJ}, \\ \boldsymbol{J}'' = \nabla\times\ \boldsymbol{A}'', & \boldsymbol{E}' = -\nabla\Phi' & \\ \boldsymbol{n}\cdot\boldsymbol{J}'' = J_0'' \ \text{ on } \ S_2 & \Phi' = \Phi_0' \ \text{ on } \ S_1 & \end{array} \tag{3.32}$$

where

$$U_{EJ} = \int_\Omega W_{EJ}(\boldsymbol{E}', \boldsymbol{J}'')d\Omega + \int_{S_1} \boldsymbol{n}\cdot\boldsymbol{J}''\Phi_0'' dS - \int_{S_2} \Phi' J_0' dS \tag{3.33}$$

and

$$W_{EJ}(\boldsymbol{E}', \boldsymbol{J}'') = \frac{1}{2}\left(\, \boldsymbol{E}' \quad \boldsymbol{J}'' \,\right)\cdot \begin{pmatrix} \boldsymbol{\sigma}' + \boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}'' & -(\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}'' \\ -\boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1} & (\boldsymbol{\sigma}')^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{J}'' \end{pmatrix} \tag{3.34}$$

The first variation of the functional (??) has a form

$$\delta U_{EJ} = \int_\Omega [\delta\boldsymbol{E}'\cdot\boldsymbol{J}' + \delta\boldsymbol{J}''\cdot\boldsymbol{E}''] - \int_{S_1} \boldsymbol{n}\cdot\delta\boldsymbol{J}''\Phi_0'' dS - \int_{S_2} \delta\Phi' J_0' dS = 0, \tag{3.35}$$

where

$$\boldsymbol{J}' = \frac{\partial W_{EJ}(\boldsymbol{E}', \boldsymbol{J}'')}{\partial \boldsymbol{E}'} = (\boldsymbol{\sigma}' + \boldsymbol{\sigma}''(\boldsymbol{\sigma}')\boldsymbol{\sigma}'')^{-1}\boldsymbol{E}' - (\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}''\boldsymbol{J}'', \tag{3.36}$$

$$\boldsymbol{E}'' = \frac{\partial W_{EJ}(\boldsymbol{E}', \boldsymbol{J}'')}{\partial \boldsymbol{J}''} = -\boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1}\boldsymbol{E}' + (\boldsymbol{\sigma}')^{-1}\boldsymbol{J}'', \tag{3.37}$$

(compare with (??) - (??)). Following the arguments of the paragraph 2.1.1 for the pair of the fields $\boldsymbol{E}'$ and $\boldsymbol{J}'$ and the arguments of the paragraph 2.1.2 for the pair of the fields $\boldsymbol{J}''$ and $\boldsymbol{E}''$ we conclude again that the Euler equations for the functional (??) coincide with the system of original equations in the form (??) - (??) and the boundary conditions (??), (??).

One could see also that the second variation of this functional is equal

$$\delta^2 U_{EJ} = \int_\Omega W_{EJ}(\delta\boldsymbol{E}', \delta\boldsymbol{J}'')d\Omega = \frac{1}{2}\int_\Omega (\delta\boldsymbol{E}' \ \ \delta\boldsymbol{J}'')\cdot\boldsymbol{D}_{EJ}\begin{pmatrix} \delta\boldsymbol{E}' \\ \delta\boldsymbol{J}'' \end{pmatrix} d\Omega. \tag{3.38}$$

It is positive due to the positiveness of the matrix $\boldsymbol{D}_{EJ}$, see (??); it can be checked directly as well. Indeed, as follows from (??)

$$W_{EJ}(\boldsymbol{E}', \boldsymbol{J}'') = \frac{1}{2}(\boldsymbol{J}' \cdot \boldsymbol{E}' + \boldsymbol{J}'' \cdot \boldsymbol{E}'') = W_{JE}(\boldsymbol{J}', \boldsymbol{E}') \tag{3.39}$$

which is positive when $\boldsymbol{\sigma}' \geq 0$.

Note also that

$$W_{JE}(\boldsymbol{J}, \boldsymbol{E}) = W_{EJ}(\boldsymbol{E}, \boldsymbol{J}) \tag{3.40}$$

identically. This feature is specific for the considered problem; usually we meet two different variational principles of minimization of the potential energy and the complementary energy (for example, the Dirichlet's and Thomson's principles in electrostatic).

3.3 Discussion: the minimax and minimal variational principles, conjugate functionals and Legandre transformation

One can check that the pairs of the variational problems (??) and (??), (??) and (??) are conjugate ones[7]. The matrices associated with the quadratic forms (??) and (??), (??) and (??) are reciprocally inverse ones, i.e.

$$\boldsymbol{D}_{EE} = \boldsymbol{D}_{JJ}^{-1}, \quad \boldsymbol{D}_{EJ} = \boldsymbol{D}_{JE}^{-1}. \tag{3.41}$$

One could pass from the first integrand in each pair to the second one by taking the appropriate Legandre transformation.

To find the relation between the minimax and minimal variational principles we refer to the well-known procedure of convex analysis: any saddle function $f(x, y)$ of two variables x and y, corresponds through the Legandre transformation x^* over the first variable x to the convex function $f_x^*(x^*, y)$ of the arguments x^*, y.

For example, the saddle-type function

$$f(x, y) = \frac{a}{2}x^2 - \frac{b}{2}y^2 \tag{3.42}$$

is conjugate upon the variable x to the convex function

$$f_x^*(x^*, y) = \max_x[xx^* - f(x, y)] = \frac{1}{2a}(x^*)^2 + \frac{b}{2}y^2. \tag{3.43}$$

By using the similar idea, we can take the Legandre transformation of the functional (??) over one of its variables (namely, over $\boldsymbol{E}''$) and receive the minimal variational principle (??). Similarly we can take the Legandre transformation of the functional (??) over the variable $\boldsymbol{J}''$ and come to the minimum variational principle (??).

The relations between four described variational problems are illustrated by the following scheme (we omit in the notations the differential restrictions on the fields and the boundary conditions): The minimax variational principle

$$\min_{\boldsymbol{E}''}\max_{\boldsymbol{E}'} U_{EE}(\boldsymbol{E}', \boldsymbol{E}''), \tag{3.44}$$

where the energy U_{EE} is defined by the relations (??) is transformed by the Legandre transformation over the variable $\boldsymbol{E}''$ into the minimum variational principle (??)

$$\min_{\boldsymbol{J}''}\min_{\boldsymbol{E}'} U_{EJ}(\boldsymbol{E}', \boldsymbol{J}''). \tag{3.45}$$

The next Legandre transformation over the variable $\boldsymbol{E}'$ leads to the minimax variational problem

$$\min_{\boldsymbol{J}'}\max_{\boldsymbol{J}''} -U_{JJ}(\boldsymbol{J}', \boldsymbol{J}''), \tag{3.46}$$

which is equivalent to (??). The next transformation over the variable $\boldsymbol{J}''$ leads to the maximization problem

$$\max_{\boldsymbol{J}'}\max_{\boldsymbol{E}''} -U_{JE}(\boldsymbol{J}', \boldsymbol{E}'') \tag{3.47}$$

which is equivalent to (??). If we take one more Legandre transformation over the variable $\boldsymbol{J}'$ we come to the problem that coincides with the one that we start, see (??).

The same method can be used to formulate the minimization problem for other problems described by the equations with complex coefficients. For example, the equations of torsion oscillation of the bar made of viscoelastic materials coincide with (??) - (??) with some changes in the definitions of the fields and moduli[8,9]. The other important example of the complex moduli problem is given by viscoelasticity equations that we discuss in the next section.

4 Viscoelasticity problem

In this section we give the variational formulation of the viscoelasticity problem. It is known[8,9,10] that harmonic oscillations of a viscoelastic body can be described (in quasistatic limit) by the equations

$$\nabla \cdot \boldsymbol{\tau} = \boldsymbol{p}, \qquad \boldsymbol{\tau} = \boldsymbol{\tau}^T, \tag{4.1}$$

$$\boldsymbol{\epsilon} = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T). \tag{4.2}$$

$$\boldsymbol{\tau} = \boldsymbol{C}\boldsymbol{\epsilon}. \tag{4.3}$$

Here $\boldsymbol{\tau}$ and $\boldsymbol{\epsilon}$ are complex Fourier coefficients of the second order tensors of the stress and strain fields respectively, $\boldsymbol{p}$ is a Fourier coefficient of the external bulk load, $\boldsymbol{u}$ is a Fourier coefficient of the displacement vector. $\boldsymbol{C} = \boldsymbol{C}(\omega)$ is a complex rigidity tensor, i.e. the fourth order tensor depending on the frequency of oscillations ω. The product in (??) means the contraction of two last indices of the first tensor and two first indices of the second tensor in the product.

External forces and displacements on the boundary $S = S_1 \cup S_2$ of the body Ω are described as

$$\boldsymbol{n} \cdot \boldsymbol{\tau} = \boldsymbol{f}_0 \quad \text{on} \quad S_1, \tag{4.4}$$

$$\boldsymbol{u} = \boldsymbol{u}_0 \quad \text{on} \quad S_2, \tag{4.5}$$

where $\boldsymbol{n}$ is an external normal to S.

Let us assume that the body is loaded by the harmonic (sinusoidal) load $\boldsymbol{p}^s$

$$\boldsymbol{p}^s = (\boldsymbol{p}e^{i\omega t})' = \boldsymbol{p}' \cos \omega t - \boldsymbol{p}'' \sin \omega t, \tag{4.6}$$

by the surface load $\boldsymbol{f}_0^s$

$$\boldsymbol{f}_0^s = (\boldsymbol{f}_0 e^{i\omega t})' = \boldsymbol{f}_0' \cos \omega t - \boldsymbol{f}_0'' \sin \omega t, \tag{4.7}$$

and that the boundary displacement $\boldsymbol{u}_0^s$ is also harmonic

$$\boldsymbol{u}_0^s = (\boldsymbol{u}_0 e^{i\omega t})' = \boldsymbol{u}_0' \cos \omega t - \boldsymbol{u}_0'' \sin \omega t, \tag{4.8}$$

The stress field $\boldsymbol{\tau}^s$ and strain field $\boldsymbol{\epsilon}^s$ fields are also harmonic in this case and can be expressed as following in terms of its complex Fourier coefficients $\boldsymbol{\tau}$ and $\boldsymbol{\epsilon}$

$$\boldsymbol{\epsilon}^s = (\boldsymbol{\epsilon} e^{i\omega t})' = \boldsymbol{\epsilon}' \cos \omega t - \boldsymbol{\epsilon}'' \sin \omega t, \tag{4.9}$$

$$\boldsymbol{\tau}^s = (\boldsymbol{\tau} e^{i\omega t})' = \boldsymbol{\tau}' \cos \omega t - \boldsymbol{\tau}'' \sin \omega t, \tag{4.10}$$

similar to (??) - (??). To get the variational principles for the viscoelasticity problem we may use almost straightforward analogue with the conductivity problem described earlier with obvious interchanging the symbols $\boldsymbol{E}$, $\boldsymbol{J}$ and $\boldsymbol{\sigma}$ for the conductivity by symbols $\boldsymbol{\tau}$, $\boldsymbol{\epsilon}$ and $\boldsymbol{C}$ for the viscoelasticity problem.

Namely, the complex Hook's law (??) can be rewritten as a system of real equations by four different ways

$$\begin{pmatrix} \boldsymbol{\tau}' \\ \boldsymbol{\tau}'' \end{pmatrix} = \boldsymbol{C}_{\epsilon\epsilon} \begin{pmatrix} \boldsymbol{\epsilon}'' \\ \boldsymbol{\epsilon}' \end{pmatrix}, \qquad \boldsymbol{C}_{\epsilon\epsilon} = \begin{pmatrix} -\boldsymbol{\mathcal{C}}'' & \boldsymbol{\mathcal{C}}' \\ \boldsymbol{\mathcal{C}}' & \boldsymbol{\mathcal{C}}'' \end{pmatrix}, \tag{4.11}$$

(compare with (??) - (??)); or

$$\begin{pmatrix} \boldsymbol{\epsilon}'' \\ \boldsymbol{\epsilon}' \end{pmatrix} = \boldsymbol{C}_{\tau\tau} \begin{pmatrix} \boldsymbol{\tau}' \\ \boldsymbol{\tau}'' \end{pmatrix}, \quad \boldsymbol{C}_{\tau\tau} = \begin{pmatrix} -(\boldsymbol{\mathcal{C}}'' + \boldsymbol{\mathcal{C}}'(\boldsymbol{\mathcal{C}}'')^{-1}\boldsymbol{\mathcal{C}}')^{-1} & (\boldsymbol{\mathcal{C}}' + \boldsymbol{\mathcal{C}}''(\boldsymbol{\mathcal{C}}')^{-1}\boldsymbol{\mathcal{C}}'')^{-1} \\ (\boldsymbol{\mathcal{C}}' + \boldsymbol{\mathcal{C}}''(\boldsymbol{\mathcal{C}}')^{-1}\boldsymbol{\mathcal{C}}'')^{-1} & (\boldsymbol{\mathcal{C}}'' + \boldsymbol{\mathcal{C}}'(\boldsymbol{\mathcal{C}}'')^{-1}\boldsymbol{\mathcal{C}}')^{-1} \end{pmatrix}, \tag{4.12}$$

(compare with (??) - (??)); or

$$\begin{pmatrix} \boldsymbol{\epsilon}'' \\ \boldsymbol{\tau}'' \end{pmatrix} = \boldsymbol{C}_{\tau\epsilon} \begin{pmatrix} -\boldsymbol{\tau}' \\ \boldsymbol{\epsilon}' \end{pmatrix}, \qquad \boldsymbol{C}_{\tau\epsilon} = \begin{pmatrix} (\boldsymbol{\mathcal{C}}'')^{-1} & (\boldsymbol{\mathcal{C}}'')^{-1}\boldsymbol{\mathcal{C}}' \\ \boldsymbol{\mathcal{C}}'(\boldsymbol{\mathcal{C}}'')^{-1} & \boldsymbol{\mathcal{C}}'' + \boldsymbol{\mathcal{C}}'(\boldsymbol{\mathcal{C}}'')^{-1}\boldsymbol{\mathcal{C}}' \end{pmatrix}, \tag{4.13}$$

(compare with (??) - (??)); or

$$\begin{pmatrix} -\boldsymbol{\tau}' \\ \boldsymbol{\epsilon}' \end{pmatrix} = \boldsymbol{C}_{\epsilon\tau} \begin{pmatrix} \boldsymbol{\epsilon}'' \\ \boldsymbol{\tau}'' \end{pmatrix}, \qquad \boldsymbol{C}_{\epsilon\tau} = \begin{pmatrix} \boldsymbol{\mathcal{C}}'' + \boldsymbol{\mathcal{C}}'(\boldsymbol{\mathcal{C}}'')^{-1}\boldsymbol{\mathcal{C}}' & -(\boldsymbol{\mathcal{C}}'')^{-1}\boldsymbol{\mathcal{C}}' \\ -\boldsymbol{\mathcal{C}}'(\boldsymbol{\mathcal{C}}'')^{-1} & (\boldsymbol{\mathcal{C}}'')^{-1} \end{pmatrix}, \tag{4.14}$$

(compare with (??) - (??)).

Remark 5. *Note that matrices $\boldsymbol{C}_{\tau\tau}$ and $\boldsymbol{C}_{\epsilon\epsilon}$, and also $\boldsymbol{C}_{\epsilon\tau}$ and $\boldsymbol{C}_{\tau\epsilon}$ are reciprocally inverse ones*

$$\boldsymbol{C}_{\epsilon\epsilon} = \boldsymbol{C}_{\tau\tau}^{-1}, \quad \boldsymbol{C}_{\epsilon\tau} = \boldsymbol{C}_{\tau\epsilon}^{-1}. \tag{4.15}$$

The matrices $\boldsymbol{C}_{\epsilon\tau}$ and $\boldsymbol{C}_{\tau\epsilon}$ are unimodal, i.e.

$$\det \boldsymbol{C}_{\epsilon\tau} = \det \boldsymbol{C}_{\tau\epsilon} = 1 \tag{4.16}$$

(compare with (??) - (??) for the conductivity problem)

These four forms of the Hook's law correspond to four variational principles, namely there are two minimax principles:

$$\min_{\substack{\boldsymbol{\epsilon}' \\ \boldsymbol{\epsilon}' = \frac{1}{2}(\nabla \boldsymbol{u}' + (\nabla \boldsymbol{u}')^T), \\ \boldsymbol{u}' = \boldsymbol{u}_0' \text{ on } S_1}} \quad \max_{\substack{\boldsymbol{\epsilon}'' \\ \boldsymbol{\epsilon}'' = \frac{1}{2}(\nabla \boldsymbol{u}'' + (\nabla \boldsymbol{u}'')^T), \\ \boldsymbol{u}'' = \boldsymbol{u}_0'' \text{ on } S_1}} \quad U_{\epsilon\epsilon}, \tag{4.17}$$

where

$$U_{\epsilon\epsilon} = \int_\Omega [W_{\epsilon\epsilon}(\boldsymbol{\epsilon}'', \boldsymbol{\epsilon}') + \boldsymbol{p}'' \cdot \boldsymbol{u}' + \boldsymbol{p}' \cdot \boldsymbol{u}''] d\Omega - \int_{S_2} [\boldsymbol{u}' \cdot \boldsymbol{f}_0'' + \boldsymbol{u}'' \cdot \boldsymbol{f}_0'] dS, \tag{4.18}$$

$$W_{\epsilon\epsilon}(\boldsymbol{\epsilon}'', \boldsymbol{\epsilon}') = \frac{1}{2} \begin{pmatrix} \boldsymbol{\epsilon}'' & \boldsymbol{\epsilon}' \end{pmatrix} \cdot \boldsymbol{C}_{\epsilon\epsilon} \begin{pmatrix} \boldsymbol{\epsilon}'' \\ \boldsymbol{\epsilon}' \end{pmatrix} \tag{4.19}$$

and the matrix $\boldsymbol{C}_{\epsilon\epsilon}$ is defined by (??); and

$$\max_{\substack{\boldsymbol{\tau}' \\ \nabla\cdot\boldsymbol{\tau}'=\boldsymbol{p}',\boldsymbol{\tau}'=(\boldsymbol{\tau}')^T, \\ \boldsymbol{n}\cdot\boldsymbol{\tau}'=\boldsymbol{f}_0' \text{ on } S_2}} \quad \min_{\substack{\boldsymbol{\tau}'' \\ \nabla\cdot\boldsymbol{\tau}''=\boldsymbol{p}'',\boldsymbol{\tau}''=(\boldsymbol{\tau}'')^T, \\ \boldsymbol{n}\cdot\boldsymbol{\tau}''=\boldsymbol{f}_0'' \text{ on } S_2}} U_{\tau\tau}, \tag{4.20}$$

where

$$U_{\tau\tau}=\int_\Omega W_{\tau\tau}(\boldsymbol{\tau}',\ \boldsymbol{\tau}'')d\Omega-\int_{S_1}[\ \boldsymbol{n}\cdot\boldsymbol{\tau}''\boldsymbol{u}_0'+\ \boldsymbol{n}\cdot\boldsymbol{\tau}'\boldsymbol{u}_0'']dS, \tag{4.21}$$

$$W_{\tau\tau}(\boldsymbol{\tau}',\ \boldsymbol{\tau}'')=(\,\boldsymbol{\tau}'\quad\boldsymbol{\tau}''\,)\cdot\boldsymbol{C}_{\tau\tau}\begin{pmatrix}\boldsymbol{\tau}'\\ \boldsymbol{\tau}''\end{pmatrix} \tag{4.22}$$

and matrix $\boldsymbol{C}_{\tau\tau}$ is defined by (??). The other two variational principles are the minimal ones:

$$\min_{\substack{\boldsymbol{\tau}' \\ \nabla\cdot\boldsymbol{\tau}'=\boldsymbol{p}',\boldsymbol{\tau}'=(\boldsymbol{\tau}')^T, \\ \boldsymbol{n}\cdot\boldsymbol{\tau}'=\boldsymbol{f}_0' \text{ on } S_2}} \quad \min_{\substack{\boldsymbol{\epsilon}' \\ \boldsymbol{\epsilon}'=\frac{1}{2}(\nabla\boldsymbol{u}'+(\nabla\boldsymbol{u}')^T), \\ \boldsymbol{u}'=\boldsymbol{u}_0' \text{ on } S_1}} U_{\tau\epsilon}, \tag{4.23}$$

where

$$U_{\tau\epsilon}=\int_\Omega[W_{\tau\epsilon}(\boldsymbol{\tau}',\boldsymbol{\epsilon}')+\boldsymbol{p}''\boldsymbol{u}']d\Omega+\int_{S_1}\boldsymbol{n}\cdot\boldsymbol{\tau}'\boldsymbol{u}_0''dS-\int_{S_2}\boldsymbol{u}'\cdot\boldsymbol{f}_0''dS, \tag{4.24}$$

$$W_{\tau\epsilon}(\boldsymbol{\tau}',\boldsymbol{\epsilon}')=\frac{1}{2}\,(\,-\boldsymbol{\tau}'\quad\boldsymbol{\epsilon}'\,)\cdot\boldsymbol{C}_{\tau\epsilon}\begin{pmatrix}-\boldsymbol{\tau}'\\ \boldsymbol{\epsilon}'\end{pmatrix}, \tag{4.25}$$

and matrix $\boldsymbol{C}_{\tau\epsilon}$ is defined by (??); and

$$\min_{\substack{\boldsymbol{\tau}'' \\ \nabla\cdot\boldsymbol{\tau}''=\boldsymbol{p}'',\boldsymbol{\tau}''=(\boldsymbol{\tau}'')^T, \\ \boldsymbol{n}\cdot\boldsymbol{\tau}''=\boldsymbol{f}_0'' \text{ on } S_2}} \quad \min_{\substack{\boldsymbol{\epsilon}'' \\ \boldsymbol{\epsilon}''=\frac{1}{2}(\nabla\boldsymbol{u}''+(\nabla\boldsymbol{u}'')^T) \\ \boldsymbol{u}''=\boldsymbol{u}_0'' \text{ on } S_1}} U_{\epsilon\tau}, \tag{4.26}$$

where

$$U_{\epsilon\tau}=\int_\Omega[W_{\epsilon\tau}(\boldsymbol{\epsilon}'',\boldsymbol{\tau}'')-\boldsymbol{p}'\cdot\boldsymbol{u}'']d\Omega-\int_{S_1}\boldsymbol{n}\cdot\boldsymbol{\tau}''\boldsymbol{u}_0'dS+\int_{S_2}\boldsymbol{u}''\cdot\boldsymbol{f}_0'dS, \tag{4.27}$$

$$W_{\epsilon\tau}(\boldsymbol{\epsilon}'',\boldsymbol{\tau}'')=\frac{1}{2}\,(\,\boldsymbol{\epsilon}''\quad\boldsymbol{\tau}''\,)\cdot\boldsymbol{C}_{\epsilon\tau}\begin{pmatrix}\boldsymbol{\epsilon}''\\ \boldsymbol{\tau}''\end{pmatrix}, \tag{4.28}$$

and matrix $\boldsymbol{C}_{\epsilon\tau}$ is defined by (??).

Note that the functionals (??) and (??) are proportional to the energy dissipation over the period of oscillation[11]. Indeed,

$$\frac{\omega}{2\pi}\int_t^{t+\frac{\omega}{2\pi}} \boldsymbol{\tau}^s \cdot \frac{\partial \boldsymbol{\epsilon}^s}{\partial t} d\Omega = \frac{\omega}{2}(\boldsymbol{\tau}\cdot\boldsymbol{\epsilon}^*)'' = \frac{\omega}{2}(\boldsymbol{\tau}''\cdot\boldsymbol{\epsilon}' - \boldsymbol{\tau}'\cdot\boldsymbol{\epsilon}'') = \omega W_{\tau\epsilon}(\boldsymbol{\tau}',\boldsymbol{\epsilon}') = \omega W_{\epsilon\tau}(\boldsymbol{\epsilon}'',\boldsymbol{\tau}'') \tag{4.29}$$

One can also check that

$$W_{\tau\epsilon}(\boldsymbol{\tau}',\boldsymbol{\epsilon}') = \frac{1}{2}(\boldsymbol{\epsilon}'\cdot\boldsymbol{\mathcal{C}}'\boldsymbol{\epsilon}' + \boldsymbol{\epsilon}''\cdot\boldsymbol{\mathcal{C}}'\boldsymbol{\epsilon}'') \tag{4.30}$$

see (??), (??), (??) and compare with (??). This quantity is positive due to the condition $\boldsymbol{\mathcal{C}}' \geq 0$ that is equivalent to the positiveness of the energy dissipation in the material. Similar to the conductivity problem we see that

$$W_{\tau\epsilon}(\boldsymbol{\tau}',\boldsymbol{\epsilon}') = W_{\epsilon\tau}(\boldsymbol{\epsilon}'',\boldsymbol{\tau}'') \tag{4.31}$$

and

$$W_{\tau\epsilon}(\boldsymbol{\tau},\boldsymbol{\epsilon}) = W_{\epsilon\tau}(\boldsymbol{\epsilon},\boldsymbol{\tau}) \tag{4.32}$$

identically. The relations between these variational principles are the following (in this schematic picture we omit the differential restrictions and boundary conditions):

The minimax variational principle

$$\min_{\boldsymbol{\epsilon}'}\max_{\boldsymbol{\epsilon}''} U_{\epsilon\epsilon}, \tag{4.33}$$

where the energy $U_{\epsilon\epsilon}$ is defined by the relations (??)-(??), is transformed after the Legandre transformation over the variable $\boldsymbol{\epsilon}'$ into the minimum variational principle (??)

$$\min_{\boldsymbol{\tau}''}\min_{\boldsymbol{\epsilon}''} U_{\epsilon\tau}, \tag{4.34}$$

The next Legandre transformation over the variable $\boldsymbol{\epsilon}''$ leads to the minimax variational problem

$$\min_{\boldsymbol{\tau}'}\max_{\boldsymbol{\tau}''} -U_{\tau\tau}, \tag{4.35}$$

which is equivalent to (??). The next transformation over the variable $\boldsymbol{\tau}''$ leads to the maximization problem

$$\max_{\boldsymbol{\tau}'}\max_{\boldsymbol{\epsilon}'} -U_{\tau\epsilon}, \tag{4.36}$$

that is equivalent to (??) If we take one more Legandre transformation over the variable $\boldsymbol{\tau}'$ we come to the problem which coincide with the one that we start, see (??). Pairs of the functionals $U_{\epsilon\epsilon}$ and $U_{\tau\tau}$, and also $U_{\epsilon\tau}$ and $U_{\tau\epsilon}$ are conjugate.

As we see, the viscoelasticity problem also can be described by the minimal variational principles.

5 Application to composites. The problem of bounds.

Let us consider the composite material which is a space-periodic mixture assembled of several materials which differ by their complex moduli $\mathcal{D}^l$ and are taken in the prescribed volume fractions f^l. (l is the index of the material in the mixture). Let us consider the homogenization problem, i.e. the problem of describing the media average properties. It is well known[12] that the average behavior of the mixture is described by the homogenized equations of the same type as for the original materials. For the complex conductivity problem the homogenized equations are

$$\langle \boldsymbol{E} \rangle = -\nabla \langle \phi \rangle, \qquad \langle \boldsymbol{J} \rangle = \boldsymbol{\sigma}^0 \langle \boldsymbol{E} \rangle, \tag{5.1}$$

where $\boldsymbol{\sigma}^0 = (\boldsymbol{\sigma}^0)' + \ i \ (\boldsymbol{\sigma}^0)''$ is by definition the complex effective conductivity tensor of a composite. The symbol $\langle \ \cdot \ \rangle$ denotes the averaging throughout the cell of periodicity Ω:

$$\langle (\cdot) \rangle = \frac{1}{vol\Omega} \int_\Omega (\cdot) d\Omega \tag{5.2}$$

The effective properties tensor $\boldsymbol{\sigma}^0$ is in general an anisotropic second order tensor which corresponds to an anisotropic conducting media with arbitrary symmetry. It depends on the properties of initial material as well as on geometrical structure of the composite and it is independent of loading.

The equivalent representation of the effective tensor of the structure may be obtained from the consideration of the variational principle (??). Indeed, the density of the energy

$$W^0_{JE}(\boldsymbol{J}'_0, \boldsymbol{E}''_0) = (\ \boldsymbol{J}'_0 \quad \boldsymbol{E}''_0\ ) \cdot \boldsymbol{D}^0_{JE} \begin{pmatrix} \boldsymbol{J}'_0 \\ \boldsymbol{E}''_0 \end{pmatrix} \tag{5.3}$$

stored in the composite cell when the last one is placed in the uniform external fields $\boldsymbol{J}'_0, \boldsymbol{E}''_0$ is known to be equal (compare with (??)- (??)) to the energy stored in an equivalent homogeneous material:

$$W^0_{JE}(\boldsymbol{J}'_0, \boldsymbol{E}''_0) = \min_{\substack{\boldsymbol{J}' \\ \boldsymbol{J}' = \nabla \times \boldsymbol{A}' \\ \langle \boldsymbol{J}' \rangle = \boldsymbol{J}'_0}} \ \min_{\substack{\boldsymbol{E}'' \\ \boldsymbol{E}'' = -\nabla \Phi'' \\ \langle \boldsymbol{E}'' \rangle = \boldsymbol{E}''_0}} \int_\Omega W_{JE}(\boldsymbol{J}'(\boldsymbol{x}), \boldsymbol{E}''(\boldsymbol{x})) d\Omega. \tag{5.4}$$

Here $\boldsymbol{E}''(\boldsymbol{x})$, $\boldsymbol{J}'(\boldsymbol{x})$ are the local fields, $\boldsymbol{D}_{JE}(\boldsymbol{x})$ is defined by (??) and represent the dependence of the material properties as a function of space coordinates $\boldsymbol{x} = (x_1, x_2, \ x_3)$, and $\boldsymbol{D}^*_{JE}$ is the corresponding effective tensor that can be expressed in terms of the complex effective tensor $\boldsymbol{\sigma}^0$ similar to (??).

The definition (**??**) that follows from the variational principle (**??**) allows us to use the known variational methods to derive the bounds on the complex effective properties tensor $\boldsymbol{\sigma}^0$. For example, the simplest bounds can be obtained by using the constant trial fields in the equation (**??**). It leads to the Reuss-Voigt type bound[13],

$$\boldsymbol{D}^0_{JE} \le \langle \boldsymbol{D}_{JE}(\boldsymbol{x}) \rangle = \sum_{(l)} f^l \boldsymbol{D}^l_{JE}, \tag{5.5}$$

where $\boldsymbol{D}^l_{JE}$ are the matrices of the properties of the phases.

The dual to (**??**) variational principle (**??**) which is produced by the quadratic form with the matrix $\boldsymbol{D}_{EJ} = \boldsymbol{D}^{-1}_{JE}$ leads to the bound,

$$(\boldsymbol{D}^*_{JE})^{-1} \le \langle \boldsymbol{D}^{-1}_{JE}(\boldsymbol{x}) \rangle = \sum_{(l)} f^l (\boldsymbol{D}^l_{JE})^{-1} \tag{5.6}$$

Due to the relation $\boldsymbol{D}^{-1}_{JE} = \boldsymbol{D}_{EJ}$ (see (**??**)) the bounds (**??**) and (**??**) coincide as can be easily see comparing the expressions (**??**) and (**??**) for the matrices $\boldsymbol{D}_{JE}$ and $\boldsymbol{D}_{EJ}$ respectively.

Although we end up with the same inequality for both Reuss and Voigt type bounds, this inequality gives in fact lower and upper coupled bounds for the real and the imaginary parts of complex-valued effective tensor $\boldsymbol{\sigma}^0$. Indeed, the inequality (**??**) restricts the diagonal components of the block matrix $\boldsymbol{D}^0_{JE}$ independently of one another, and also requires the condition

$$\det(\langle \boldsymbol{D}_{JE}(\boldsymbol{x}) \rangle - \boldsymbol{D}^*_{JE}) \ge 0, \tag{5.7}$$

which leads to the coupled bounds[1,2] on the effective properties tensor $\boldsymbol{\sigma}^0$.

Similar to the Reuss-Voigt bounds, one can apply the Hashin-Shtrikman method[14], the translation method[1,2,15,16], or any other methods to derive more detailed bounds on the effective properties tensor. For all of them it is also sufficient to study only the variational problem (**??**); the consideration of the dual functional gives exactly the same bounds.

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