STRUCTURAL OPTIMIZATION AND BIOLOGICAL "DESIGNS"

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Abstract. The amazing rationality of biological "constructions" excites the interest to modelling them by using the mathematical tools developed in the theory of structural optimization. The structural optimization solves a geometrical problem of the "best" displacements of different materials in a given domain, under certain loadings. Of course, this approach simplifies the real biological problem, because the questions of the mechanism of the building and maintaining of structures are not addressed. The main problem is to guess a functional for the optimization of a living organism. The optimal designs are highly inhomogeneous; their microstructures may be geometrically different, but possess the same effective properties. Therefore the comparing of the various optimal geometries is not trivial. We show, that the variety of optimal geometries shares the same characteristics of the stress tensor in any optimal structure. Namely, special norm of this tensor stay constant within each phase of the optimal mixture.

The paper also addresses the uncertainty of the ultimate load in biological "structures". We discuss the corresponding min-max formulation of the optimal design problem. The design problem is formulated as minimization of the stored energy of the project under the most unfavorable loading. The problem is reduced to minimization of Steklov eigenvalues. Several stable solutions of various optimal design problems are demonstrated; among them are the optimal structure of a structure stable to variations to a main loading, the optimal specific stiffness of an uncertainly loaded beam, and the stable design of an optimal wheel.

1. Criteria of Optimality of a Structure of Bones

1.1. SEARCH FOR AN APPROPRIATE FUNCTIONAL

This meeting is devoted to a challenge problem of an explanation of the obvious rationality of the living organism. We mean an explicit optimality of a natural "structure" rather than a general reference to the evolution that perfects organisms.

The problem of bone structure provides a perfect object for such study. Indeed, a bone is a mechanical construction, made of composites with variable parameters that adapts itself to the working conditions. It performs a clear mechanical task of supporting the organism. These features are similar to man-made composite constructions of masts, bridges, towers, domes, etc. Therefore it is natural to apply optimization methods developed for the engineering constructions to the bone structures.

However, the two problems are not the same. In engineering problems, the aim is the minimization of a given functional which is not a subject of a search or even a discussion. The problem is to find the structure that minimizes a functional prescribed by a designer. On the contrary, the structure of a bone is known, its properties are measurable. But it is not clear, in what sense the bone structure is optimal.

This problem can mathematically be formulated as the search for an goal functional of an optimization problem, if the solution of that problem is known. This problem is not enough investigated, to our knowledge. Our forthcoming paper (Cherkaev & Cherkaeva, 1998[2]) discusses the subject in details. The simplest examples show that the problem can be underdetermined, and the functional is not unique.

1.2. OPTIMIZATION OF STIFFNESS AND THE CONSTANCY OF A NORM OF STRESS TENSOR

It seems that the evolutionary beneficial functional deals with the survival capacity (strength) of a bone, because the break of it would mean almost certain death of an animal. However, the criteria of the bone strength are complex: they must deal with long-term strength as well as with impact strength, with not well defined loading, etc. Presently, it is difficult to formulate a reliable mathematical optimization problem accounting for all the complexity of the problem.

There are numerous attempts to compare the bone structure with the structures, optimal with respect to the stiffness of an elastic construction (see the papers in this volume, for example). The resulting structures seem to be similar. In our opinion, this similarity does not mean that the stiffness is the functional that the nature "wants" to optimize. Indeed, it is hardly explainable what evolutionary advantage has an animal with a bit stiffer bones. Recall, that the greater stiffness means only a smaller deflection of the loading surface. Consider a leg bone, for example. Its contraction under the animal weight is definitely less than the rate of motion of the bones in joints. Therefore it is not clear why the evolution needs stiffer bones.

In view of this remark, the question remains why structures optimizing the stiffness, are reminiscent to the bones structures? A possible answer comes from the study of the local optimality condition. One can see, that the optimality with respect to the overall stiffness requires *the constancy of a norm of the stress in the material in each point of the structure.* These conditions show that the optimal structure adapts itself to the loading by varying the geometry of the structure, fraction of the strong material in the structure, but a norm of the stress remains constant everywhere:

$$\mathcal{N}(\boldsymbol{\sigma}) = |\sigma_1| + |\sigma_2| + |\sigma_3| = \text{ constant in all points of structure} \quad (1)$$

(The derivation of this criterion is based on the consideration of necessary conditions of Weierstrass type; it is similar to the derivation in (Cherkaev, 1998).)

The constancy of the norm of the stress in each point of the structure means decreasing of its maximum, which is directly related to the strength of the whole structure (we leave out the discussion of the exact form of the stress norm that is responsible for the strength). These remarks motivate us to consider the optimality of elastic behavior of constructions. We hope that these constructions are also optimal with respect of their strength.

2. Optimization under Uncertain Loading

Uncertain Loading The other serious problem in bio-structures is the uncertainty in the loading conditions. Applied to a bone forces are varying in time in the natural environment and they are not completely predictable. One have to formulate the problem to account possible variations and uncertainties in loading.

However, in rather extensive literature on optimal design major attention is paid so far to optimization of constructions that are subject to a fixed loading. The optimality requirement forces the structure to concentrate its resistivity against an applied loading, since its abilities to resist other loadings are limited. This high sensitivity to the loading restricts the applicability of most optimal designs. One can foresee a significant change in the optimal structure if the loading is not completely known, and below we demonstrate this change.

The problem of Optimal Design The overall compliance of an elastic construction is characterized by the mechanical work produced by an applied loading. This work is equal to the total energy stored in the loaded construction. It is found from the following variational problem

$$H(\boldsymbol{p}, \boldsymbol{f}) = \min_{\boldsymbol{\sigma} \in \Sigma} \left\{ \int_\Omega W(\boldsymbol{p}, \boldsymbol{\sigma}) \right\}, \tag{2}$$

where W is the (doubled) elastic energy

$$W(\boldsymbol{p}, \boldsymbol{\sigma}) = \boldsymbol{\sigma} : \boldsymbol{S}(\boldsymbol{p}, \boldsymbol{x}) : \boldsymbol{\sigma}. \tag{3}$$

$\boldsymbol{\sigma}$ is the stress tensor, the set Σ is (see (2), (3))

$$\Sigma = \left\{ \boldsymbol{\sigma} : \quad \nabla \cdot \boldsymbol{\sigma} = 0 \text{ in } \Omega, \quad \boldsymbol{\sigma} = \boldsymbol{S}^{-1} (\nabla u)^s, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T, \quad \boldsymbol{n} \cdot \boldsymbol{\sigma} = \boldsymbol{f} \text{ on } \partial\Omega \right\}. \tag{4}$$

$\boldsymbol{f}$ is the vector of applied boundary forces and $\boldsymbol{u}$ is the vector of deflection, $(\nabla \boldsymbol{u})^s = \boldsymbol{\epsilon}$ is the strain field, that is given by a symmetrized part of the gradient of $\boldsymbol{u}$, $(\nabla \boldsymbol{u})^s = (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)/2$. $\boldsymbol{S}(\boldsymbol{p}, \boldsymbol{x})$ is the tensor of elastic compliance: a fourth order symmetric positive tensor, which depends on the point $\boldsymbol{x}$ in Ω and on the structural parameter $\boldsymbol{p}$ that defines the material's properties. The symbol (:) denotes the contraction by two indices.

The stored energy H is a quadratic functional of the loading $\boldsymbol{f}$ that depends on the layout of the material's properties $\boldsymbol{p}$, called the design variables.

Consider the typical problem of optimal design: minimize H with respect to layout $\boldsymbol{p}$:

$$\min_{\boldsymbol{p} \in \mathcal{P}} H(\boldsymbol{p}, \boldsymbol{f}), \tag{5}$$

where $\mathcal{P}$ is the admissible set of design variables. There are many possible settings for the set $\mathcal{P}$: it can be defined as the set of effective moduli of the composite (Gibiansky & Cherkaev, 1987), or it could describe the shape of the body, the thickness of a thin construction, and so on.

Problem 1. Instabilities in the Optimal Design Problems. The following example demonstrates the instability of the optimal structure and suggests ways of reformulating the problem in order to stabilize the design.

Suppose that a square domain $a\,b\,c\,d$ filled with a composite material, is loaded by a uniaxial loading. Suppose for simplicity, that the composite is assembled from the material with unit compliance tensor $\boldsymbol{S}_1 = \boldsymbol{I}$ (the Poisson ratio is equal to zero and the Young modulus is equal to one) and from the void with infinite compliance: $\boldsymbol{S}_2 = \infty$. Suppose also, that the fractions m_1 of the material and m_2 of the void are equal to one half each:

$$m_1 = m_2 = \frac{1}{2}. \tag{6}$$

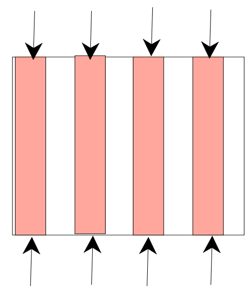

Figure 1. The optimal composite under the homogeneous axial loading.

Let the domain be loaded by a uniaxial loading

$$\boldsymbol{f}_0 = \begin{cases} \boldsymbol{i}_1 & \text{on} \quad a\,b \\ 0 & \text{on} \quad b\,c \\ -\boldsymbol{i}_1 & \text{on} \quad c\,d \\ 0 & \text{on} \quad a\,d \end{cases} . \tag{7}$$

The optimal design is obviously homogeneous. The loading $\boldsymbol{f}_0$ creates a stress field $\boldsymbol{\sigma}_1$,

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{or} \quad \boldsymbol{\sigma}_1 = \boldsymbol{i}_1 \otimes \boldsymbol{i}_1 \tag{8}$$

inside the domain.

The problem is to find the composite that minimizes the energy of the project under the loading $\boldsymbol{f}_0$. Obviously, the best structure is a simple laminate, with layers oriented along the loading (see figure 1). see ())

The effective compliance s_{1111} in the direction $\boldsymbol{i}_1$ of the loading is equal to the harmonic mean of the (unit) material's complaince $s_m = 1$ and the (infinite) complaince of the void $s_v = \infty$:

$$s_{1111} = \left(\frac{m_1}{s_m} + \frac{m_2}{s_v} \right)^{-1} = 2 \tag{9}$$

The minimal energy and the problem cost are:

$$W(\boldsymbol{\sigma}_1) = \boldsymbol{\sigma}_1 : \boldsymbol{S}_* : \boldsymbol{\sigma}_1 = s_{1111} \boldsymbol{\sigma}_{11}^2 = 2, \quad H(\boldsymbol{\sigma}_1) = 2 \tag{10}$$

This solution, however, is not satisfactory from a viewpoint of a common sense . Indeed, the laminate structure is extremely unstable, and its compliance tensor is singular. The laminate structure cannot resist any loading but the prescribed one. Its compliance is infinitely large for all other loadings. Simply speaking, the structure falls apart under any infinitesimally small applied stress that has either shear component, or a component along the axis $\boldsymbol{i}_2$.

Remark 2.1 *The described instability is typical for the projects that are designed to optimally resist to a prescribed loading, at the expense of the resistivity in other directions.*

Formulation of the Problem of Stable Optimal Design. Let us consider a problem of energy optimization of an elastic body Ω loaded by unknown forces $\boldsymbol{f}$ applied on the boundary $\partial\Omega$. In this paper, we focus on the dependence of the optimal project on the loading that belongs to a set $\mathcal{F}$: $\boldsymbol{f} \in \mathcal{F}$. We define the compliance Λ of a construction as the maximum of compliances upon all admissible loadings

$$\Lambda(\boldsymbol{p}) = \max_{\boldsymbol{f}\in\mathcal{F}} \; H(\boldsymbol{p}, \boldsymbol{f}), \tag{11}$$

and we formulate the problem of the optimal design against the "worst" loading:

$$\min_{\boldsymbol{p}\in\mathcal{P}} \Lambda = \min_{\boldsymbol{p}\in\mathcal{P}} \left\{ \max_{\boldsymbol{f}\in\mathcal{F}} \; H(\boldsymbol{p}, \boldsymbol{f}) \right\}. \tag{12}$$

To impose constraints on acting forces, we formulate a problem for a design that offers a minimal compliance in a class of loadings.

Integral Constraints for the Loading. Let the set of the loadings $\mathcal{F}$ be characterized by an integral constraint. It is convenient to consider the constraints as a quadratic form of the loading: this form leads to rather simple equations and possesses a needed generality and flexibility. Suppose that an unknown loading by normal forces $\boldsymbol{f} \in \mathcal{F}$ is constrained as following:

$$\mathcal{F} = \left\{ \boldsymbol{f} : \;\; \oint_{\partial\Omega} \boldsymbol{f} \cdot \Psi(S) \boldsymbol{f} \leq 1 \right\}, \tag{13}$$

where $\Psi(S)$ is a positively defined weight matrix,

$$\Psi(S) > 0, \quad \forall \; S \in \partial\Omega. \tag{14}$$

The introduced here weight function Ψ expresses a priori assumptions about the unknown loading. For instance, the case when all loadings are equally possible, corresponds to $\Psi = const(S)$.

The compliance of the design, introduced in (11), is given by the solution of the problem of maximization of the stored in the design energy with respect to the applied loadings $\boldsymbol{f} \in \mathcal{F}$:

$$\Lambda = \max_{\boldsymbol{f}} H(\boldsymbol{p}, \boldsymbol{f}) = \max_{\boldsymbol{f}} \min_{\boldsymbol{\sigma}} \frac{\int_\Omega W(\boldsymbol{p}, \boldsymbol{\sigma})}{\oint_{\partial\Omega} \boldsymbol{f} \cdot \Psi(S)\boldsymbol{f}}. \tag{15}$$

The energy H is a quadratic functional of $||\boldsymbol{f}||$, and (15) is the Rayleigh ratio of two quadratic forms of $\boldsymbol{f}$. Therefore problem (15) is reduced to an eigenvalue problem for a linear differential operator. The value $\Lambda(\boldsymbol{p})$ corresponds to the first eigenfunction or to the set of the eigenfunctions, that generate the most "dangerous" loading(s) from the considered class. Hence we formulate the stable optimal design problem as a problem of eigenvalue optimization:

$$J_* = \min_{\boldsymbol{p} \in \mathcal{P}} \Lambda(\boldsymbol{p}) = \min_{\boldsymbol{p} \in \mathcal{P}} \max_{\boldsymbol{f} \in \mathcal{F}} \min_{\boldsymbol{\sigma} \in \Sigma} \frac{\int_\Omega W(\boldsymbol{p}, \boldsymbol{\sigma})}{\oint_{\partial\Omega} \boldsymbol{f} \cdot \Psi \boldsymbol{f}}. \tag{16}$$

2.1. AN EIGENVALUE PROBLEM

Saddle Point Case. The question of whether or not the multiple eigenvalue case is taking place depends on the power of the control. It the control $\boldsymbol{p}$ is "weak", that is if the control cannot change the sequence of eigenvalues, then we are dealing with a saddle point situation. In this case, the minimal upon the control $\boldsymbol{p}$ eigenvalue corresponds to a unique eigenfunction $\boldsymbol{f}(\boldsymbol{p})$. The example below illustrates this situation.

In this case the functional Λ (15) is a saddle function of the arguments, and the operations of max with respect to $\boldsymbol{f}$ and min with respect to $\boldsymbol{\sigma}$ can be switched. Then varying the functional, we find the Euler equations for the most dangerous loading. Let us find this loading. Variation of (15) with respect to $\boldsymbol{f}$ gives:

$$\delta \Lambda_{\boldsymbol{f}} = -\left(\oint_{\partial\Omega} \boldsymbol{f} \cdot \Psi(S) \boldsymbol{f} \right)^{-1} (\boldsymbol{u} - \Lambda \Psi \boldsymbol{f})\ \delta \boldsymbol{f}, \tag{17}$$

which implies the relation point-wise between the optimal loading and the boundary deflection

$$\boldsymbol{f}(S) = \frac{1}{\Lambda} \Psi^{-1} \boldsymbol{u}(S), \quad \forall S \in \partial\Omega. \tag{18}$$

It is also easy to see that the stationary condition corresponds to the *maximum* not the minimum of the functional using the second variation technique.

The problem of the most dangerous loading $\boldsymbol{f}_0$ becomes an eigenvalue problem

$$\frac{1}{\Lambda} = \min_{\boldsymbol{\sigma}} \frac{\int_\Omega W(\boldsymbol{p}, \boldsymbol{\sigma})}{\oint_{\partial\Omega} \boldsymbol{u} \cdot \Psi(S)^{-1} \boldsymbol{u}}. \tag{19}$$

The cost Λ corresponds to the minimal eigenvalue given by the Rayleigh ratio (19), and the most "dangerous" loading corresponds to the first eigenfunction of this problem.

Remark 2.2 *One can consider also the problem of the most "favorable" loading, that is*

$$\Lambda_- = \min_{\boldsymbol{f}} \min_{\boldsymbol{\sigma}} \frac{\int_\Omega W(\boldsymbol{p}, \boldsymbol{\sigma})}{\oint_{\partial\Omega} \boldsymbol{f} \cdot \Psi \boldsymbol{f}}. \tag{20}$$

However, Λ_- is zero. Clearly, the spectrum of the operator is clustered at zero. A minimizing sequence is formed from often oscillating forces.

Euler Equations. The Euler equations (with respect to $\boldsymbol{\sigma}$) are

$$\begin{array}{ll} \nabla \cdot \boldsymbol{\sigma} = 0, \quad \boldsymbol{\sigma} = \boldsymbol{S}^{-1}(\boldsymbol{p}) : (\nabla \boldsymbol{u})^s & \text{in } \ \Omega, \\ \boldsymbol{u} = \Lambda \Psi \boldsymbol{\sigma} \cdot \boldsymbol{n} & \text{on } \partial\Omega. \end{array} \tag{21}$$

They describe the vibration of the body with inertial elements concentrated on $\partial\Omega$.

The problem admits the following physical interpretation: the optimal loading forces are equal to a distribution of inertial elements (concentrated masses) on the boundary component $\partial\Omega$. The specific inertia is described by the tensor Ψ, so it could include the resistance to the turning as well. The vibration of such loaded system excites the forces that are proportional to the deflection $\boldsymbol{u}$. The compliance is proportional to the eigenfrequency of vibrations. One can see that the introduced quantity Λ characterizes the domain or the construction itself, it represents the maximum of possible stored energy under any loading from the class $\mathcal{F}$.

These equations form an eigenvalue problem that possesses infinitely many solutions. We pick up the pair $\{\Lambda_1, \ \boldsymbol{\sigma}_1\}$ that corresponds to the maximal eigenvalue $\Lambda_1 = \max \{\Lambda_k\}$.

The problem (19) with unit matrix Ψ is called the Steklov eigenvalue problem, which considers the ratio of integrals of different dimensionality. The corresponding Euler equation (21) has an eigenvalue in the boundary condition. Similar optimality conditions were derived in (Cherkaeva, 1997) for the optimal boundary sources in electrical tomography problem.

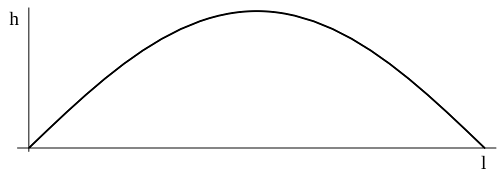

Figure 2. The stiffness $p(x)$ of the optimal beam.

Problem 2. Optimal Design of a Beam. The problems for beams and bending plates admit the loading distributed in the whole domain of the definition: on the interval in the case of the beam, and in the plane domain in the case of the bending plate or shell. In these problems, the loaded surface $\partial\Omega$ coincide with the domain Ω itself.

Consider an elastic beam whose energy density is

$$W = p(w'')^2 - 2fw, \tag{22}$$

where $p \geq 0$ is a material's stiffness, that can be varied from point to point. The stiffness is subject to the integral constraint

$$\int_0^l p\, dx = V \tag{23}$$

which expresses the limits on resources; f is the intensity of the normal loading, subject to the constraint

$$\int_0^l f^2\, dx = 1. \tag{24}$$

Consider an optimization problem of choosing a stiffness $p(x)$ that maximally resists to the most dangerous loading f:

$$\min_{p\geq 0, p\in(23)} \max_f \min_w \mu, \quad \mu = \frac{\int_0^l (p(w'')^2 - 2fw)\, dx}{\int_0^l (f^2)\, dx}. \tag{25}$$

The stationary conditions are:

$$\delta\, w: \quad (p\, w'')'' - f = 0,\ \forall x \in (0,\ 1), \quad pw''|_{x=0} = pw''|_{x=l} = 0, \tag{26}$$

$$\delta\, f: \quad f + \frac{w}{\mu} = 0,\ \forall x \in (0,\ 1), \tag{27}$$

$$\delta\, p: \quad (w'')^2 = \gamma,\ \forall x \in (0,\ 1), \tag{28}$$

where γ is the Lagrange multiplier for the constraint (23). This system admits a solution

$$w = -\mu f = -\gamma x(l-x)/2, \tag{29}$$

$$f = \frac{\sqrt{\gamma}}{\mu} x(l-x)/2, \tag{30}$$

$$p_0 = \frac{1}{\mu}\frac{x}{24}(x-l)\left((x-\frac{l}{2})^2 - \frac{5l}{4}\right). \tag{31}$$

Accounting for the constraints, we get

$$\mu = \frac{l^5}{5!V}, \tag{32}$$

$$p(x) = \frac{5V}{4l^5} x(l-x)\left(5l^2 - (2x-l)^2\right), \tag{33}$$

$$w(x) = -\frac{1}{V}\sqrt{\frac{l^5}{5!}}\frac{x(l-x)}{2}. \tag{34}$$

The optimal stiffness $p(x)$ of the beam is shown on figure 2. Interestingly, that the optimal solution is found analytically.

2.2. MULTIPLE EIGENVALUES

Eigenvalue Optimization. We return to the discussion of the project that minimizes the functional Λ, or minimizes the stored energy in the most unfortunate situation. The problem has the form (16). The specific effect of the min-max problem is the possibility of appearance of multiple eigenvalues. The mechanism of this phenomenon is the following. Minimization of the maximal eigenvalue likely leads to the situation when its value meets the second eigenvalue of the problem. In this case, both eigenvalues must be minimized together, until their common value reaches the third eigenvalue, and so on. The multiplicity means that two or more loadings give the same value of the problem. We will bring below an example demonstrating this phenomenon: the resistance of the construction to five different loadings in this example of the stable optimal design is the same. Similar min-max problem with multiple eigenvalues was considered in (Cherkaev & Cherkaeva, 1995) for nondestructive testing of the worst possible damage by applying optimal boundary currents.

There is an extended literature on eigenvalue optimization. It was understood in a different setting: the maximization of the fundamental frequency. We refer to the recent review papers (Cox & Overton, 1992; Seyranian *et al.*, 1994) and references therein.

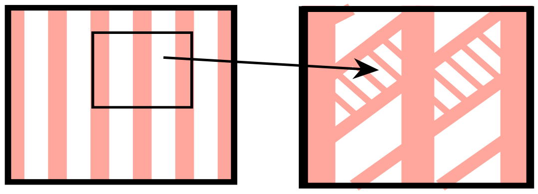

Figure 3. The schematic picture of the composite of the third rank.

Optimal Composite Structures. Consider the following problem of structural optimization. A domain made of a two-phase composite material of an arbitrary structure is loaded by an uncertain loading f_0. We want to find the most resistant structure of the composite, that is to minimize the functional (11). Here $\boldsymbol{p}$ is a vector of parameters that defines the tensor $\boldsymbol{S}_*$ of the effective compliance of the composite. For definiteness, consider the two-dimensional elasticity problem.

We do not know a priori, how many loadings should be taken in consideration. But clearly, it is sufficient to enlarge the set of admissible composites to those which minimize the sum of elastic energies caused by any number of different loadings. These composites are described in the paper by Avellaneda (Avellaneda, 1987): in two-dimensional elasticity, they form the class of the so-called matrix laminates of the third rank (see figure 3).

The effective property tensors of these composites admit an analytical expression through their structural parameters. To describe the class of the effective tensors of these anisotropic structures, we use the natural tensor basis

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{e}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{35}$$

Any stress and strain matrices are represented as vectors in their basis, and the effective compliance $\boldsymbol{S}_*$ of matrix laminates is given by the 3×3 matrix (see (Gibiansky & Cherkaev, 1987))

$$\boldsymbol{S}_* = \boldsymbol{S}_1 + m_2 \left((\boldsymbol{S}_2 - \boldsymbol{S}_1)^{-1} + \frac{2m_1}{E_1} \boldsymbol{N} \right)^{-1}, \tag{36}$$

where $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are the compliance matrices of the first and the second materials, m_1 and m_2 are the volume fractions, E_1 is the Young modulus of the first material (which forms the envelope). The matrix $\boldsymbol{N}$ depends on the structural parameters: on the angles θ_i between the tangent to the laminates and the axis $\boldsymbol{i}_1$, and on the relative thickness α_i (see figure 3):

$$\boldsymbol{N} = \sum_i^3 \alpha_i \begin{pmatrix} \cos^4\theta_i & \cos^2\theta_i\sin^2\theta_i & \cos^3\theta_i\sin\theta_i \\ \cos^2\theta_i\sin^2\theta_i & \sin^4\theta_i & \cos\theta_i\sin^3\theta_i \\ \cos^3\theta_i\sin\theta_i & \cos\theta_i\sin^3\theta_i & \cos^2\theta_i\sin^2\theta_i \end{pmatrix}, \quad \sum_i^3 \alpha_i = 1, \quad \alpha_i \geq 0. \tag{37}$$

α_i and θ_i form the control vector $\boldsymbol{p}$.

Problem 1 Revisited. Unstable Design for a Uniaxial Loading. Discussing the instabilities of the optimal project in the problem 1 below, we considered the optimization problem

$$\min_{\boldsymbol{p}\in\mathcal{P}} H(\boldsymbol{p}, \boldsymbol{f}_0), \tag{38}$$

where $\boldsymbol{f}_0$ is given by (7), and the set $\mathcal{P}$ constrains the parameters of the composite α_i and θ_i. The solution of the problem is a laminate, that is easily found from (36), (37). It corresponds to the parameters $\alpha_1 = 1,\ \theta_1 = 0,\ \alpha_2 = \alpha_3 = 0$. This structure is shown on figure 1, below we discussed the instabilities of this solution. Indeed, the compliance tensor $\boldsymbol{S}_*$ of a third rank composite becomes ($m_1 = m_2 = 1/2$)

$$\boldsymbol{S}_* = \boldsymbol{S}_1 + \frac{1}{2}\left(\boldsymbol{N}\right)^{-1}. \tag{39}$$

For the optimal choice of the parameters $\alpha_i,\ \theta_i$ the matrix $\boldsymbol{N}$ (see (37)) has two zero eigenvalues, and the two eigenvalues of $\boldsymbol{S}_*$ corresponding to the shear loading and the loading in the direction $\boldsymbol{i}_2$, are infinite (see (36)). Hence the compliance of the structure is infinitely large for any loading that has a projection on these two eigenvectors.

Problem 3. Stable Design for a Uniaxial Loading. Now we reformulate the design problem (38) to obtain a stable project. Suppose that the loading is not exactly known. Namely, the loading field $\boldsymbol{\sigma}$ can take one of the following six values $\boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i, \quad i = 1, \ldots 6$, where $\boldsymbol{\sigma}_1$ is given by (8) and

$$\begin{aligned} \boldsymbol{\tau}_{1,2} &= \pm r\boldsymbol{e}_1 = \begin{pmatrix} \pm r & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{\tau}_{3,4} = \pm r\boldsymbol{e}_2 = \begin{pmatrix} 0 & 0 \\ 0 & \pm r \end{pmatrix}, \\ \boldsymbol{\tau}_{5,6} &= \pm r\boldsymbol{e}_3 = \begin{pmatrix} 0 & \pm r/\sqrt{2} \\ \pm r/\sqrt{2} & 0 \end{pmatrix}. \end{aligned} \tag{40}$$

Here, $r > 0$ is a real parameter. The additional loadings of the magnitude r corresponding to the cases 1, 3, 5 are shown on figure 4. The 'twin' loadings correspond to the reverse directions of the forces.

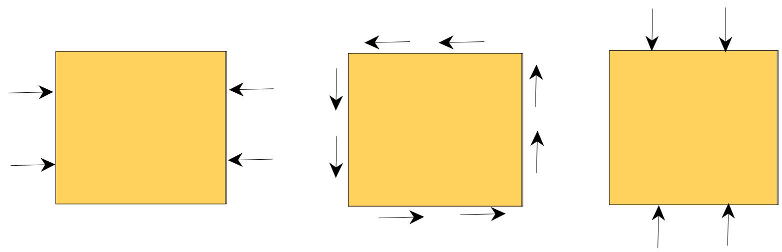

Figure 4. The additional loadings of the magnitude r.

Assume, in addition, that r is smaller than the magnitude of the "main" loading, which in our example is equal to one. The six loadings are viewed as small perturbations of the main loading, that correspond to all linearly independent directions of the symmetric tensor $\boldsymbol{\sigma}$. In spite of the smallness of r, the perturbation of the functional (38) is infinitely large, if $\boldsymbol{S}_*$ is optimally chosen. This characterizes the instability of the optimal project to those perturbations.

Let us reformulate the optimization problem. We are looking for a structure of a composite that minimizes the *maximum* of compliances $H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i)$ upon all considered loadings.

$$\min_{\boldsymbol{p}=\{\alpha_i,\ \theta_i\}} \left\{ \max_{i=1..6} H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i) \right\}. \tag{41}$$

The obtained min-max problem asks for the minimal compliance in the case of the "most dangerous" loading. To construct the solution of the optimization problem, we introduce a variable z that is greater than any of $H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i)$,

$$z \geq H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i), \quad i = 1, ...6. \tag{42}$$

The problem (41) can be formulated as follows (see (Demjanov & Malozemov, 1972)):

$$\min_{\boldsymbol{p}} \left\{ z + \sum_{i}^{6} \lambda_i^2 (z - H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i)) \right\}, \tag{43}$$

where λ_i^2 are the non-negative Lagrange multipliers by the constraints (42). The Lagrange multiplier is equal to zero, if this relation is satisfied as a strong inequality, and is non-zero, if it is satisfied as an equality (Demjanov & Malozemov, 1972):

$$\lambda_i^2 = \begin{cases} = 0 & \text{if} \quad z > H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i) \\ > 0 & \text{if} \quad z = H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i) \end{cases}. \tag{44}$$

The problem requires minimization of the weighted sum of energies of the 'dangerous' loadings $(\boldsymbol{\tau}_i),\ i \in I$. Here I is the set of such 'dangerous' loadings. Other loadings lead to the smaller energies $H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_j)$: $H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_j) < H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i)$, if $i \in I,\ j \notin I$, and therefore to $\lambda_j = 0$. This leads to the equalities

$$\begin{array}{l} z = H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i), \quad \text{if} \ \ i \in I, \\ z > H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i), \quad \text{if} \ \ i \notin I \end{array}. \tag{45}$$

Applying to the problem (43), we argue that the set of dangerous loadings in this example consists of five elements, $I = \{1, 3, 4, 5, 6\}$:

$$H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_1) > H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_2), \tag{46}$$

$$\begin{array}{c} H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_3) = H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_4), \\ H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_5) = H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_6). \end{array} \tag{47}$$

The inequality (46) is explained by the observation that an additional loading, if codirected with the main load, will either increase or decrease its magnitude independently of the composite structure. Clearly, the energy of the more intensive loading is greater.

The symmetry of the loadings # 3 and #4 and of the loadings # 5 and # 6 together with the symmetry of the set of admissible structural tensors $\mathcal{P}$ suggests that the "twin" loadings lead to the same cost of the problem. In other words, the same project $\boldsymbol{p}$ minimizes both $H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_3)$ and $H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_4)$, keeping them equal to each other; the same for the other pair of loadings. To achieve the equalities (47), we require the symmetry of the would be optimal tensor $\boldsymbol{S}_*$ (see (36)):

$$\alpha_1 = 1 - a, \quad \alpha_2 = \alpha_3 = a/2, \quad \theta_1 = 0, \quad \theta_2 = -\theta_3 = \theta, \tag{48}$$

where a and θ are two parameters. Physically, we require the orthotropy of $\boldsymbol{S}_*$.

Under the conditions (48), the matrix $\boldsymbol{N}$ (see (37)) takes the form

$$\boldsymbol{N} = (1-a) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a \begin{pmatrix} \cos^4\theta & \cos^2\theta \sin^2\theta & 0 \\ \cos^2\theta \sin^2\theta & \sin^4\theta & 0 \\ 0 & 0 & \cos^2\theta \sin^2\theta \end{pmatrix} \tag{49}$$

and, from (39)

$$\begin{aligned} \boldsymbol{S}_* \ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2a(1-a)} \begin{pmatrix} a & -a\cot\theta & 0 \\ -a\cot\theta & T & 0 \\ 0 & 0 & (1-a)\csc^2\theta \sec^2\theta \end{pmatrix}, \\ T \ &= \frac{1}{8}(\, 8 - 5\,a + 4\,a\, \cos 2\,\theta + a\, \cos 4\,\theta \,)\, \csc^4\theta. \end{aligned} \tag{50}$$

Remark 2.3 *Note, that the matrix becomes singular when $a \to 0$, which corresponds to unstable design.*

The described class of symmetric composites is defined by two parameters θ and a. The symmetry of the project eliminates the necessity to compare the loadings except from those with numbers 1, 3, 5. It turns out that these loadings are equally "dangerous":

$$H(a,\theta;\boldsymbol{\sigma}_1+\boldsymbol{\tau}_1)=H(a,\theta;\boldsymbol{\sigma}_1+\boldsymbol{\tau}_3)=H(a,\theta;\boldsymbol{\sigma}_1+\boldsymbol{\tau}_5). \tag{51}$$

Two equalities (51) allow to compute the optimal values of θ and a. One can easily see that the problem is always solvable. The optimal values of the parameters θ and a correspond to the solution of the min-max problem:

$$J(a,\theta)=\min_{a,\theta}\left\{\max\left\{H(a,\theta;\boldsymbol{\sigma}_1+\boldsymbol{\tau}_1),\ H(a,\theta;\boldsymbol{\sigma}_1+\boldsymbol{\tau}_3),\ H(a,\theta;\boldsymbol{\sigma}_1+\boldsymbol{\tau}_5)\right\}\right\}. \tag{52}$$

Note, that the project (50) is not optimal for any single loading but it is optimal for the set of them. The solution provides an example of a mixed strategy in the game: loadings versus design.

Illustration. Set $r=0.1$. The graph of the function $J(a,\theta)$ is shown on figure 5. The optimal values of the parameters are $\theta=0.889,\ a=0.0496,\ J=2.483$. We see that the compliance is bigger than the compliance of the construction optimal for a single load. On the other hand, the found construction is stable to all loadings, unlike the original design.

The picture of the optimal structure is shown on the figure 3. Note that a part of the material is removed from the laminates that resist the main load. This material is placed in "reinforcements" that reduce the compliance in all directions.

2.3. INVARIANCE OF LOADING AND SYMMETRY OF THE DESIGN

An interesting statement follows from the previous consideration is an analog of the Noether theorem for an optimal design problem.

If the restrictions on the loading and the boundary conditions are invariant to rotation than the optimal design could be rotational symmetric. Indeed, the symmetry of the loading restrictions implies that the ratio (16) t possesses a symmetric set of eigenfunctions with a common eigenvalue.

Symmetry. An Optimal Wheel. The next example demonstrates the appearance of symmetric projects in a min-max optimal design problem.

Problem 4. Consider the problem of a design of an optimal wheel. A circular domain is loaded by a non-axisymmetric loading that consists of

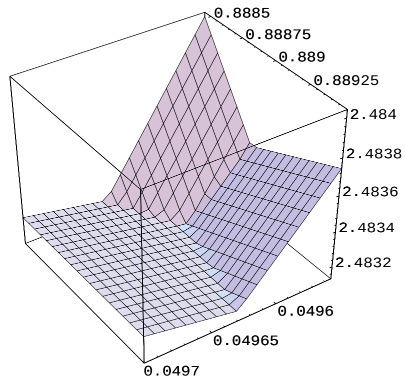

Figure 5. Graph of the function $J(a, \theta)$ in a proximity of the minimum.

a pair of radial forces applied to the rim and to the hub. These forces can move circumferential, which corresponds to the revolution of the wheel. If a loading $\boldsymbol{f}(S)$ is admissible, then any shifted loading $\boldsymbol{f}(S+\theta)$ is admissible too. Here, S is the circumferential coordinate and θ is an arbitrary real number.

Consider an optimal design problem. Suppose, that it is required to minimize the maximal compliance of the wheel in a class of forces. The design which minimizes the maximal compliance is obviously axisymmetric even if a particular loading is not. The symmetry comes from the min-max requirement of the equal resistance to all forces $\boldsymbol{f}(S+\theta)$: the project is independent of the angle θ.

The optimal axisymmetric layout of the composite properties $\boldsymbol{S}_*(\rho)$ in any particular point ρ minimizes the integral over θ of the energy distribution. The solution locally is again the third rank laminate, symmetric with respect to angular coordinate θ. The properties of the structure vary with the radius.

In the large, it can be represented as a periodic system of radii and two symmetric spirals (see figure 6). The period of the spirals is infinitesimal, and the thickness of the materials varies with radius.

Generally, the uncertainty in the direction of possible impact leads to cylindrical of spherical shapes of optimally designed structures. Would this explain the shapes of skulls, eggs, and the cylindrical shapes of bones and of bamboo stalks?

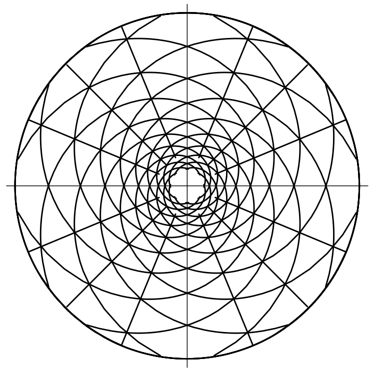

Figure 6. The cartoon of the optimal structure of the wheel.

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