

Principles of optimization of structures against an impact

Andrej Cherkaev¹, Elena Cherkaev², Seubpong Leelavanichkul³

¹ Department of Mathematics, University of Utah, Salt Lake City, Utah, USA, cherk@math.utah.edu

² Department of Mathematics, University of Utah, Salt Lake City, Utah, USA, elena@math.utah.edu

³ Department of Mechanical Engineering, University of Utah, Salt Lake City, Utah, USA, sleelava@eng.utah.edu

1. Abstract

We discuss principles of design of optimal structure subjected to a dynamical load. The goal is to design a structure which is capable of sustaining an impact of a dynamical load (collision or blast) by absorbing and redistributing the energy of the collision and keeping the structural integrity.

An unstructured material can absorb energy until it melts. However, a tiny portion of this energy yields to a construction's disintegration due to instability of the damage that leads to energy concentration. Appearance of concentrated damage zones, such as cracks or delaminating, destroys the construction. The remains of the disintegrated construction still can absorb a lot of energy. The goal of optimal design is to maximally use the ability of a damageable material to absorb energy.

We develop a theory of macro- and micro behavior of damageable structures. Damage is understood as a process of irreversible phase transition from initial to damaged state. The wave of phase transition is described and the structural parameters are optimized. We propose a concept of a replaceable protective structure design. Such structure should dissipate maximum energy, be able to spread energy of damage, and the damage process should be as stable as possible. An optimized structure should trigger stress waves propagation and thereby spread the stress more evenly; it must also contain multiple energy absorbers. This can be achieved by the use of a composite with a special structure made of a strong frame, and an absorbing filler material. The absorbing rate and rate of damage propagation are times amplified by waiting-link structures that we describe below.

The optimization strategy requires a proper choice of design parameters. They are: the shape of the whole protective construction, location of beams or frames and waiting links, and the choice of materials in the structure and in the outer layers.

2. Keywords: optimization, fracture, bistable, waiting links, wave of damage

3. Introduction

We investigate a propagation of structural damage, develop a technique for optimization of the impact-protecting structures, and present models of a protective structure with improved characteristics. The designed structures must sustain a sudden impact: they absorb the energy while sustain their structural integrity. The goal is in contrast with the conventional optimal design of constructions which sustain a given static load. Examples of protective structures are shown in Figure 1. Commonly used protective devices include car bumpers, helmets, tempered glass, climbing "screamer," and woven baskets of hot balloons. These very different constructions are all designed to be damaged or destroyed in a collision saving the protected object - main vehicle or its passengers. These constructions are used only once, they absorb the collision (impact) energy shielding the protected object and keeping the structural integrity. Both features are equally important. In a collision, a properly designed bumper will be badly damaged, thereby absorbing the energy of impact, and saving the vehicle and its passengers. A strong and stiff bumper that stays undamaged while the car is ruined does not fit its purpose. Likewise, a fragile bumper does not fit because it is easily destroyed and does not absorb enough energy to protect the vehicle. The optimally designed structure realizes the known recommendation of defensive driving instructions "Choose to hit something that will give way (such as brush or shrubs) rather than something hard." The principle of design of protective structure is in contrast with the conventional optimal design of constructions of maximal stiffness, which should stay undamaged under a given static load.

The total fracture energy is an important factor when determining the capability of a structure to withstand a dynamical loading. In most structures, however, the energy density is uneven. Localized yielding occurs at a particular location while the total energy associated with fracture in the structure is relatively small. To increase the total fracture energy, the concept of a bistability has been introduced [1]. In bistable structures, yielding is produced in a larger volume, thus increasing the total fracture energy. A bistable structure is designed with localized redundant load paths. When one load path yields and subsequently fails, the remaining load path can carry the load and retain structural integrity. A bistable structure may be developed by forming a chain or lattice of bistable structural elements [2-4]. Multiple regions of such structure exhibit yielding prior to ultimate failure. Thus, the



Figure 1: Examples of protective structures

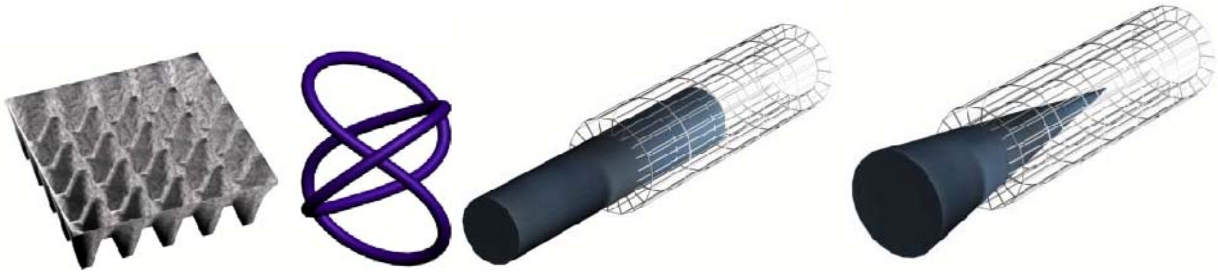


Figure 2: Structures that absorb energy and can sustain damage

bistable structures allow for the delocalization of yielding in a structure.

This paper reviews the results from our previous developments of the bistable structures and their applications. We emphasize their design principles and advantages in comparison to conventional structures. The structure of the paper is as follows. In Section 4, we discuss the bistability and its features [2, 3]; in Section 6 and 7, we review the dynamics of bistable lattices from elastic-brittle material [5], and from elastic-plastic materials [6], respectively. In Section 8, new morphology of spiraling bistable structures [7, 8] is discussed.

4. Elements and features of an impact resistive structure

In this section, we analyze local damage behavior. “Local” means that the load is assumed to be uniform everywhere and the problem becomes to design a strip that absorbs maximal energy and has large and stable deformation. In all unstructured material, stress rate decreases when strain increases, due to development of imperfection, micro-cracks opening, dislocation concentration, and similar mechanisms. This phenomenon leads to stress concentration and instability that eventually destroys the structure. We are looking for structures that become stiffer and stronger when damaged, thanks to their morphology. Such structures would evenly distribute the stress and will be stable even when partially destroyed (Figure 2). Such structures can experience multiple inner breakages. They can sustain large deformation without destruction as the damage is transported away from the impacted region because of stable structural transform.

Strong elastic waves and waves of damage propagate throughout the construction, reflecting from its boundaries and interfering with each other. This process leads to a high concentration of stresses that destroys the construction. On the other hand, the elastic waves can be redirected, thereby sending energy away from the zone of contact as in the lattice shown in Figure 3.

Initiation and redirection of waves can be effectively done by waiting-link structures. Waiting link is an assembly of roughly parallel two brittle-elastic or elastic-plastic rods, one of which is longer than the other. When the shorter (basic) link fails, the load is assumed by the second (waiting) element that was initially inactive due to its morphology. The structures are able to consume kinetic energy of the projectile; because of their local instabilities; they excite intensive waves which evenly distribute the “partial damage” and relax the structure by allowing for large local deformations of the elements.

The bistable cellular structures (lattices) containing “waiting elements” naturally excite and transmit waves of damage, acting as a domino train. These features of bistable structures from waiting elements could be utilized in

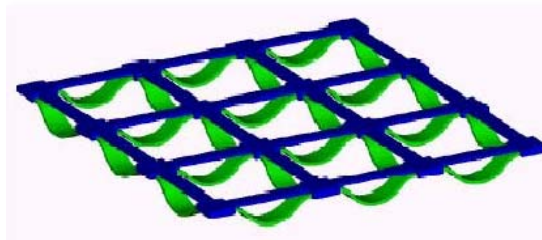


Figure 3: Structure that generates elastic wave across the direction of an impact

protective structures. Bistable waiting links lead to large but stable pseudo-plastic strains and thus increase the resistivity. The structure with bistable links transforms the energy of an impact into fast traveling waves that distribute energy throughout a larger area where it dissipates.

Waves in locally unstable systems are controllable. The possibility to redirect the transition wave in elongated strip-like structures was demonstrated in [5]. Depending on a tuning of structural parameters, the damage may either start at the end where the impact is applied and propagate toward the opposite end where the structure is fasten, or it may start at the fasten end and propagate toward the end where the force is applied, or two transition waves may be initiated at both ends simultaneously. The process is controlled by the parameters of the structure; it is possible to create structures which direct waves in desirable direction along the protective plate orthogonal to the direction of the impact.

5. Effectiveness Criteria

An integral criterion that is not sensitive to the details of the damage is suggested by [9] where the variation of the impulse of the projectile is measured. It is assume that the projectile hits the structure flying into it vertically down.

To evaluate the effectiveness, the ratio R in the vertical component $p_v : \mathbf{p}_p = [p_n, p_v]^T$ of the impulse \mathbf{p}_p of the projectile before and after the impact:

$$R = \frac{p_v(T_{\text{final}})}{|p_v(T_0)|} , \quad (1)$$

where T_0 and T_{final} are the initial and the final moments of the observation, respectively. The variation of impulse of the projectile R shows how much of it is transformed to the motion of structural elements. Parameter R evaluates the structure's performance using the projectile as the measuring device without considering the energy dissipated in each element of the structure; it does not vary when the projectile is not in contact with the structure.

Other criteria compare the state of the structure before and after the collision. These criteria are applicable only if the structure (or its pieces) reaches a steady state after the collision. This is why we need the dissipation factor in the model. Without this factor, the elastic waves never stop and their interferences may cause additional damage to the structure any time after the collision. We register two criteria:

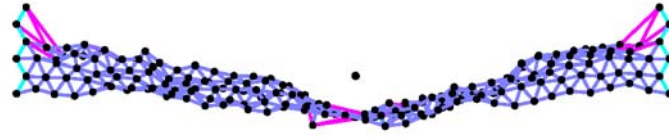
1. The percentage of partially damaged links.
2. The percentage of destroyed links.

The first number shows how effectively the damage is spread, and the second one shows how badly the structure is damaged. Ideally, we wish to have a structure in which all elements are partially damaged, but no element is completely destroyed. The number of destroyed elements is a rough quality criterion. It ignores a significant factor – the positions of the destroyed links.

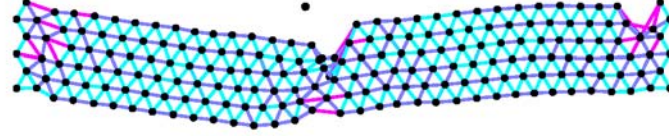
Numerical examples of a structure constructed with waiting elements are shown in Figure 4. Mathematical details are given in the following sections. Intact waiting elements (both links are undamaged) are aquamarine; partially damaged elements (the short link is destroyed, the longer one is undamaged) are blue; destroyed elements (both links are damaged) are purple. The fraction of material used for the first (shorter) rod in the waiting link is given by a parameter α . The structures with $\alpha = 0.5$ and $\alpha = 1.00$ (conventional structure) soon develop cracks and fall apart allowing the projectile to go through (see Figure 4c, and 4d) while the structures with $\alpha = 0.10$ and $\alpha = 0.25$ preserve the structural integrity by dissipating energy and taking the stress away from the point of impact; this results in the rejection of the projectile (see Figure 4a and 4b). Notice that the final time T_{final} is twice as small in the last two examples.

The propagation of the damage in the bridge is due to several factors:

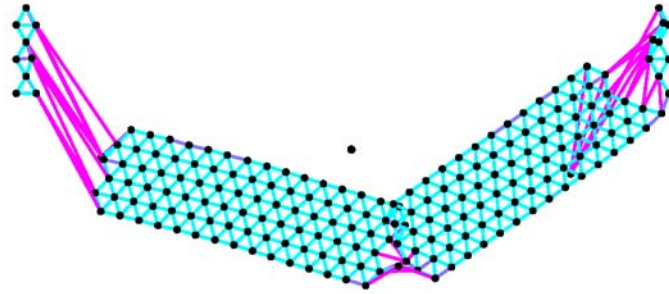
- (i) local instabilities of the part of the network that contains a damaged link,
- (ii) the increase of force on neighboring links which allows the damage to spread,
- (iii) the waves that propagate through the network and initiate damage in areas remote from the point of collision.



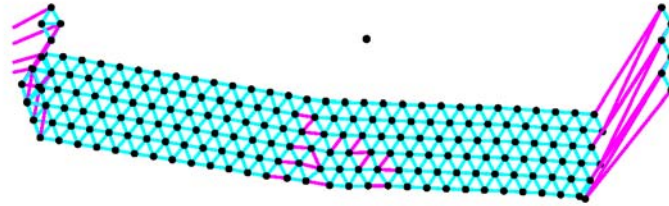
(a) $\alpha = 0.10$, $T_{\text{final}} = 500$, projectile is rejected.



(b) $\alpha = 0.25$, $T_{\text{final}} = 500$, projectile is rejected.



(c) $\alpha = 0.50$, $T_{\text{final}} = 250$, projectile goes through.



(d) $\alpha = 1.00$, $T_{\text{final}} = 250$, projectile goes through.

Figure 4: The final stages of the impacted bridges. The designs use the same amount of the materials, but differ in the parameter α : fraction of material used for the first (shorter) rod in the waiting link.

6. Elastic-brittle waiting elements

Consider structural elements-waiting links that significantly increase the resistivity of the structure due to their morphology. These elements and their quasistatic behavior are described in [5]. The link is an assembly of two elastic-brittle rods, lengths L and Δ ($\Delta > L$) joined by their ends. The longer bar is initially slightly curved to fit. When an increasing external elongation stretches the link, only the shortest rod resists in the beginning. If the elongation exceeds the critical value, this rod breaks and the next (longer) rod then assumes the load replacing the broken one (Figure 5).

Assume that a unit amount of material is used for both rods. This amount is divided between the shorter and longer rod: The amount α is used for the shorter (first) rod and the amount $1 - \alpha$ is used for the longer (second) one. The cross-sections s_1 and s_2 of rods are

$$s_1(\alpha) = \frac{\alpha}{L} \quad \text{and} \quad s_2(\alpha) = \frac{1-\alpha}{\Delta}. \quad (2)$$

The force-displacement relationship in the shorter rod is

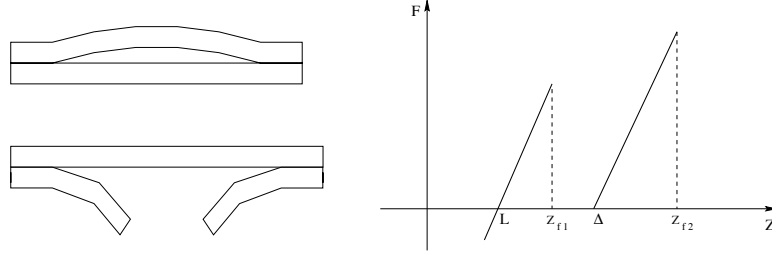


Figure 5: The waiting link mechanism and its the force-displacement response

$$F_1(z) = ks_1(\alpha) \left(\frac{z}{L} - 1 \right) (1 - c_1), \quad (3)$$

where $c_1 = c_1(z, t)$ is the damage parameter for this rod; it satisfies the following equation

$$\frac{dc_1(z, t)}{dt} = \begin{cases} v_d & \text{if } z \geq z_{f1} \text{ and } c_1(z, t) < 1 \\ 0 & \text{otherwise} \end{cases}, \quad (4)$$

where $z_{f1} = L(1 + \varepsilon_f)$, $c_1(z, 0) = 0$, and v_d is the speed of damage.

The longer rod starts to resist when the elongation z is large enough to straighten this rod. After the rod is straight, the force-displacement relationship is similar to that for the shorter rod:

$$F_2(z) = \begin{cases} ks_2(\alpha) \left(\frac{z}{\Delta} - 1 \right) (1 - c_2) & \text{if } z \geq \Delta \\ 0 & \text{if } z < \Delta \end{cases}. \quad (5)$$

Here F_2 is the resistance force and $c_2 = c_2(z, t)$ is the damage parameter for the second rod:

$$\frac{dc_2(z, t)}{dt} = \begin{cases} v_d & \text{if } z \geq z_{f2} \text{ and } c_2(z, t) < 1 \\ 0 & \text{otherwise} \end{cases}, \quad (6)$$

where $c_2(z, 0) = 0$. These equations are similar to (3) and (4), where the cross-section $s_1(\alpha)$ is replaced by $s_2(\alpha)$ and the critical elongation z_{f1} by $z_{f2} = \Delta(1 + \varepsilon_f)$. The difference between the two rods is that the longer (slack) rod starts to resist only when the elongation is large enough. The total resistance force $F(z)$ in the waiting link is the sum of $F_1(z)$ and $F_2(z)$:

$$F(z) = F_1(z) + F_2(z). \quad (7)$$

The graph of this force-displacement response for the monotonic external elongation is shown in Figure 5 where the damage parameters jump from zero to one at the critical point z_{f1} .

6.1. A two-dimensional assembly of the waiting links.

Consider a unilateral triangular grid: each inner point has six equally distant neighbors (see Figure 6). The distance between neighboring knots is equal to L . The knots in the boundary (including corners) have lower number of neighbors. The waiting links connect the neighboring knots. Initially, the system is in equilibrium and does not have any inner stress. If the strain is small everywhere and each link is strained less than the critical value z_f , the system is linear and it models a linear elastic material. After the first rod in a link breaks and is replaced with a longer one, the net experiences an “irreversible phase transition.”

After each break, the network changes its elastic properties and its equilibrium position. The dynamics of the damage and failure of the net can be viewed as a series of these transitions. In contrast with the conventional structure, the waiting links become stronger after first break. Another advantage is that the additional slackness that is added after the break helps to spread the damage across the structure.

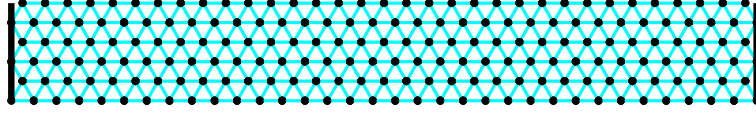


Figure 6: A two-dimensional assembly of the waiting links represented by a unilateral triangular grid

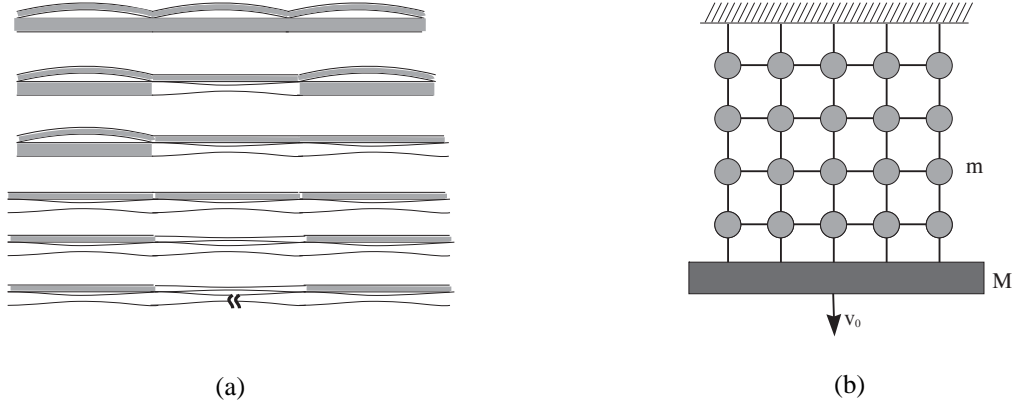


Figure 7. Elastic-plastic lattices with waiting elements

6.2. Kinematic of waiting links

It is assumed that the inertial masses m_i are concentrated in the knots, joined by the inertia-less waiting links (nonlinear springs), therefore the dynamics of the structure is described by ordinary differential equations of motion of the knots. We assume that the links are elastic-brittle, as described above.

Additionally, we assume that the space between the knots is filled with a viscous substance with the dissipation coefficient γ . Without the viscosity, the system never reaches a steady state. On the other hand, even a slow external excitation leads to intensive waves in the system caused by breaking of the links. These waves, reflected from the boundaries, cause intensive vibration of the structure and the viscosity eventually absorbs the energy.

The motion of the i th knot satisfies the equation

$$m_i \ddot{\mathbf{z}}_i + \gamma \dot{\mathbf{z}}_i = \sum_{j \in N(i)} \frac{F_{ij}(|\mathbf{z}_i - \mathbf{z}_j|)}{|\mathbf{z}_i - \mathbf{z}_j|} (\mathbf{z}_i - \mathbf{z}_j). \quad (8)$$

where \mathbf{z}_i is the vector of coordinates of the i th knot, $|\cdot|$ is the length of the vector, $N(i)$ is the set of knots neighboring the knot i , and m_i is the mass of the i th knot. The force F_{ij} in the ij th link depends on the damage parameters $c_{ij,1}$ and $c_{ij,2}$ as in (7). The set of neighbor knots depends on the geometric configuration.

7. Elastic-plastic lattices with waiting elements

A chain or lattice of bistable links from an elastic-plastic material is considered here as presented in [6]. Each bistable link consists of the straight main link and the longer initially curved waiting link. The main and waiting links are made of the same elastic-plastic material and joined at the ends. An example of a one-dimensional chain is shown in Figure 7a.

Initially, the resistance of the assembly is mainly due to the elastic response of the main link. The resistance of the curved waiting link is relatively small. As the elongation increases, the main link enters the plastic zone and develops a neck which reduces its resistance. At that moment, the waiting link is straightened and its resistance sharply increases. The sum of the resistances of the two links becomes bistable, if the parameters of these links are properly chosen: The total resistance increases, then decreases, then increases again. The increasing regimes correspond to the locally stable state, and the decreasing regime to the unstable state. The passage between these states is the phase transition.

The properly designed bistable chain is much more resistant than the conventional chain. In a conventional elongated bar of elastic-plastic material, the necking corresponds to the beginning of a fatal instability; the damage is accumulated in the region of the neck and the sample breaks. Only one neck is developed in the sample independently of its length. In a bistable assembly, the activated waiting link arrests the development of the irreversible plastic deformation and prevents the failure of the necked main link. When the elongation of the whole chain increases, another neck is formed in a neighbor assembly instead, and the damage propagates. This sequentially

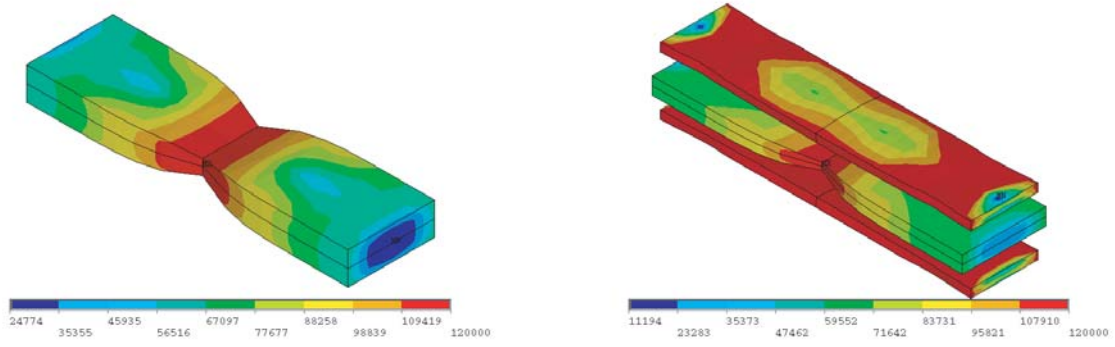


Figure 8: Contour plot of von Mises stress at the final load step. Right: conventional solid link. Left: Bistable link

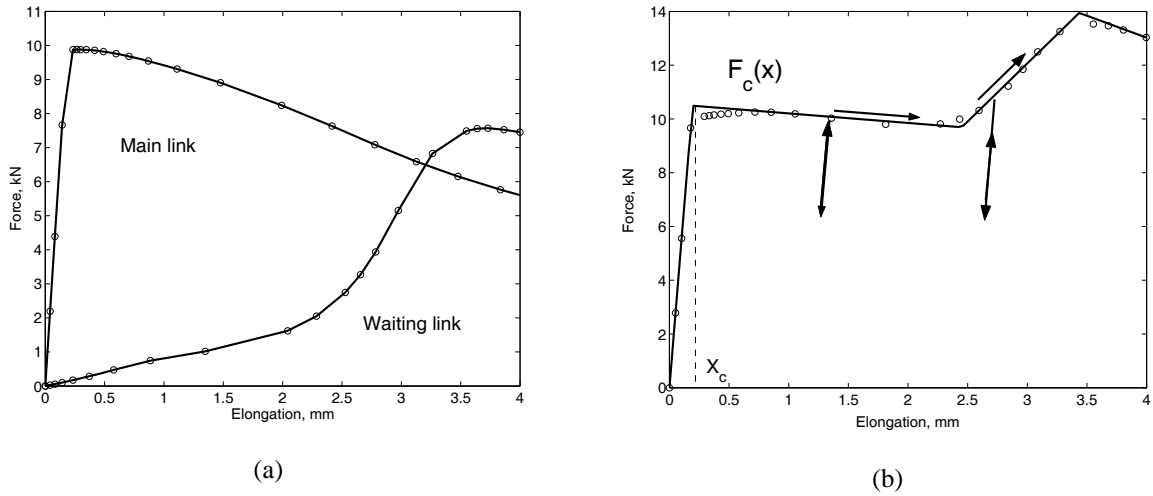


Figure 9: Force–displacement diagram for the main and waiting links

occurred transformation from one stable state to another creates a transition wave that propagates along the chain. The process continues until all links experience the transition, then the deformations in the waiting links also become plastic and the chain breaks.

The bistable link is compared with the solid link of the same length, mass, and material, see Figure 8. The solid link also develops a neck, but lacks the second stable interval of elongation. Notice that the elongation of a rectangular lattice under plain normal impact is equivalent to the elongation of the chain (Figure 7b). The links that are perpendicular to the loading are not stressed. They enter the equations only by the addition of their masses at the joints. The static behavior is analogous to the behavior of a chain.

7.1. Dynamics of a chain

Addressing the dynamics of bistable chains (lattices), we use a simplified model as suggested by Vinogradov et al. [6] The elastic–plastic deformation of the link is described by the equation

$$F(t, x, \dot{x}) = F_c(x_{pl}(t) + x_c) - c(x_{pl}(t) + x_c - x(t)). \quad (9)$$

$$\dot{x}_{pl}(t) = \begin{cases} 0 & \text{if } x \leq x_c \text{ or } F < F_c \\ 0 & \text{if } F = F_c \quad x(t) \leq 0. \\ \dot{x}(t) & \text{if } F = F_c \quad x(t) > 0 \end{cases} \quad (10)$$

where x is the elongation of the link, x_c is the elongation of the elastic limit, x_{pl} is the plastic (irreversible) elongation, and c is the elastic constant. The critical forces $F_c(x)$ for the bistable and solid links are shown in Figure 9. Notice that $F_c(x) = 0$ if the elongation exceeds a threshold, which corresponds to the failure. The initial condition $x_{pl}(0) = 0$ assumes that initially there is no plastic deformation.

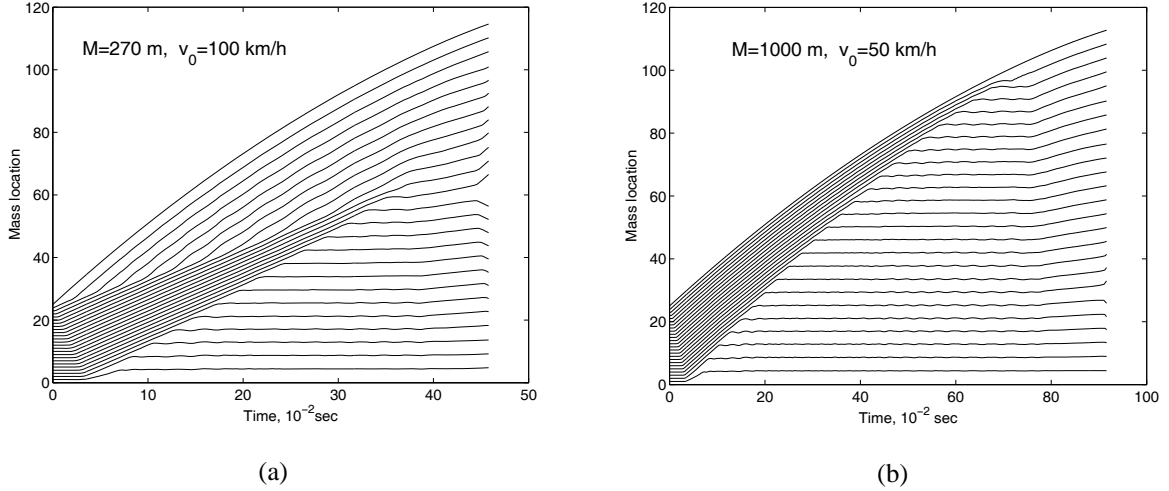


Figure 10. Waves in elastic–plastic chain of 25 bistable links. (a) Observe the fast sonic wave, two waves of transition (damage) and the

Consider a chain fixed at one end and impacted by the projectile of mass M at the other end. After the impact, the mass M stays attached to the free end of the chain. The dynamics of the chain is described by the set of differential equations:

$$m\ddot{z}_n = F(t, x_n, \dot{x}_n) - F(t, x_{n-1}, \dot{x}_{n-1}), \quad (11)$$

$$x_{n-1} = z_{n+1} - z_n, \quad n = 1, \dots, N-1, \quad (12)$$

$$(M + m)\ddot{z}_N = -F(t, x_{N-1}, \dot{x}_{N-1}), \quad z_0 = 0, \quad (13)$$

where z_n is the position of n th knot, x_n is the elongation of n th link, N is the number of links in the chain, and z_0 is the position of the fixed end of the chain. Mass m of the knot is equal to the mass of the link. In the lattice model, the mass m is equal to the double mass of the link, so that the masses of the unstressed links in the lattice are taken into account. This system is integrated with the initial conditions

$$z_n = na, \quad \dot{z}_n = 0, \quad n = 1, \dots, N-1, \quad (14)$$

$$z_N = Na, \quad \dot{z}_N = Kv_0, \quad K = \frac{M}{M + m}, \quad (15)$$

where a is the length of the unstressed link, and v_0 is the speed of the impact. This impact delivers the impulse Mv_0 to the last mass in the chain. The impulse corresponds to the initial speed Kv_0 of that mass m and the attached mass M of the projectile.

7.2. Waves of transition

The typical picture of the motion of the masses in a bistable chain is shown in Figure 10. The impact initiates an elastic wave which propagates along the chain. When the wave reflects from the support, its amplitude doubles. If the magnitude of the wave is large enough, it transfers the first link into the plastic unstable state. After this, the wave of the transition is initiated and propagates toward the impacted end. Simultaneously, the impact originates another transitional wave which propagates toward the support. The chain breaks when these two waves meet. If the speed of the impact is larger, the chain fails immediately after the impact when the first link breaks, as shown in Figure 10b. If the speed of the impact is smaller, the transition waves do not break the chain when they meet. The even smaller speed corresponds to the absence of one of the transitional waves (Figure 10b) or both of them.

7.3. Energy absorption

Let us compare the state of the links in the broken conventional and bistable chains. All the links of the broken solid chain are undamaged but one. Only the damaged link experiences the plastic deformation and absorbs the energy. Contrary to this, many or all the links in the bistable chain experience a partial damage before breakage. Each of the partially damaged links is plastically deformed and absorbs the energy. Therefore, the bistable chain

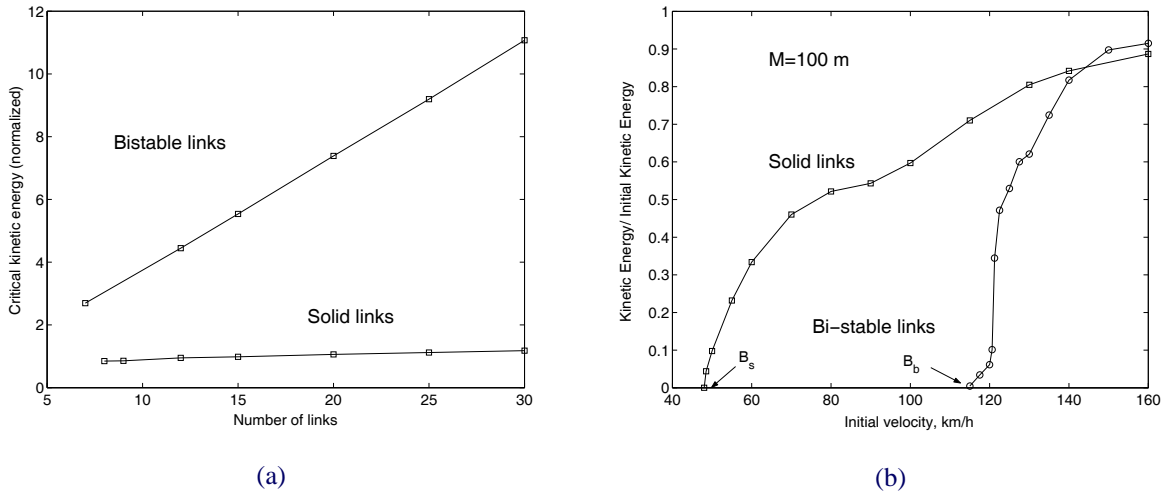


Figure 11. Comparison of kinetic energy needed to break the bistable and conventional chains

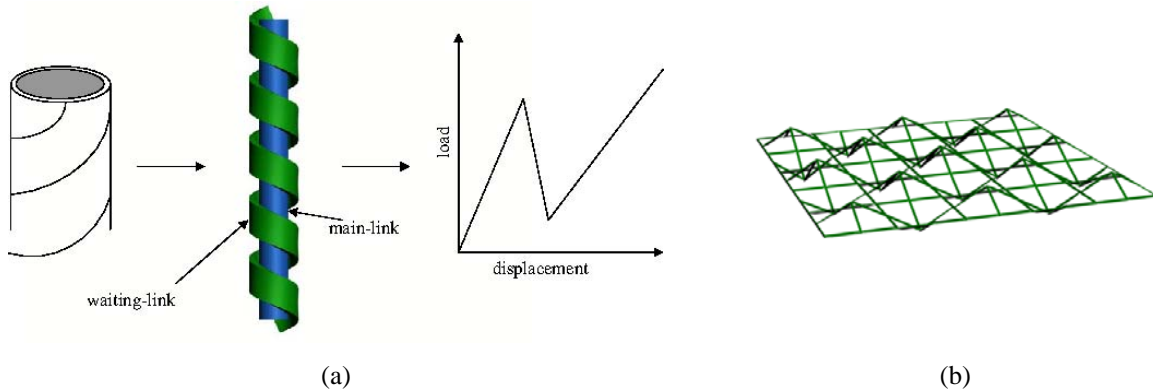


Figure 12. (a) Concept sketch of helicoidal bistable structure; (b) Deformation in 3D space as a result of rotation after impact.

absorbs larger energy before the breakage. In the optimally designed chain all the links are partially damaged and absorb the energy. The bistability leads to an unexpected result: the maximal absorbed energy in the bistable chain is proportional to its volume, not to the cross-section as in the conventional chain, as illustrated in Figure 11a.

Figure 11b compares the resistance of the bistable and conventional chains. The picture shows the kinetic energy of the projectile that breaks the chain. The energy is computed at the instance of the breakage and it is zero if the projectile is stopped. The bistable chain of the same mass and length is able to capture a faster projectile. Even if it is broken, the projectile has lower residual speed. When the speed of the projectile is very large, both the bistable and conventional chains are broken by the projectile of the same energy; in that range, the link closest to the impact is broken and the other links do not contribute.

8. Energy absorption of a helicoidal bistable structure

A bistable structure presented in the previous section requires that the waiting-element is sufficiently long so that it remains inactive upon the initial loading. More space is needed as the waiting element becomes longer and curvier. In this section, we suggest another configuration, a helicoid, (see [7, 8]), which improves the spacing requirement. To avoid confusion in the terminology, the “main element/main-link” and “waiting element/waiting-link” previously used will be replaced with “main-strip” and “waiting-strip” to reflect the geometry of the structure in this section.

A helicoidal bistable link, consisting of a “main-strip” and “waiting-strip,” is shown in Figure 12. The center core, a solid cylinder, is treated as the main-strip and it is made of an isotropic material. The outer layer serves as the waiting-strip, and consists of an orthotropic material having its primary direction of the orthotropy spiraling about the main-strip (central core). By enveloping the inner cylinder with a helicoidal anisotropic shell, the structure maintains symmetry about its cylindrical axis.

Under an axial tensile load, both components of the helicoidal bistable link elongate until the main-strip yields. As yielding continues, an increasing portion of the applied load is transferred to the outer helicoidal waiting-strip.

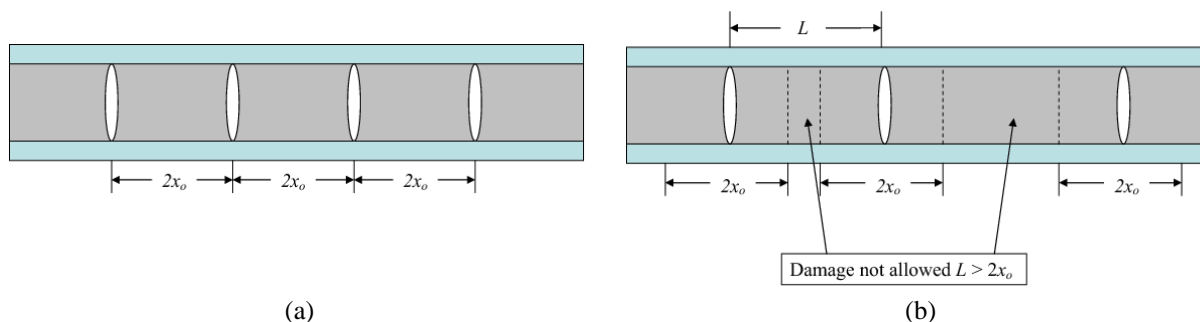


Figure 13: Representative model of the breakages in the main-strip. (a) breakages occur sequentially at fixed distance apart. (b) locations of breakages are random with minimum distance of $2x_0$ apart.

The paper [8] investigates an energy absorption concept of a helicoidal bistable structure using a simplified model where the central main-strip exhibits linear elastic material behavior to failure (brittle failure). The outer helicoidal waiting-strip is assumed to remain nonactive (unloaded) until the inner brittle main-strip fails. Thus, for a chain of bistable links, each link transitions from a lower strain state to a higher-strain state as the main-strip (inner cylinder) fails. The main-strip failure produces a transition wave that propagates along the chain Figure 13. Unlike the conventional structures in which the strain is concentrated near the impact zone, the structure with waiting-strip distributes the strain over a large area.

The spiraling allows for a longer “waiting” inactive distance of the waiting-strip while additional spacing is not required as the original bistable structure presented in Section 7. Unlike that structure, the helicoidal bistable structure can be designed to sustain a smaller elongation but still achieves similar energy release. The additional “twisting” degrees of freedom of the strip correspond to the additional energy released, in the process of the elongation. The location where the breakages occur does not affect the amount of the energy absorbed [8].

9. Conclusion

The bistable structure can absorb more energy in comparison to the conventional structure by distributing the damage along its length. This behavior is due to special morphology of “waiting links.” The elastic-brittle and elastic-plastic materials have been studied; the curved and spiraling waiting links have been examined. The wave of damage propagation has been described. Optimization of parameters was performed.

10. References

- [1] A. V. Cherkaev and L. I. Slepyan, Waiting Element Structures and Stability Under Extension, *Journal of Applied Mechanics*, 4 (1), 58-82, 1995.
- [2] L. I. Slepyan, A. V. Cherkaev, and E. Cherkaev, Transition waves in bistable structures. I. Delocalization of damage, *Journal of the mechanics and physics of solids*, 53, 383-405, 2005.
- [3] L. I. Slepyan, A. V. Cherkaev, and E. Cherkaev, Transition waves in bistable structures. II. Analytical solution: Wave speed and energy dissipation, *Journal of the mechanics and physics of solids*, 53, 407-436, 2005.
- [4] L. I. Slepyan, A. V. Cherkaev, E. Cherkaev, and V. Vinogradov, Transition waves in controllable cellular structures with high structural resistance, *ISBN 83-89687-01-1*, 53, 2005.
- [5] A. V. Cherkaev and L. Zhornitskaya, Dynamics of damage in two-dimensional structures with waiting links, *Asymptotics, Singularities and Homogenisation in Problems of Mechanics*, 273-284, 2004.
- [6] V. Vinogradov, A. Cherkaev, and S. Leelavanichkul, The wave of damage in elastic-plastic lattices with waiting links: design and simulation, *Mechanics of Materials*, 38, 748-756, 2006.
- [7] S. Leelavanichkul, *Helicoidal morphology in engineering structures*, University of Utah, Salt Lake City, 2007.
- [8] S. Leelavanichkul, A. Cherkaev, D. Adams, and F. Solzbacher, Energy absorption of a helicoidal bistable structure, *Journal of Mechanics of Materials and Structures*, 2009.
- [9] A. Cherkaev and L. Zhornitskaya, Protective Structures with Waiting Links and Their Damage Evolution, *Multibody System Dynamics*, 13 (1), 53-67, 2004.