COUPLED BOUNDS FOR THE BULK AND SHEAR MODULI OF A TWO-DIMENSIONAL ISOTROPIC ELASTIC COMPOSITE. (DRAFT)[1]

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Contents

Abstract

We improve the classical Hashin-Shtrikman and Walpole estimates of the effective properties for an isotropic mixture assembled of two isotropic elastic materials. The planar elasticity problem is considered. Unlike the prior estimates which bound the bulk and shear moduli independently, our estimates are coupled and more restrictive. The set of the bulk modulus - shear modulus pairs turn to be bounded in the plane of the values of these moduli by two straight lines (the Hashin - Shtrikman or Walpole bulk modulus estimates) and also by two curves given by a fractional linear transformations.

To obtain the new estimates we use the translation method, which provides a general approach to both Hashin - Shtrikman (well ordered materials) and Walpole (badly ordered materials) cases; the method also provides the estimates for anisotropic mixtures.

1 INTRODUCTION.

A priori estimates of the effective properties for elastic composites are useful for many applications. Such bounds allow to predict the properties of composites with unknown structures, and show the limits for the improvement of the mechanical properties of a mixture by changing the disposition of the components.

In this paper we suggest new estimates for the effective properties of an isotropic composite made from two isotropic elastic materials with known properties. The materials are supposed to be mixed in an arbitrary way but with fixed volume fractions. The obtained estimates improve the classical Hashin-Shtrikman and Walpole bounds; they are the most restrictive among the known ones. The corresponding results for anisotropic composites are also obtained, we refer it briefly.

To get the bounds we use the "translation method" suggested by Lurie and Cherkaev in the papers [27, 28] (see also [30]), Murat and Tartar in the papers [39, 44, 45], and developed in a number of papers (see, for example, Kohn and Strang [21], Francfort and Murat [9], Kohn and Milton [20], Milton [34, 36, 37], and the authors [6, 11, 12]). We have discussed this approach in details in our recent paper [6] where the exact coupled bounds for the electromagnetic properties of an anisotropic composite have been found.

Here we adopt the translation method for an estimating of the effective rigidity tensor for the planar elasticity. Then we give an elementary proof of the known Hashin–Shtrikman and Walpole estimates and apply the same method to prove the new coupled bounds for the shear and bulk moduli of an isotropic elastic composite; we also discuss some estimates for the effective properties of anisotropic mixtures. The history of the problem and known results are outlined in the next section after introducing basic notations.

We consider the plane problem of the elasticity. The obtained results are also valid for the effective moduli of a transversally isotropic three-dimensional composite with arbitrary cylindrical inclusions.

2 KNOWN BOUNDS FOR THE MODULI OF THE ELASTIC COMPOSITES.

In this section we describe the equations of the plane elasticity, introduce the notations, give the statement of the problem, and outline known results.

2.1 Basic equations and notations.

We deal with the plane problem of the elasticity. Let $\mathbf{x} = (x_1, x_2)$ be the Cartesian coordinates, $\mathbf{u} = (u_1, u_2)$ be the displacement vector, ϵ be the strain tensor, σ be the stress tensor. The state of an isotropic body is characterized by the following system (see for example [23]):
the compatibility equations

$$\epsilon_{11} = u_{1,x_1}, \quad \epsilon_{12} = \epsilon_{21} = \frac{1}{2}(u_{1,x_2} + u_{2,x_1}), \quad \epsilon_{22} = u_{2,x_2}, \tag{2.1}$$

the equilibrium equations

$$\sigma_{11,x_1} + \sigma_{12,x_2} = 0, \quad \sigma_{21,x_1} + \sigma_{22,x_2} = 0, \quad \sigma_{12} = \sigma_{21}, \tag{2.2}$$

and the equations of the Hooke law

$$\sigma_{11} = (K + \mu)\epsilon_{11}, \quad \sigma_{22} = (K + \mu)\epsilon_{22}, \quad \sigma_{12} = 2\mu\epsilon_{12}. \tag{2.3}$$

Here $K = K(x)$, $\mu = \mu(x)$ are the bulk and shear moduli of the plane elasticity (plane strain problem) in the point $\mathbf{x} = (x_1, x_2)$.

Remark 2.1: The bulk modulus of the planar elasticity K is expressed as

$$K = k + \mu/3, \tag{2.4}$$

in terms of the bulk modulus k of the three dimensional theory. The planar shear modulus μ doesn't differ from the one used in the three dimensional theory.

We will use the following tensor form of the equations (2.1)–(2.3)

$$\begin{gathered} \epsilon = \tfrac{1}{2}(\nabla \otimes \mathbf{u} + (\nabla \otimes \mathbf{u})^T), \\ \nabla \cdot \sigma = 0, \quad \sigma = \sigma^T, \\ \sigma = \mathbf{C}(K, \mu) \cdot \cdot \epsilon, \end{gathered} \tag{2.5}$$

where ∇ is the two-dimensional Hamilton operator, $\mathbf{C}(K(x), \mu(x))$ is the tensor of rigidity of the isotropic material - the fourth order symmetric positively defined tensor, $\otimes$ is the sign of the tensor product, $(\cdot)$ and $(\cdot\cdot)$ are the convolutions on one and two indices respectively. We will use also the representation for the stress tensor σ through the Airy stress function ϕ (see [23])

$$\sigma = \begin{pmatrix} \phi_{,x_2x_2} & -\phi_{,x_2x_1} \\ -\phi_{,x_1x_2} & \phi_{,x_1x_1} \end{pmatrix} = \mathbf{R}^T \cdot \nabla^2\phi \cdot \mathbf{R}, \tag{2.6}$$

where

$$\nabla^2\phi = \nabla \otimes \nabla\phi = \begin{pmatrix} \phi_{,x_1x_1} & \phi_{,x_1x_2} \\ \phi_{,x_2x_1} & \phi_{,x_2x_2} \end{pmatrix} \tag{2.7}$$

and

$$\mathbf{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.8}$$

is the tensor of rotation on the right angle.

It is convenient (see for example [11, 25, 26, 30]) to introduce the following orthonormolized basis in the space of the second order tensors :

$$\begin{array}{cc} \mathbf{a}_1 = (\mathbf{i}\otimes\mathbf{i}+\mathbf{j}\otimes\mathbf{j})/\sqrt{2}, & \mathbf{a}_2 = (\mathbf{i}\otimes\mathbf{i}-\mathbf{j}\otimes\mathbf{j})/\sqrt{2}, \\ \mathbf{a}_3 = (\mathbf{i}\otimes\mathbf{j}+\mathbf{j}\otimes\mathbf{i})/\sqrt{2}, & \mathbf{a}_4 = (\mathbf{i}\otimes\mathbf{j}-\mathbf{j}\otimes\mathbf{i})/\sqrt{2}, \\ \multicolumn{2}{c}{\mathbf{a}_i\cdot\cdot\mathbf{a}_j = \delta_{ij},} \end{array} \tag{2.9}$$

where $\mathbf{i}$ and $\mathbf{j}$ are the orts of the Cartesian axis x_1 and x_2, δ_{ij} is the Kroneker symbol. The subspace of the symmetric second order tensors has the orthonormolized basis $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. Symmetric strain and stress tensors have the following representation in this basis

$$\epsilon = \sum_{i=1}^{3} \epsilon_i \mathbf{a}_i \ , \qquad \sigma = \sum_{i=1}^{3} \sigma_i \mathbf{a}_i \ , \tag{2.10}$$

where the coefficients ϵ_i , σ_i are equal

$$\epsilon_1 = \epsilon\cdot\cdot\mathbf{a}_1 = \tfrac{1}{\sqrt{2}}(\epsilon_{11}+\epsilon_{22}), \qquad \epsilon_2 = \epsilon\cdot\cdot\mathbf{a}_2 = \tfrac{1}{\sqrt{2}}(\epsilon_{11}-\epsilon_{22}), \qquad \epsilon_3 = \epsilon\cdot\cdot\mathbf{a}_3 = \sqrt{2}\epsilon_{12},$$
$$\sigma_1 = \sigma\cdot\cdot\mathbf{a}_1 = \tfrac{1}{\sqrt{2}}(\sigma_{11}+\sigma_{22}), \quad \sigma_2 = \sigma\cdot\cdot\mathbf{a}_2 = \tfrac{1}{\sqrt{2}}(\sigma_{11}-\sigma_{22}), \quad \sigma_3 = \sigma\cdot\cdot\mathbf{a}_3 = \sqrt{2}\sigma_{12}. \tag{2.11}$$

In this basis the Hooke law has the form (see [23])

$$\sigma_1 = 2K\epsilon_1\ , \qquad \sigma_2 = 2\mu\epsilon_2\ , \qquad \sigma_3 = 2\mu\epsilon_3 \tag{2.12}$$

(compare with (2.3)), the rigidity isotropic tensor $C(K, \mu)$ is represented by the diagonal matrix

$$\mathbf{C}(K, \mu) = \begin{pmatrix} 2K & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & 2\mu \end{pmatrix}. \tag{2.13}$$

The density of the elastic energy can be represented either as a quadratic form of strains

$$W_\epsilon = \epsilon \cdot \cdot \mathbf{C} \cdot \cdot \epsilon \tag{2.14}$$

or as a quadratic form of stresses

$$W_\sigma = \sigma \cdot \cdot \mathbf{S} \cdot \cdot \sigma, \tag{2.15}$$

where

$$\mathbf{S} = \mathbf{C}^{-1} = \begin{pmatrix} \frac{1}{2K} & 0 & 0 \\ 0 & \frac{1}{2\mu} & 0 \\ 0 & 0 & \frac{1}{2\mu} \end{pmatrix} \tag{2.16}$$

is the compliance tensor.

It is convenient to introduce the nonsymmetric matrices $\zeta = \nabla \otimes \mathbf{u}$ of the gradient of displacement vector $\mathbf{u}$. These matrices have the following representation in the basis (2.9)

$$\zeta = \nabla \otimes \mathbf{u}, \quad \zeta = \sum_{i=1}^{4} \zeta_i \mathbf{a}_i, \quad \zeta_i = \zeta \cdot \cdot \mathbf{a}_i, \quad i = 1, 2, 3, 4. \tag{2.17}$$

Note that the first three components of the tensors ζ and ϵ coincide

$$\zeta_i = \epsilon_i, \quad i = 1, 2, 3, \tag{2.18}$$

and the fourth component of ζ ($\zeta_4 \neq \epsilon_4 = 0$) doesn't affect the equation of elasticity. Namely, the density of the elastic energy (2.14) as a function of the tensor ζ becomes

$$W(\zeta) = \zeta \cdot \cdot \mathbf{C}' \cdot \cdot \zeta, \tag{2.19}$$

where the (4×4) matrix $\mathbf{C}'$ has the diagonal form

$$\mathbf{C}' = \begin{pmatrix} \frac{1}{2K} & 0 & 0 & 0 \\ 0 & \frac{1}{2\mu} & 0 & 0 \\ 0 & 0 & \frac{1}{2\mu} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.20}$$

in the introduced basis $\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3,\ \mathbf{a}_4$. Zero value of the fourth modulus of the matrix $\mathbf{C}'$ corresponds to the zero value of the energy associated with the rotational degree of freedom of the material points and provide the frame independence of the energy.

2.2 Homogenization.

Let us consider a double-periodical structure in the $(x_1,\ x_2)$ plane. The element of periodicity Ω is supposed to be divided into two parts Ω_1 and Ω_2 with the relative squares m_1 and m_2 respectively.

$$\begin{gathered} \Omega_1 \cup \Omega_2 = \Omega, \\ (vol\ \Omega_1)/(vol\ \Omega) = m_1, \quad (vol\ \Omega_2)/(vol\ \Omega) = m_2, \\ m_1 + m_2 = 1\ , \end{gathered} \tag{2.21}$$

where the symbol $vol(\cdot)$ means the square of the domain $(\cdot)$

Assume that these two parts are occupied by two isotropic elastic materials with the moduli (K_1, μ_1) and $(K_2,\ \mu_2)$ respectively and investigate the homogenization problem i.e. the problem of describing the medium average properties. It is well known [7, 43] that the average behavior of a mixture is described by the homogenized equations of elasticity

$$\begin{aligned} &< \epsilon >=< \tfrac{1}{2}(\nabla \otimes \mathbf{u} + (\nabla \otimes \mathbf{u})^{\mathbf{T}}) >, \qquad && \nabla\cdot < \sigma >= 0, \\ &< \sigma >=< \sigma >^T, && < \sigma >= \mathbf{C}_0 \cdot\cdot < \epsilon > \ . \end{aligned} \tag{2.22}$$

Here the symbol $< (.) >$ denotes averaging upon the element of periodicity Ω

$$< (.) >= \frac{1}{vol\Omega} \int_\Omega (.) d\Omega \tag{2.23}$$

and the tensor $\mathbf{C}_0$ connecting the averaged stresses and straines is by the definition the effective rigidity tensor of a composite. This tensor is in general an anisotropic positively defined fourth order symmetric tensor. It depends on the properties of initial materials, their volume fractions as well as on a geometrical structure of the composite and it is independent of loading. The energy density W stored in the composite is known to be equal

$$W_\epsilon = < \epsilon > \cdot\cdot \mathbf{C}_0 \cdot\cdot < \epsilon > \tag{2.24}$$

or

$$W_\sigma = < \sigma > \cdot\cdot \mathbf{S}_0 \cdot\cdot < \sigma >, \tag{2.25}$$

where the effective compliance tensor $\mathbf{S}_0$ is determined as $\mathbf{S}_0 = \mathbf{C}_0^{-1}$ (see for example [7, 43]) .

2.3 The known estimates.

The problem of bounds for the elastic moduli has a long history. Several estimates were found for the three-dimensional problem and all of them had natural two-dimensional analogues; isotropy in the planar elasticity corresponds to transversal isotropy in the three-dimensional problem for the structures with cylindrical arrangement of pure materials.

Here we refer the known estimates for the two-dimensional problem and simplify it by using some new notations.

2.3.1 The Hill's estimates.

The first bounds of the effective moduli K_0, μ_0 were found by Hill [15] who applied the Reuss [41] and Voigt [47] estimates (see also Wiener [50]) to the elasticity. Hill's bounds have the form

$$< K^{-1} >^{-1} \le K_0 \le < K >, \quad < \mu^{-1} >^{-1} \le \mu_0 \le < \mu > \tag{2.26}$$

Note that the functions $K(x)$, $\mu(x)$ are piece-wise constant; any function $d(x)$ of such type is averaged as

$$< d(x) > = m_1 d_1 + m_2 d_2, \tag{2.27}$$

where d_1 and d_2 are the values of this function in the subdomains Ω_1 and Ω_2 respectively.

The estimates (2.26) can be represented in the following form

$$y(K_0) \geq 0, \qquad y(\mu_0) \geq 0, \tag{2.28}$$

where the fraction linear function y is determined by the relation

$$y(d_0) = \underline{y}(d_0,\ d_1,\ d_2) = -\frac{d_0 - < d^{-1} >^{-1}}{d_0 - < d >}\ d_1\ d_2\ < d^{-1} > \tag{2.29}$$

Remark 2.2: This fraction linear "Y-transformation" turns out to play an important role in the problem of the bounds for the effective properties, see, for example, Bergman [3], Berryman [4], Milton and Golden [35], Milton [32, 37], Cherkaev and Gibiansky [6].

Here we use two representations of the "Y-transformation". The first one is a function $y(d_0)$ of one argument d_0 with implicit the dependence of the parameters $d_1,\ d_2$ and m_1; it maps one-to-one d_0 into $y(d_0)$. The second one is an explicit function $\underline{y}(d_0,\ d_1,\ d_2)$ of three arguments $d_0,\ d_1,\ d_2$ and one implicit parameter m_1.

It is easy to verify that functions $y(d_0)$ and $\underline{y}(d_0,\ d_1,\ d_2)$ have the following properties:

$$\begin{array}{c} y(d_1) = \underline{y}(d_1,\ d_1,\ d_2) = -d_1, \\ y(d_2) = \underline{y}(d_2,\ d_1,\ d_2) = -d_2, \\ \underline{y}(\alpha d_0,\ \alpha d_1,\ \alpha d_2) = \alpha \underline{y}(d_0,\ d_1,\ d_2), \\ \underline{y}(d_0^{-1},\ d_1^{-1},\ d_2^{-1}) = \underline{y}^{-1}(d_0,\ d_1,\ d_2). \end{array} \tag{2.30}$$

2.3.2 The Hashin–Shtrikman and Walpole estimates.

Hashin and Shtrikman [13] suggested the variational method which allows them to take into account differential restrictions on stress and strain fields; they found new bounds of the elastic moduli of an isotropic mixture made from isotropic materials.

First the Hashin–Shtrikman bounds were formulated for isotropic three dimensional mixtures; however, in the paper [14] they were formulated for the transversal isotropic composites with cylindrical inclusions as well. The above problem is just the case we will study here.

The initial materials were supposed to be "well ordered". This means that both bulk and shear moduli of the first material are bigger than of the second one

$$\mu_1 \geq \mu_2, \quad K_1 \geq K_2. \tag{2.31}$$

The obtained estimates have the form (see [14])

$$K^l_{HS} \le K_0 \le K^u_{HS}, \quad \mu^l_{HS} \le \mu_0 \le \mu^u_{HS}, \tag{2.32}$$

where

$$K^l_{HS} = K_2 + \frac{m_1}{\frac{1}{K_1 - K_2} + \frac{m_2}{K_2 + \mu_2}}, \tag{2.33}$$

$$K^u_{HS} = K_1 + \frac{m_2}{\frac{1}{K_2 - K_1} + \frac{m_1}{K_1 + \mu_1}}, \tag{2.34}$$

$$\mu^l_{HS} = \mu_2 + \frac{m_1}{\frac{1}{\mu_1 - \mu_2} + \frac{m_2\ (K_2 + 2\mu_2)}{2\mu_2\ (K_2 + \mu_2)}}, \tag{2.35}$$

$$\mu^u_{HS} = \mu_1 + \frac{m_2}{\frac{1}{\mu_2 - \mu_1} + \frac{m_1\ (K_1 + 2\mu_1)}{2\mu_1\ (K_1 + \mu_1)}}. \tag{2.36}$$

By using the introduced "Y-transformation" the above values K^l_{HS}, K^u_{HS} and μ^l_{HS}, μ^u_{HS} can be defined as the unique solutions of the equations

$$\begin{array}{ll} y(K^l_{HS}) = \mu_2, & y(K^u_{HS}) = \mu_1, \\ y(\mu^l_{HS}) = \frac{K_2\mu_2}{K_2 + 2\mu_2}, & y(\mu^u_{HS}) = \frac{K_1\mu_1}{K_1 + 2\mu_1}. \end{array} \tag{2.37}$$

The Hashin–Shtrikman estimates are more restrictive than the Hill's ones. Moreover, they describe the minimal rectangular in the bulk modulus - shear modulus plane which contains the set of possible values of these moduli. It is proved that two opposite corner points A and C

$$A = (K^l_{HS},\ \mu^l_{HS}), \quad C = (K^u_{HS},\ \mu^u_{HS}) \tag{2.38}$$

of this rectangular (see Fig.1) corresponding to maximal or minimal values of both moduli are attainable by special microstructures (i.e. there exist microstructures with such properties, see Francfort and Murat [9], Lurie and Cherkaev [29], Milton [33], Norris [40]).

Walpole [48, 49] considered the opposite case of "badly ordered" initial materials when

$$\mu_1 \ge \mu_2, \quad K_1 \le K_2. \tag{2.39}$$

He obtained the bounds for the effective moduli of an isotropic mixture by using similar variational method. The two-dimensional Walpole's estimates we are dealing with also have the simple representation in terms of the "Y-transformation":

$$K_W^l \le K_0 \le K_W^u, \qquad \mu_W^l \le \mu_0 \le \mu_W^u, \tag{2.40}$$

where the parameters $K_W^l,\ K_W^u$ and $\mu_W^l,\ \mu_W^u$ satisfy the equations:

$$\begin{array}{ll} y(K_W^l) = \mu_2, & y(K_W^u) = \mu_1, \\ y(\mu_W^l) = \frac{K_1\mu_2}{K_1+2\mu_2}, & y(\mu_W^u) = \frac{K_2\mu_1}{K_2+2\mu_1}. \end{array} \tag{2.41}$$

Remark 2.3: Note, that the cases (2.31) and (2.39) cover all possible relations between the elastic moduli of two initial materials, because one can numbered the materials so, that

$$\mu_1 \ge \mu_2. \tag{2.42}$$

All mentioned estimates provide the independent intervals of the bulk and shear moduli variation. They bound the set of these moduli by a rectangular domain in the (K_0, μ_0) plane (see Fig.1 – Fig.4).

Remark 2.4: One can check that if $K_1 = K_2 = K$ then the Hashin–Shtrikman and Walpole rectangulars degenerate into the interval on the straight line

$$K_0 = K, \quad \mu_{HS}^l = \mu_W^l \le \mu_0 \le \mu_{HS}^u = \ \mu_W^u. \tag{2.43}$$

If $\mu_1 = \mu_2 = \mu$ then the Hashin–Shtrikman and Walpole rectangulars degenerate into the point

$$\mu_0 = \mu, \quad K_{HS}^l = K_W^l = K_0 = K_{HS}^u = K_W^u. \tag{2.44}$$

The mentioned particular cases were studied by Hill [16, 17], by Lurie, Cherkaev and Fedorov [25], by Lurie and Cherkaev [26, 30], by Lipton [24] and by Frankfort and Tartar [10].

2.3.3 The Milton–Phan-Thien–Kublanov estimates.

The next step was taken by Milton and Phan-Thien [32] and Berryman and Milton [5] who used a different variational method to obtain bounds for the effective properties of isotropic three dimensional mixtures; Milton and Kublanov [22] applied this method for the plane case we are interested here. The authors obtained an impressive result - they found the coupled bulk-shear moduli estimates. In other words, the rectangular domain of the permitted values of the moduli was replaced by a smaller domain lying in this rectangular and bounded by some curves.

In the well-ordered case (2.31) these estimates have the form:

$$K^l_{HS} \leq K_0 \leq K^u_{HS}, \quad \mu^l_{MK+}(K_0) \leq \mu_0 \leq \mu^u_{MK+}(K_0), \tag{2.45}$$

where the values of $\mu^l_{MK+}(K_0)$ and $\mu^u_{MK+}(K_0)$ are defined by the equations

$$y(\mu^l_{MK+}(K_0)) = \frac{\mu_2 K_2}{2\mu_2 + K_2 - 2\frac{K_1-K_2}{\mu_1-\mu_2}\frac{\mu_2}{K_1}(y(K_0)-\mu_2)}, \tag{2.46}$$

$$y(\mu^u_{MK+}(K_0)) = \frac{\mu_1(K_1 - \frac{K_1-K_2}{\mu_1-\mu_2}\frac{\mu_2}{y(K_0)}(\mu_1 - y(K_0)))}{2\mu_1 + K_1 - \frac{K_1-K_2}{\mu_1-\mu_2}\frac{\mu_2}{y(K_0)}(\mu_1 - y(K_0))}. \tag{2.47}$$

In the badly-ordered case (2.39) the estimates can be presented in the form:

$$K^l_W \leq K_0 \leq K^u_W, \quad \mu^l_{MK-}(K_0) \leq \mu_0 \leq \mu^u_{MK-}(K_0), \tag{2.48}$$

where $\mu^l_{MK-}(K_0)$ and $\mu^u_{MK-}(K_0)$ are the solutions of the equations

$$y(\mu^l_{MK-}(K_0)) = \frac{\mu_2 K_2}{2\mu_2 + K_2 - 2\frac{K_1-K_2}{\mu_1-\mu_2}\frac{\mu_1\mu_2}{K_1 y(K_0)}(y(K_0)-\mu_2)}, \tag{2.49}$$

$$y(\mu^u_{MK-}(K_0)) = \frac{\mu_1(K_1 - \frac{K_1-K_2}{\mu_1-\mu_2}(\mu_1 - y(K_0)))}{2\mu_1 + K_1 - \frac{K_1-K_2}{\mu_1-\mu_2}(\mu_1 - y(K_0))}. \tag{2.50}$$

The Milton–Kublanov estimates are illustrated by the curves SS and SS on the Fig.1, by the curves SS and SS on the Fig.3. It is remarkable, that they pass through Hashin - Shtrikman points A, C and through Walpole points

$$B = (K^l_W,\ \mu^u_W), \quad D = (K^u_W,\ \mu^l_W). \tag{2.51}$$

Therefore the Hashin - Shtrikman and Walpole points A,B,C,D turn out to be important for both well and badly ordered cases.

Note, that we have presented all the estimates in terms of Y-transformation and in such representation the estimates turn out to be independent of the volume fractions of the initial materials.

3 NEW ESTIMATES.

In this paper we suggest the estimates that are sharper than the known ones. The results are given by the following statement:

Theorem

All the pairs $(y(K_0),\ y(\mu_0))$ of values of the Y-transformation of the bulk K_0 and shear μ_0 moduli for an elastic isotropic composite are restricted by the inequalities (Hashin–Shtrikman–Walpole estimates of bulk modulus):

$$\mu_2 \leq y(K_0) \leq \mu_1 \tag{3.1}$$

and by the inequalities:

$$F_1(y(K_0)) \leq y(\mu_0) \leq F_2(y(K_0)) \tag{3.2}$$

in the well-ordered case (2.31),
or by the inequalities

$$F_3(y(K_0)) \leq y(\mu_0) \leq F_4(y(K_0)) \tag{3.3}$$

in the badly ordered case (2.39).

Here $F_1(y(K_0))$ is defined as

$$F_1(y(K_0)) = -\mu_1 + \frac{\alpha_1(y(K_0)^{-1} + K_1^{-1})}{\beta_1(y(K_0)^{-1} + K_1^{-1}) - \gamma_1}, \tag{3.4}$$

where

$$\alpha_1 = (\mu_2^{-1} - \mu_1^{-1})(K_1\mu_1 + K_1\mu_2 + 2\mu_1\mu_2)(K_2\mu_1 + K_2\mu_2 + 2\mu_1\mu_2), \tag{3.5}$$

$$\begin{aligned}\beta_1 = (K_1\mu_1 + K_1\mu_2 + 2\mu_1\mu_2)(K_2 + 2\mu_2)(K_1^{-1} + \mu_2^{-1})\\ -(K_2\mu_1 + K_2\mu_2 + 2\mu_1\mu_2)(K_1 + 2\mu_2)(K_1^{-1} + \mu_1^{-1}),\end{aligned} \tag{3.6}$$

$$\gamma_1 = 2(K_1 - K_2)\mu_2^2(K_1^{-1} + \mu_2^{-1})(K_1^{-1} + \mu_1^{-1}). \tag{3.7}$$

The function $F_2(y(K_0))$ is equal to

$$F_2(y(K_0)) = -\mu_1 + \frac{\alpha_2 y(K_0)}{\beta_2 y(K_0) - \gamma_2}, \tag{3.8}$$

where

$$\alpha_2 = 2\mu_1(\mu_2 - \mu_1)(K_1 + \mu_1)(K_2 + \mu_1), \tag{3.9}$$

$$\beta_2 = (K_1 + \mu_1)(K_2 + 2\mu_1)\mu_2 - \mu_1(K_2 + \mu_1)(K_1 + 2\mu_1), \tag{3.10}$$

$$\gamma_2 = \mu_2\mu_1^2(K_1 - K_2). \tag{3.11}$$

$F_3(y(K_0))$ *is equal to*

$$F_3(y(K_0)) = -\mu_2 + \frac{\alpha_3 y(K_0)}{\beta_3 y(K_0) - \gamma_3}, \tag{3.12}$$

where

$$\alpha_3 = 2\mu_2(\mu_1 - \mu_2)(K_2 + \mu_2)(K_1 + \mu_2), \tag{3.13}$$

$$\beta_3 = (K_2 + \mu_2)(K_1 + 2\mu_2)\mu_1 - \mu_2(K_1 + \mu_2)(K_2 + 2\mu_2), \tag{3.14}$$

$$\gamma_3 = \mu_1\mu_2^2(K_2 - K_1). \tag{3.15}$$

$F_4(y(K_0))$ *is equal to*

$$F_4(y(K_0)) = -\mu_2 + \frac{\alpha_4(y(K_0)^{-1} + K_2^{-1})}{\beta_4(y(K_0)^{-1} + K_2^{-1}) - \gamma_4}, \tag{3.16}$$

where

$$\alpha_4 = (\mu_1^{-1} - \mu_2^{-1})(K_1\mu_1 + K_1\mu_2 + 2\mu_1\mu_2)(K_2\mu_1 + K_2\mu_2 + 2\mu_1\mu_2), \tag{3.17}$$

$$\begin{array}{c} \beta_4 = (K_2\mu_1 + K_2\mu_2 + 2\mu_1\mu_2)(K_1 + 2\mu_1)(K_2^{-1} + \mu_1^{-1}) - \\ (K_1\mu_1 + K_1\mu_2 + 2\mu_1\mu_2)(K_2 + 2\mu_1)(K_2^{-1} + \mu_2^{-1}) \ , \end{array} \tag{3.18}$$

$$\gamma_4 = 2(K_2 - K_1)\mu_1^2(K_2^{-1} + \mu_1^{-1})(K_2^{-1} + \mu_2^{-1}); \tag{3.19}$$

Note, that the limiting curves $y(\mu_0) = F_i(y(K_0)), \quad i = 1,2,3,4$ in the inequalities (3.2),(3.3) are fraction linear functions. As any fraction linear curve each of them can be uniquely determined by the coordinates of any three points it passes through. There are some remarkable points of the curves. Namely the curve $y(\mu_0) = F_1(y(K_0))$ (line EAD, Fig. 1) passes the points

$$\begin{array}{c} A = (y(K^l_{HS}),\ y(\mu^l_{HS})), \\ D = (y(K^u_W),\ y(\mu^l_W)), \\ E = (y(K_1),\ y(\mu_1)) = (-K_1,\ -\mu_1); \end{array} \tag{3.20}$$

the curve $F_2(y(K_0))$ (line FBC, Fig. 1) passes the points

$$\begin{array}{c} B = (y(K^l_W),\ y(\mu^u_W)), \\ C = (y(K^u_{HS}),\ y(\mu^u_{HS})), \\ F = (y(< K^{-1} >^{-1}),\ y(\mu_1)) = (0,\ -\mu_1); \end{array} \tag{3.21}$$

the curve $F_3(y(K_0))$ (line ADG, Fig. 2) passes the points

$$\begin{array}{c} A = (y(K^l_{HS}),\ y(\mu^l_{HS})), \\ D = (y(K^u_W),\ y(\mu^l_W)), \\ G = (y(< K^{-1} >^{-1}),\ y(\mu_2)) = (0,\ -\mu_2); \end{array} \tag{3.22}$$

the curve $F_4(y(K_0))$ (line BCH, Fig. 2) passes the points

$$\begin{array}{c} B = (y(K^l_W),\ y(\mu^u_W)), \\ C = (y(K^u_{HS}),\ y(\mu^u_{HS})), \\ H = (y(K_2),\ y(\mu_2)) = (-K_2,\ -\mu_2). \end{array} \tag{3.23}$$

The **Y**-transformation (2.29) itself is also the fraction-linear function of the effective moduli. Therefore, the corresponding bounds in the plane of the effective moduli $(K_0,\ \mu_0)$ are given also by fraction-linear functions.

Let illustrate the results by some numerical calculations. Fig. 3 shows all mentioned bounds in the terms of the Y-transformation $y(K_0),\ y(\mu_0)$ when the moduli of the components are equal

$$K_1 = 9, \quad \mu_1 = 9, \quad K_2 = 1, \quad \mu_2 = 1 \tag{3.24}$$

(well ordered materials). On this figure $AB'CD'$ is the Hashin–Shtrikman rectangular (bounds (2.32), (2.37)), lines B-1-C and A-4-D correspond to the Milton-Kublanov bounds (2.45)-(2.47), and the lines B-2-C and A-3-D show the estimates (3.2), (3.4)-(3.11) of the Theorem. The same pictures on $(K_0,\ \mu_0)$ plane for the volume fractions $m_1 = m_2 = 0.5$ are presented on the Fig. 5.

The case of a bedly ordered materials for the parameters

$$K_1 = 6, \quad \mu_1 = 3, \quad K_2 = 9, \quad \mu_2 = 1 \tag{3.25}$$

is illustrated by the Fig.4 (on the $(y(K_0),\ y(\mu_0)$ plane) and by the Fig. 6 (on the $(K_0,\ \mu_0)$ plane for the same volume fractions $m_1 = m_2 = 0.5$). On these figures $A'BC'D$ is the Walpole rectangular (bounds (2.40)-(2.41)), lines A-4-D and B-1-C correspond to the Milton-Kublanov bounds (2.48)-(2.50), and the lines A-3-D and B-2-C represent the estimates (3.3), (3.12)-(3.19) of the Theorem. Note, that the lines A-3-D and A-4-D practically coinside for our choice of the parameters.

One can check that the obtained estimates permit the negative Poison ratio ν_0

$$\nu_0 = \frac{K_0 - \mu_0}{K_0 + \mu_0} \tag{3.26}$$

for a mixture, prepared from the materials with the positive ones. This seemed unnatural until the paper [38] by Milton appeared where such mixtures (in particular, laminated composites) were constructed. Moreover, these mixtures possess the Poison ratio close to the minimal value permitted by our estimates as it was shown in paper [38]. On the Fig. 7 line 1 shows the function $\nu_0(h)$ for the Poisson ratio attainable by the Milton's structures, lines 2 and 3 represent the lower bounds of ν_0 following from the Hashin-Shtrikman estimates (2.32) - (2.37) and from the estimates (3.2), (3.4) - (3.11) respectively. It is assumed here that the Poisson ratios $\nu_1,\ \nu_2$ of the initial materials have the same value and h is a ratio of the rigidities of the first and second materials

$$\nu_i = \frac{K_i - \mu_i}{K_i + \mu_i} = \frac{1}{3} \quad i = 1, 2\ , \qquad h = \frac{K_1}{K_2} = \frac{\mu_1}{\mu_2}. \tag{3.27}$$

4 THE TRANSLATION METHOD

To prove the Theorem we use the variant of the translation method described in details in our paper [6]; we outline it in this section.

Note that although we concentrate mainly on the bounds for isotropic mixtures, the inequalities of the type (4.35) that we are going to obtain are valid for the anisotropic effective tensor as well.

The method [6, 30] is based on the lower estimate of the functional I

$$I = \sum_{i=1}^{N} W_i. \tag{4.1}$$

this functional is equal to the sum of the values of elastic energy W_i stored in the element of periodicity of a composite which is exposed sequenceally into the N linearly independent external stress or strain fields with fixed mean values. The energy functional is used because its value is equal to energy stored by the equivalent homogeneous medium in the uniform field. The equivalent medium is characterized by the effective tensor of the composite, and the uniform external field coincides with the mean value of the field in the composite.

Clearly, the lower estimate of the functional (4.1) provides also the estimates of the effective tensor we are interest in. Below we get the estimates of the functional (4.1) independent of the microstructure of a mixture and extract the geometrically independent estimates for the effective moduli from them.

4.1 Functionals

Now let us specify the functional of the type (4.1) which attains minimal values at the boundary of the set of pairs (K_0, μ_0). We discuss here functionals providing estimates for various components of the boundary.

To obtain the lower estimate for the bulk modulus one can expose the composite into the external hydrostatic strain field $\epsilon_h = \epsilon_1 \mathbf{a}_1$ (see (2.10)) because the energy of an isotropic composite under the action of this field is proportional to the effective bulk modulus K_0 (see (2.13), (2.14)).

It is convenient for the following procedure to express the energy as a function of the gradient $\zeta = \nabla \otimes \mathbf{u}$ of the displacement vector $\mathbf{u}$ (see (2.17), (2.18)). The corresponding functional has the form (see (2.19), (2.20)):

$$I^\zeta =< \ \zeta(x)\cdot\cdot\mathbf{C}'(x)\cdot\cdot\zeta(x) \ >= 2K_0\epsilon_1^2,$$

$$if \quad \zeta(x) \in (2.18),\ (2.5), \tag{4.2}$$

$$< \zeta(x) > \cdot\cdot\, \mathbf{a}_1 = \epsilon_1, \quad < \zeta(x) > \cdot\cdot\, \mathbf{a}_2 = 0, \quad < \zeta(x) > \cdot\cdot\, \mathbf{a}_3 = 0,$$

where ϵ_1 is a given constant. It is clear that the lower estimate of the functional (4.2) gives the lower estimate of the effective bulk modulus K_0, because the amplitude ϵ_1 of the hydrostatic strain field is assuming to be fixed.

To get the upper estimate of this modulus we need to expose the composite into the hydrostatic stress field $\sigma_h = \sigma_1\ \mathbf{a}_1$; this makes the stored energy proportional to $1/K_0$. The lower estimate of the corresponding functional give us the upper estimate of K_0. So, this time we minimize the functional

$$\begin{gathered} I^\sigma =< \ \sigma(x)\cdot\cdot S(x)\cdot\cdot\sigma(x) \ >= (2K_0)^{-1}\sigma_1^2, \\ \sigma(x) \in (2.5), \quad < \sigma(x) >= \sigma_1\ \mathbf{a}_1, \end{gathered} \tag{4.3}$$

where σ_1 is a given constant.

We will see that the exact estimates of these functionals provide the Hashin - Shtrikman estimates for the bulk modulus.

Similarly, to obtain the lower estimate for the shear modulus of a mixture one can examine the energy stored in the composite exposed into the shear-type trial strain field. This way we obtain an estimate on the any of two shear moduli of the mixture which is anisotropic in general. However, the other shear modulus can have arbitrary value and the energy functionals of the types (4.2), (4.3) are not sensitive to its value. To provide the isotropy of the mixture we should also care about the reaction of the composite on the orthogonal shear field. So, to estimate the shear modulus of an isotropic composite we should minimize the functional equal to the sum of two values of energy stored by the medium under the action of two trial orthogonal shear fields $\epsilon = \epsilon_2\ \mathbf{a}_2$ and $\epsilon' = \epsilon_3\ \mathbf{a}_3$. And again we express the energy in terms of the corresponding gradients $\zeta = \nabla\otimes\mathbf{u}$ and $\zeta' = \nabla\otimes\mathbf{u}'$ of the trial displacement vectors $\mathbf{u}$ and $\mathbf{u}'$

$$I^{\zeta\zeta} =< \ \zeta(x) \cdot \cdot \mathbf{C}'(x) \cdot \cdot \zeta(x) + \zeta'(x) \cdot \cdot \mathbf{C}'(x) \cdot \cdot \zeta'(x) \ >$$

$$= 2\mu_0(\epsilon_2^2 + \epsilon_3^2),$$

$$if \quad \zeta(x), \ \zeta'(x) \in (2.18), \ (2.5), \tag{4.4}$$

$$< \zeta(x) > \cdot \cdot \mathbf{a}_1 = 0, < \zeta(x) > \cdot \cdot \mathbf{a}_2 = \epsilon_2, \quad < \zeta(x) > \cdot \cdot \mathbf{a}_3 = 0,$$

$$< \zeta'(x) > \cdot \cdot \mathbf{a}_1 = 0, \quad < \zeta'(x) > \cdot \cdot \mathbf{a}_2 = 0, < \zeta'(x) > \cdot \cdot \mathbf{a}_3 = \epsilon_3,$$

where ϵ_2 and ϵ_3 are the fixed constants. To find the upper estimate of the shear modulus we use the functional equal to the sum of two energies stored by the composite exposed into two orthogonal shear fields of the stresses $\sigma = \ \sigma_2 \ \mathbf{a}_2$ and $\sigma' = \ \sigma_3 \ \mathbf{a}_3$

$$I^{\sigma\sigma} =< \ \sigma(x) \cdot \cdot \mathbf{S}(x) \cdot \cdot \sigma(x) + \sigma'(x) \cdot \cdot \mathbf{S}(x) \cdot \cdot \sigma'(x) \ >$$

$$= \tfrac{1}{2\mu_0}(\sigma_2^2 + \sigma_3^2), \tag{4.5}$$

$$if \quad \sigma, \ \sigma' \in (2.5), \quad < \sigma >= \sigma_2 \ \mathbf{a}_2, \quad < \sigma' >= \sigma_3 \ \mathbf{a}_3.$$

Here σ_2 and σ_3 are the given constants. We show below that the lower estimates of these functionals lead to the Hashin - Shtrikman and the Walpole bounds for the shear modulus.

In order to get coupled shear-bulk estimates one can expose the composite into three different fields: the hydrostatics and two orthogonal shear ones. We have a choice between stress and strain trial fields (two shear fields are supposed to be of the same nature to provide isotropy of the mixture). Therefore the following functionals should be considered:

$$I^{\zeta\zeta\zeta} = I^{\zeta} + I^{\zeta\zeta}, \tag{4.6}$$

$$I^{\sigma\sigma\sigma} = I^{\sigma} + I^{\sigma\sigma}, \tag{4.7}$$

$$I^{\sigma\zeta\zeta} = I^{\sigma} + I^{\zeta\zeta}, \tag{4.8}$$

$$I^{\zeta\sigma\sigma} = I^{\zeta} + I^{\sigma\sigma}. \tag{4.9}$$

The lower estimate of each of these functionals gives some component of the boundary.

We should distinguish the case of well and badly ordered materials.

The lower estimates of the functionals $I^{\zeta\zeta\zeta}$ and $I^{\sigma\sigma\sigma}$ provide the lower and upper estimates of the convex combination of the effective bulk and shear moduli because these functionals linearly depend on these moduli (the functional $I^{\zeta\zeta\zeta}$) or on their inverse values (the functional $I^{\sigma\sigma\sigma}$).

In the well ordered case (2.31), however, the points of maximal and minimal values of both moduli (the Hashin - Shtrikman points A and C on Fig.1) are attainable by special microstructures (see [9, 29, 33, 40]); and it is clear that the estimates of these functionals cannot improve the classical inequalities (2.32), (2.37).

On the other hand, minimization of the functionals $I^{\sigma\zeta\zeta}$ or $I^{\zeta\sigma\sigma}$ demands to minimize one of the moduli and maximize the other one. We show below that for well ordered materials it leads to the coupled estimates of the moduli which are more restrictive then the Hashin - Shtrikman ones.

In the badly ordered case (2.39) we face the opposite situation: the estimates of the functionals $I^{\zeta\zeta\zeta}$ and $I^{\sigma\sigma\sigma}$ improve the Walpole bounds, and the estimates of the functionals $I^{\sigma\zeta\zeta}$ and $I^{\zeta\sigma\sigma}$ leads to known ones.

4.2 The estimates

In this paragraph we outline the ideas of the estimating procedure (for details see [6, 30, 34, 37]) and give the final formulae.

First, note, that each of the functionals described above is a quadratic form of fields. It can be represented in the form

$$I = < \mathbf{e} \cdot \mathbf{D}(x) \cdot \mathbf{e} >, \tag{4.10}$$

$$\mathbf{e} \in EK, \tag{4.11}$$

where $\mathbf{e}$ is a vector composed from the coefficients of tensors of gradients ζ or stresses σ in the basis (2.9); EK is a set of the double periodic vectors with prescribed mean values and satisfying some differential restrictions. For components of stresses these restrictions express the equilibrium equations $\nabla \cdot \sigma = 0$ and for the gradients ζ they follow the compatibility conditions $\zeta = \nabla \otimes \mathbf{u}$. The matrix $\mathbf{D}$ is a piece-wise constant block diagonal matrix composed from the coefficients of the tensors of the properties in the mentioned basis (2.9).

For example, the matrix $\mathbf{D}$ and the vector $\mathbf{e}$ for the functional $I^{\zeta\zeta\zeta}$ have the form

$$\mathbf{D} = \mathbf{D}^{\zeta\zeta\zeta} = \begin{pmatrix} \mathbf{C}' & 0 & 0 \\ 0 & \mathbf{C}' & 0 \\ 0 & 0 & \mathbf{C}' \end{pmatrix}, \tag{4.12}$$

$$\begin{gathered} \mathbf{e} = [\zeta,\ \zeta',\ \zeta''] \\ = [(\zeta_1,\ \zeta_2,\ \zeta_3,\ \zeta_4),\ (\zeta_1',\ \zeta_2',\ \zeta_3',\ \zeta_4'),\ (\zeta_1'',\ \zeta_2'',\ \zeta_3'',\ \zeta_4'')], \\ EK = \{\mathbf{e} : \zeta = \nabla\mathbf{u}, \quad \zeta' = \nabla\mathbf{u}', \quad \zeta'' = \nabla\mathbf{u}'', \\ <\mathbf{e}>= \mathbf{e}_0, \mathbf{e} \ - \ \Omega - periodic.\}. \end{gathered} \tag{4.13}$$

Here the vectors $\mathbf{u}$, $\mathbf{u}'$, $\mathbf{u}''$ correspond to the three different displacement fields; the tensors ζ, ζ', ζ'' suppose to be resolved in the basis (2.9). Similarly, the matrix $\mathbf{D}$ and the vector $\mathbf{e}$ for the functional $I^{\zeta\sigma\sigma}$ have the form

$$\mathbf{D} = \mathbf{D}^{\zeta\sigma\sigma} = \begin{pmatrix} \mathbf{C}' & 0 & 0 \\ 0 & \mathbf{S} & 0 \\ 0 & 0 & \mathbf{S} \end{pmatrix}, \tag{4.14}$$

$$\begin{gathered} \mathbf{e} = [\zeta,\ \sigma,\ \sigma'] \\ = [(\zeta_1,\ \zeta_2,\ \zeta_3,\ \zeta_4),\ (\sigma_1,\ \sigma_2,\ \sigma_3),\ (\sigma_1',\ \sigma_2',\ \sigma_3')], \\ EK = \{\mathbf{e} : \zeta = \nabla\mathbf{u}, \quad \nabla\sigma = \mathbf{0}, \quad \nabla\sigma' = 0, \\ <\mathbf{e}>= \mathbf{e}_0,\ \mathbf{e} \ - \ \Omega - periodic.\}. \end{gathered} \tag{4.15}$$

and so on. Our goal is to find the infimum of the functional I over all fields $\mathbf{e} \in EK$ and over all microstructures of the composite with prescribed volume fractions of the components.

To obtain the simplest estimate of this infimum it is sufficient to note (see, for example [30]) that the integrand of the functional (4.10) is the convex function of the vector $\mathbf{e}$ and the inverse matrix $\mathbf{D}^{-1}$. It is assumed that the matrix $\mathbf{D}$ is nonnegative.

Due to the convexity of the integrand in the variables $\mathbf{e}$ and $\mathbf{D}^{-1}$ the estimate of the functional (4.10) has a form

$$I \ \geq <\mathbf{e}> \cdot <\mathbf{D}^{-1}>^{-1} \cdot <\mathbf{e}> \tag{4.16}$$

At the same time the value of the functional I is determined by the effective properties of the mixture:

$$I = < \mathbf{e} > \cdot < \mathbf{D}_0 > \cdot < \mathbf{e} > \tag{4.17}$$

Comparing (4.16) and (4.17) which are valid for any vector $< \mathbf{e} >$, we end up with the Hill's type estimates of the moduli

$$\mathbf{D}_0 \quad \geq \quad < \mathbf{D}^{-1} >^{-1} . \tag{4.18}$$

Note , that this estimating procedure doesn't take into account the differential restrictions (4.11) and the estimates are valid but, in general, not exact.

The translation method is based on the theory of quasiconvexity (see for example [8]) and the theory of compensated compactness [46]. Its main idea is to use the integral corollaries of the linear differential restrictions (4.11). Namely, it is possible to find *quasiaffine* [8] quadratic functions of the fields $\mathbf{e}$

$$\phi(\mathbf{e}) = \mathbf{e} \cdot \mathbf{T} \cdot \mathbf{e} \tag{4.19}$$

possessing the property of affine functions

$$< \phi(\mathbf{e}) >= \phi(< \mathbf{e} >) \tag{4.20}$$

for every field $\mathbf{e} \in EK$. Here $\mathbf{T}$ is a so called translation matrix which is the constant matrix depending on the differential restriction (4.11). Such matrices can possess several free parameters. Constructing of quasiaffine bilinear and quadratic functions for the problem under study (i.e. for stress and strain fields) is discussed in the next section.

Let us show the procedure of obtaining the "translation estimates" (see [6, 30, 37]). First, assuming that quasiaffine functions are already found we add such function $\phi(\mathbf{e})$ to the functional I and then subtract it:

$$I =< \mathbf{e} \cdot (\mathbf{D} - \mathbf{T})(x) \cdot \mathbf{e} > + < \mathbf{e} \cdot \mathbf{T}(x) \cdot \mathbf{e} > . \tag{4.21}$$

Then we estimate the first term by using the described above Hill's procedure and the second term by using the equality (4.20). This leads to the inequality

$$< \mathbf{e} > \cdot \mathbf{D}_0 \cdot < \mathbf{e} > \quad \geq \quad < \mathbf{e} > \cdot [< (\mathbf{D} - \mathbf{T})^{-1} >^{-1} + \mathbf{T}] \cdot < \mathbf{e} > . \tag{4.22}$$

Similarly to the simplest case, we may refer to arbitrarity of the acting fields $\mathbf{e}$ and pass to the matrix inequalities:

$$\mathbf{D}_0 \quad \geq \quad < (\mathbf{D} - \mathbf{T})^{-1} >^{-1} + \mathbf{T}. \tag{4.23}$$

As before, it is assumed here that the piece-wise constant matrix $\mathbf{D}(x) - \mathbf{T}$ is nonnegative at any point. This leads to the restrictions on the coefficients of the translation matrix $\mathbf{T}$:

$$\mathbf{D}_1 - \mathbf{T} \geq \mathbf{0}, \tag{4.24}$$

$$\mathbf{D}_2 - \mathbf{T} \geq 0. \tag{4.25}$$

To transform the estimates we use the matrix form of the described above (see (2.29)) Y-transformation introduced by Milton [37] and the authors [6]

$$\mathbf{Y}(\mathbf{D}_0) = (< \mathbf{D} > - \mathbf{D}_1 - \mathbf{D}_2) - m_1 m_2 (\mathbf{D}_1 - \mathbf{D}_2) \cdot (\mathbf{D}_0 - < \mathbf{D} >)^{-1} \cdot (\mathbf{D}_1 - \mathbf{D}_2). \tag{4.26}$$

If the matrix $(\mathbf{D}_1 - \mathbf{D}_2)$ does not degenerated then the estimates (4.22) can be represented in the surprisely simple form (see [37, 6] and Appendix)

$$\mathbf{Y}(\mathbf{D}_0) + \mathbf{T} \geq 0. \tag{4.27}$$

However, in the problem under study we face the situation, when some of the eigendirections and eigenvalues of the matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ coincide because the matrix $\mathbf{C}'$ for any material has the fourth eigenvalue equal to zero (see (2.20)). The matrix $\mathbf{C}'$ in its turn is the diagonal block of the matrices $\mathbf{D}$ used in the functionals (4.2), (4.4), (4.6), (4.8), (4.9) . Degeneration of the differences $\mathbf{D}_1 - \mathbf{D}_2$ means that these matrices satisfy the relationships

$$\begin{gathered} \mathbf{P}_\perp \cdot \mathbf{D}_1 \cdot \mathbf{P}_\perp{}^T = \mathbf{P}_\perp \cdot \mathbf{D}_2 \cdot \mathbf{P}_\perp{}^T, \\ det[\mathbf{P} \cdot (\mathbf{D}_1 - \mathbf{D}_2) \cdot \mathbf{P}^T] \neq 0. \end{gathered} \tag{4.28}$$

Here $P_\perp$ is a projector on the degenerated subspace of the matrix $(\mathbf{D}_1 - \mathbf{D}_2)$ and P is a projector on the nondegenerated subspace of this matrix which is the compliment to the projector $\mathbf{P}_\perp$

$$\mathbf{P} \oplus \mathbf{P}_\perp = \mathbf{E}, \tag{4.29}$$

$\mathbf{E}$ is a unit matrix. We assume that these subspaces are the eigensubspaces for the matrices $\mathbf{D}_1$ and $\mathbf{D}_2$

$$\mathbf{P}_\perp \cdot \mathbf{D}_1 \cdot \mathbf{P}^T = \mathbf{P}_\perp \cdot \mathbf{D}_2 \cdot \mathbf{P}^T = 0. \tag{4.30}$$

¿From (2.20) we have also

$$\mathbf{P}_\perp \cdot \mathbf{D}_1 \cdot \mathbf{P}_\perp{}^T = \mathbf{P}_\perp \cdot \mathbf{D}_2 \cdot \mathbf{P}_\perp{}^T = 0. \tag{4.31}$$

Remark 4.1: We also meet the degenerated matrix $\mathbf{D}_1 - \mathbf{D}_2$ *in homogenization problems dealing with composites made from materials with equal bulk or shear moduli. Such problems were studied in the already mentioned papers [10, 17, 24, 25, 26]. Note that these cases doesn't imply the equation (4.31)*

The homogenized problem for a mixture made from materials satisfying (4.28) was studied by Lurie and Cherkaev in the paper [26]. It was found that the effective properties tensor $\mathbf{D}_0$ possesses the properties

$$\begin{aligned} \mathbf{P}_\perp \cdot \mathbf{D}_0 \cdot \mathbf{P}_\perp{}^T = \mathbf{P}_\perp \cdot \mathbf{D}_1 \cdot \mathbf{P}_\perp{}^T = \mathbf{P}_\perp \cdot \mathbf{D}_2 \cdot \mathbf{P}_\perp{}^T, \\ \mathbf{P}_\perp \cdot \mathbf{D}_0 \cdot \mathbf{P}^T = \mathbf{P}_\perp \cdot \mathbf{D}_1 \cdot \mathbf{P}^T = \mathbf{P}_\perp \cdot \mathbf{D}_2 \cdot \mathbf{P}^T = 0 \end{aligned} \tag{4.32}$$

independentely of the microstructure of the mixture. In other words, if the property tensors of components have the same eigenvalues and eigendirections, then the effective tensor of a composite also posesses the same eigenvalues and eigendirections.

This imply that the tensor transformation Y is undefined in the whole space because it includes ratio of two zeros (see (4.26), (4.32)). However, it can be defined in the nondegenerated subspace of the difference $(\mathbf{D}_1 - \mathbf{D}_2)$.

Instead of the inequalities (4.23) we will use here (see Appendix) the system of estimates: the equality (4.32) and the inequality

$$\mathbf{P} \cdot \mathbf{D}_0 \cdot \mathbf{P}^T \quad \geq \quad \mathbf{P} < (\mathbf{D} - \mathbf{T})^{-1} >^{-1} \mathbf{P}^T + \mathbf{P} \cdot \mathbf{T} \cdot \mathbf{P}^T. \tag{4.33}$$

Note, however, that the restrictions (4.24), (4.25) on the translation matrices T should be valid in the whole space, not only on nonzero subspace of the differences $(\mathbf{D}_1 - \mathbf{D}_2)$.

To simplify the inequality (4.33) we also can use the Y-transformation; the result has the form (see Appendix)

$$\mathbf{Y}(\mathbf{P} \cdot \mathbf{D}_0 \cdot \mathbf{P}^T) + \mathbf{P} \cdot \mathbf{T} \cdot \mathbf{P}^T + (\mathbf{P} \cdot \mathbf{T} \cdot \mathbf{P}_\perp{}^T) \cdot [\mathbf{P}_\perp \cdot (\mathbf{D}_1 - \mathbf{T}) \cdot \mathbf{P}_\perp{}^T]^{-1} \cdot (\mathbf{P}_\perp \cdot \mathbf{T} \cdot \mathbf{P}^T) \geq 0, \tag{4.34}$$

where

$$\mathbf{Y}(\mathbf{P}\cdot\mathbf{D}_0\cdot\mathbf{P}^T)=\mathbf{P}\cdot(<\mathbf{D}> -\mathbf{D}_1-\mathbf{D}_2)\cdot\mathbf{P}^T-$$

$$m_1 m_2\mathbf{P}\cdot(\mathbf{D}_1-\mathbf{D}_2)\cdot\mathbf{P}^T\cdot(\mathbf{P}\cdot(\mathbf{D}_0-<\mathbf{D}>)\cdot\mathbf{P}^T)^{-1}\cdot\mathbf{P}\cdot(\mathbf{D}_1-\mathbf{D}_2)\cdot\mathbf{P}^T. \tag{4.35}$$

Remember now that $\mathbf{P}_\perp\cdot\mathbf{D}_1\cdot{\mathbf{P}_\perp}^T=\mathbf{P}_\perp\cdot\mathbf{D}_2\cdot{\mathbf{P}_\perp}^T$, see (4.28). The equality (4.31) leads to the further simplification of the estimates

$$\mathbf{Y}(\mathbf{P}\cdot\mathbf{D}_0\cdot\mathbf{P}^T)+\mathbf{P}\cdot\mathbf{T}\cdot\mathbf{P}^T-(\mathbf{P}\cdot\mathbf{T}\cdot{\mathbf{P}_\perp}^T)\cdot[\mathbf{P}_\perp\cdot\mathbf{T}\cdot{\mathbf{P}_\perp}^T]^{-1}\cdot(\mathbf{P}_\perp\cdot\mathbf{T}\cdot\mathbf{P}^T)\geq 0, \tag{4.36}$$

This relation means that all eigenvalues of the matrix in the left hand side are nonnegative. The scalar consequence of this inequality which is used for establishing the bounds has the form

$$\begin{gathered} det[\mathbf{Y}(\mathbf{P}\cdot\mathbf{D}_0\cdot\mathbf{P}^T)+\mathbf{P}\cdot\mathbf{T}\cdot\mathbf{P}^T- \\ (\mathbf{P}\cdot\mathbf{T}\cdot{\mathbf{P}_\perp}^T)\cdot[\mathbf{P}_\perp\cdot\mathbf{T}\cdot{\mathbf{P}_\perp}^T]^{-1}\cdot(\mathbf{P}_\perp\cdot\mathbf{T}\cdot\mathbf{P}^T]\geq 0, \\ \\ for\ \ all\ translation\ \ matrices\ \ \mathbf{T}: \\ \mathbf{T}\in(4.19),\ (4.20),\ (4.24),\ (4.25). \end{gathered} \tag{4.37}$$

The parameters of the symmetric matrix $\mathbf{T}$ should be chosen in order to the estimates (4.37) to be more restrictive. To do it we have to determine these parameters from the requirements that the maximum number of eigenvalues of the matrices $\mathbf{D}_1-\mathbf{T}$ and $\mathbf{D}_2-\mathbf{T}$ vanishes (see [6])

$$\begin{gathered} rank(\mathbf{D}_1-\mathbf{T})+rank(\mathbf{D}_2-\mathbf{T}) \\ is\ \ to\ \ be\ \ minimized\ \ over\ \ all\ translation\ \ matrices\ \ \mathbf{T}: \\ \mathbf{T}\in(4.19),\ (4.20),\ (4.24),\ (4.25). \end{gathered} \tag{4.38}$$

5 CONSTRUCTION OF QUASIAFFINE FUNCTIONS

To use the translation method for the problem under study we need to find the set of the bilinear quasiaffine functions of stresses and strains. In this section we determine consequencely such functions depending on two strain fields, two stress fields and the bilinear function depending on stress and strain fields.

5.1 The strain-strain quasiaffine functions

We are looking for the quasiaffine functions in the space of strains. To use the already known results we will find them in an enlarged class of gradients of displacements. Remember that the tensors of strains and gradients of displacements are connected by the relationships (2.18). So, we expand the space of strains by adding the fourth orthogonal coordinate equal to the antisymmetric part of the gradient of displacement (see (2.17)-(2.20)).

We will use the well known result about weakly convergence of the Jacobian J of two potentials w_1 and w_2

$$J = det(\nabla w_1,\ \nabla w_2) = \nabla w_1 \cdot \mathbf{R} \cdot \nabla w_2, \tag{5.1}$$

where R is a rotation on the right angle

$$R = \mathbf{i} \otimes \mathbf{j} - \mathbf{j} \otimes \mathbf{i}. \tag{5.2}$$

According to the Reshetnyak papers [42] (see also [8, 21, 45]) the Jacobian (5.1) is a quasiaffine function of it arguments:

$$<\ J\ >=<\ \nabla w_1 \cdot \mathbf{R} \cdot \nabla w_2\ >=<\ \nabla w_1\ > \cdot \mathbf{R} \cdot <\ \nabla w_2\ > . \tag{5.3}$$

To find the bilinear quasiaffine function of the gradients of displacements let us consider the bilinear form

$$\phi(\zeta,\ \zeta') = [\nabla u_1,\ \nabla u_2] \cdot \begin{pmatrix} t_1'\mathbf{R} & t_2'\mathbf{R} \\ t_3'\mathbf{R} & t_4'\mathbf{R} \end{pmatrix} \cdot [\nabla u_1',\ \nabla u_2']^T \tag{5.4}$$

of the gradients $\zeta = \nabla \otimes \mathbf{u}$, $\zeta' = \nabla \otimes \mathbf{u}'$ for any two displacement vectors

$$\mathbf{u} = (u_1 \mathbf{i} + u_2 \mathbf{j})\ , \qquad \mathbf{u}' = (u_1' \mathbf{i} + u_2' \mathbf{j}). \tag{5.5}$$

This function is an arbitrary linear combination of Jacobians of pairs of components u_1, u_2, u_1', u_2'. It is clear from the previous note that $\phi(\zeta,\ \zeta')$ is the quasiaffine function for any values of the parameters $t_i',\ i = 1, 2, 3, 4$ of this linear combination.

The expression (5.4) can be written in the form

$$\phi(\zeta,\ \zeta') = \zeta \cdot \mathbf{\Phi}^{\zeta\zeta}(t_1,\ t_2,\ t_3,\ t_4) \cdot \zeta' \ , \tag{5.6}$$

where the matrix $\mathbf{\Phi}^{\zeta\zeta}$ is represented in the basis $\mathbf{a}_1, ..., \mathbf{a}_4$ (see (2.9)) as following

$$\mathbf{\Phi}^{\zeta\zeta}(t_1,\ t_2,\ t_3,\ t_4) = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ -t_2 & -t_1 & t_4 & t_3 \\ -t_3 & -t_4 & -t_1 & -t_2 \\ -t_4 & -t_3 & t_2 & t_1 \end{pmatrix} \tag{5.7}$$

and the parameters $t_i,\ i = 1, 2, 3, 4$ are the linearly independent linear combinations of the parameters $t'_i,\ i = 1, 2, 3, 4$.

Remark 5.1: Particularly, for the estimates of the energy of a composite (the functional I^ζ) it is sufficient to use the quadratic form of gradients ζ instead of the bilinear one. This quadratic form is associated with the symmetric part $\mathbf{T}^\zeta = \mathbf{\Phi}(\zeta,\ \zeta)^S$ of the matrix $\Phi^{\zeta\zeta}$ described above. It has the form

$$\mathbf{T}^\zeta(t_1) = \Phi^{\zeta\zeta}(t_1,\ 0,\ 0,\ 0). \tag{5.8}$$

This function written in different terms was used for obtaining estimates in the papers [2, 11, 12, 21, 25, 26, 30] .

5.2 The stress-stress quasiaffine functions

Let us note first that any stress tensor has the following representation (see [23])

$$\mathbf{R}^T \cdot \sigma \cdot \mathbf{R} = \nabla \otimes \nabla\chi \tag{5.9}$$

through the Airy stress function χ; therefore we have

$$\mathbf{R}^T \cdot \sigma \cdot \mathbf{R} = \nabla \otimes \mathbf{v}, \quad \mathbf{v} = \nabla\chi. \tag{5.10}$$

On the other hand the bilinear form

$$\nabla \otimes \mathbf{v} \cdot\cdot \mathbf{\Phi}^{\zeta\zeta} \cdot\cdot \nabla \otimes \mathbf{v}' \tag{5.11}$$

is quasiaffine for any vector fields $\mathbf{v},\ \mathbf{v}'$, as it was shown in the previous paragraph. Therefore, the bilinear form of the stresses $\sigma,\ \sigma'$

$$(\mathbf{R}^T \cdot \sigma \cdot \mathbf{R}) \cdot\cdot \mathbf{\Phi}^{\zeta\zeta} \cdot\cdot (\mathbf{R} \cdot \sigma \cdot \mathbf{R}^T) = \sigma \cdot\cdot (\mathbf{R}^T \otimes \mathbf{R}^T) \cdot\cdot \mathbf{\Phi}^{\zeta\zeta} \cdot\cdot (\mathbf{R} \otimes \mathbf{R}) \cdot\cdot \sigma' \tag{5.12}$$

is also quasiaffine.

Remember now that σ is a symmetric matrix due to the equilibrium conditions, therefore $\sigma \cdot\cdot \mathbf{a}_4 = 0$ identically. It means that the quasiaffine function (5.12) can be represented as a bilinear function

$$\sigma \cdot \mathbf{\Phi}^{\sigma\sigma}(t_1,\ t_2,\ t_3) \cdot \sigma' \tag{5.13}$$

of only first three of the coefficients $\sigma_i,\ \sigma'_i$, i=1,2,3 (see (2.10)) with the matrix

$$\mathbf{\Phi}^{\sigma\sigma}(t_1,\ t_2,\ t_3) = \begin{pmatrix} t_1 & t_2 & t_3 \\ -t_2 & -t_1 & t_4 \\ -t_3 & -t_4 & -t_1 \end{pmatrix}, \tag{5.14}$$

which is the upper left (3×3) minor of the matrix $(\mathbf{R}^T \otimes \mathbf{R}^T) \cdot\cdot \mathbf{\Phi}^{\zeta\zeta} \cdot\cdot (\mathbf{R} \otimes \mathbf{R})$ in the basis $(\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3,\ \mathbf{a}_4)$.

5.3 The strain-stress quasiaffine functions

Let us consider again the bilinear quasiaffine function

$$\zeta \cdot\cdot \mathbf{\Phi}^{\zeta\zeta} \cdot\cdot \nabla \mathbf{v} \tag{5.15}$$

of the arguments ζ and $\mathbf{v} = \mathbf{R}^T \cdot \sigma \cdot \mathbf{R}$ which are the gradients of vector potentials. We are to rewrite this form as a function of the stress tensor and the tensor of the gradient of displacement. We also should take into account the condition of symmetry of the stress tensor $\sigma \cdot\cdot \mathbf{a}_4 = 0$. After obvious calculations we get the following quasiaffine bilinear form:

$$\phi^{\sigma\zeta} = \sigma \cdot \mathbf{\Phi}^{\sigma\zeta} \cdot \zeta, \tag{5.16}$$

where the (3×4) matrix $\mathbf{\Phi}^{\sigma\zeta}$ has a form

$$\mathbf{\Phi}^{\sigma\zeta} = \mathbf{\Phi}^{\sigma\zeta}(t_1,\ t_2,\ t_3,\ t_4) = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ t_2 & -t_1 & -t_4 & -t_3 \\ t_3 & t_4 & -t_1 & t_2 \end{pmatrix} \tag{5.17}$$

in the mentioned basis (2.9) . Here the three-dimensional vector σ and the four-dimensional vector ζ are the coefficients of the tensors σ and ζ in the basises $(\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3)$ and $(\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3,\ \mathbf{a}_4)$ respectively.

6 PROOF OF THE HASHIN-SHTRIKMAN AND WALPOLE ESTIMATES

In this section we get the Hashin - Shtrikman and Walpole bounds by using a regular procedure of the translation method (see also the papers [36, 37] by Milton where similar results were obtained). We prove it here for the completity of the discussion and also for demonstration of the regular procedure on the simple examples.

6.1 Bounds for the bulk modulus

6.1.1 The lower estimate.

To estimate the functional I^ζ we need the symmetric translation strain-strain matrix $\mathbf{T}^\zeta$ (see (5.8)).

To get the result we use the inequality (4.37) where we should substitute the following matrices

$$
\begin{gathered}
\mathbf{D}_i = \mathbf{D}_i^\zeta = \mathbf{C}'_i, \quad i = 1, 2, \\
\mathbf{T} = \mathbf{T}^\zeta(t), \\
\mathbf{P}^\zeta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
\mathbf{P}_\perp{}^\zeta = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}
\end{gathered}
\tag{6.1}
$$

(see (4.2), (5.8)) The restrictions (4.24), (4.25) of the positivity of the matrices $\mathbf{C}'_1 - \mathbf{T}^\zeta(t)$ and $\mathbf{C}'_2 - \mathbf{T}^\zeta(t)$ have the form

$$
\begin{pmatrix} 2K_i - t & 0 & 0 & 0 \\ 0 & 2\mu_i + t & 0 & 0 \\ 0 & 0 & 2\mu_i + t & 0 \\ 0 & 0 & 0 & -t \end{pmatrix} \geq 0, \quad i = 1,\ 2 \tag{6.2}
$$

and lead to the scalar inequalities

$$
t \leq 0, \quad t \geq -min\{\mu_1, \mu_2\} = -\mu_2. \tag{6.3}
$$

The estimate (4.36) for the isotropic matrix $\mathbf{D}_0^\zeta = \mathbf{C}_0'$ associated with the functional I^ζ has the form

$$\begin{pmatrix} y(2K_0)+t & 0 & 0 \\ 0 & y(2\mu_0)-t & 0 \\ 0 & 0 & y(2\mu_0)-t \end{pmatrix} \geq 0 \tag{6.4}$$

and leads to the inequality for the bulk modulus K_0

$$y(2K_0) \geq -t, \quad t \in (6.3). \tag{6.5}$$

The most restrictive bound corresponds to the critical value (see (6.3), (4.38))

$$t = t^* = -\mu_2 \tag{6.6}$$

of the parameter t. This bound coincides with the Hashin - Shtrikman and Walpole lower bound for the bulk modulus (see (2.32)–(2.37), (2.40)– (2.41)).

6.1.2 The upper estimate.

The upper estimate for the bulk modulus can be obtained analogously using the functional I^σ instead of I^ζ and the quasiaffine function associated with the translation matrix

$$\mathbf{T}^\sigma(t_1) = \mathbf{\Phi}^{\sigma\sigma}(t_1,\ 0,\ 0,\ 0). \tag{6.7}$$

This matrix is the symmetric part of the matrix $\mathbf{\Phi}^{\sigma\sigma}(t_1,\ t_2,\ t_3,\ t_4)$. For this functional matrices $\mathbf{D}_i,\ i = 0, 1, 2$ are equal

$$\mathbf{D}_i = \mathbf{D}_i^\sigma = \mathbf{S}_i\ , \quad i = 0, 1, 2. \tag{6.8}$$

The difference $\mathbf{D}_1 - \mathbf{D}_2$ is nondegenerated, therefore we can use the estimates in the form (4.27). Now the restrictions (4.24), (4.25) have the matrix form

$$\begin{pmatrix} \frac{1}{2K_i}-t_1 & 0 & 0 \\ 0 & \frac{1}{2\mu_i}+t_1 & 0 \\ 0 & 0 & \frac{1}{2\mu_i}+t_1 \end{pmatrix} \geq 0 \tag{6.9}$$

or scalar form

$$-\frac{1}{2\mu_1} = -min\{\frac{1}{2\mu_1},\ \frac{1}{2\mu_2}\} \leq t \leq min\{\frac{1}{2K_1},\ \frac{1}{2K_2}\}. \tag{6.10}$$

The estimate for an isotropic effective tensor $\mathbf{S}_0$ are the following

$$\begin{pmatrix} y(\frac{1}{2K_0})+t_1 & 0 & 0 \\ 0 & y(\frac{1}{2\mu_0})-t_1 & 0 \\ 0 & 0 & y(\frac{1}{2\mu_0})-t_1 \end{pmatrix} \geq 0. \tag{6.11}$$

The corresponding scalar inequality for bulk modulus K_0

$$y(\frac{1}{2K_0})+t_1 \geq 0 \tag{6.12}$$

becomes the most restrictive when the parameter t_1 is chosen as $t_1 = t_1^* = -\frac{1}{2\mu_1}$ (see (4.38), (6.10)). By using the properties of the Y-transformation (2.30) it can be represented in the form

$$y(K_0) \leq \mu_1. \tag{6.13}$$

This bound coincides with the Hashin-Shtrikman and Walpole upper bounds for the bulk modulus (see (2.32)–(2.37), (2.40)– (2.41)).

6.2 Bounds for the shear modulus

6.2.1 The lower estimate.

We begin with the functional $I^{\zeta\zeta}$ to get the lower bound for the shear modulus. We use the formulae (4.22)– (4.38), where $\mathbf{e}$ should be treated as an eight-dimensional vector

$$\mathbf{e} = \{\zeta_1,\ \zeta_2,\ \zeta_3,\ \zeta_4,\ \zeta_1',\ \zeta_2',\ \zeta_3',\ \zeta_4'\}, \tag{6.14}$$

where $\zeta_i,\ \zeta_i'$, i=1,2,3,4 are the components of gradients $\zeta,\ \zeta'$ in the basis (2.9). The matrices $\mathbf{D}_i,\ \ i=0,1,2$ have the block-diagonal form

$$\mathbf{D}_i = \mathbf{D}_i^{\zeta\zeta} = \begin{pmatrix} \mathbf{C}_i' & 0 \\ 0 & \mathbf{C}_i' \end{pmatrix}, \quad i=1,2. \tag{6.15}$$

The projector P (see (4.28)-(4.32)) on nonzero subspace of the matrix $(\mathbf{D}_1 - \mathbf{D}_2)$ and the complementary projector $\mathbf{P}_\perp$ have a block forms

$$\mathbf{P} = \mathbf{P}^{\zeta\zeta} = \begin{pmatrix} \mathbf{P}^\zeta & 0 \\ 0 & \mathbf{P}^\zeta \end{pmatrix}, \quad \mathbf{P}_\perp = \mathbf{P}_\perp{}^{\zeta\zeta} = \begin{pmatrix} \mathbf{P}_\perp{}^\zeta & 0 \\ 0 & \mathbf{P}_\perp{}^\zeta \end{pmatrix}, \tag{6.16}$$

where $\mathbf{P}^\zeta$ and $\mathbf{P}_\perp{}^\zeta$ are defined by (6.1).

Let us construct a quasiaffine quadratic form of the vector $\mathbf{e}$. Consider the quadratic form of this vector associated with the block matrix

$$\mathbf{T}^{\zeta\zeta} = \begin{pmatrix} \mathbf{\Phi}^{\zeta\zeta}(t_1^1,\ t_2^1,\ t_3^1,\ t_4^1) & \mathbf{\Phi}^{\zeta\zeta}(t_1^2,\ t_2^2,\ t_3^2,\ t_4^2) \\ \mathbf{\Phi}^{\zeta\zeta}(t_1^3,\ t_2^3,\ t_3^3,\ t_4^3) & \mathbf{\Phi}^{\zeta\zeta}(t_1^4,\ t_2^4,\ t_3^4,\ t_4^4) \end{pmatrix}, \tag{6.17}$$

where the matrices $\mathbf{\Phi}^{\zeta\zeta}$ are defined by the formula (5.7). It is clear that this function is quasiaffine as a sum of quasiaffine functions. The matrix $\mathbf{T}^{\zeta\zeta}$ depends on 16 parameters $t_i^j, i, j = 1,\ 2,\ 3,\ 4$. However, the matrix of a quadratic form has to be symmetric; it reduce the number of parameters to 6. We will also assume that $t_1^1 = t_1^4$ because of the isotropy of the mixture. It turns out to be in additional, that it is sufficient for the shear modulus estimates to put zero three of five the remaining parameters t_i^j and to deal with the matrix

$$\mathbf{T}^{\zeta\zeta}(t_1,\ t_2) = \begin{pmatrix} -t_1 & 0 & 0 & 0 & 0 & 0 & 0 & t_2 \\ 0 & t_1 & 0 & 0 & 0 & 0 & t_2 & 0 \\ 0 & 0 & t_1 & 0 & 0 & -t_2 & 0 & 0 \\ 0 & 0 & 0 & -t_1 & -t_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -t_2 & -t_1 & 0 & 0 & 0 \\ 0 & 0 & -t_2 & 0 & 0 & t_1 & 0 & 0 \\ 0 & t_2 & 0 & 0 & 0 & 0 & t_1 & 0 \\ t_2 & 0 & 0 & 0 & 0 & 0 & 0 & -t_1 \end{pmatrix}. \tag{6.18}$$

which depends only on two parameters $t_1,\ t_2$.

The restrictions (4.24), (4.25) have the matrix form

$$\mathbf{D}_i - \mathbf{T}^{\zeta\zeta}(t_1,\ t_2) =$$

$$\begin{pmatrix} 2K_i + t_1 & 0 & 0 & 0 & 0 & 0 & 0 & -t_2 \\ 0 & 2\mu_i - t_1 & 0 & 0 & 0 & 0 & -t_2 & 0 \\ 0 & 0 & 2\mu_i - t_1 & 0 & 0 & t_2 & 0 & 0 \\ 0 & 0 & 0 & t_1 & t_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_2 & 2K_i + t_1 & 0 & 0 & 0 \\ 0 & 0 & t_2 & 0 & 0 & 2\mu_i - t_1 & 0 & 0 \\ 0 & -t_2 & 0 & 0 & 0 & 0 & 2\mu_i - t_1 & 0 \\ -t_2 & 0 & 0 & 0 & 0 & 0 & 0 & t_1 \end{pmatrix} \geq 0,$$

$$i = 1,\ 2 \tag{6.19}$$

and the scalar form

$$\begin{array}{c}(2K_i + t_1)\ t_1 - t_2^2 \geq 0 \\ \qquad\qquad\qquad\qquad i = 1, 2\ , \\ (2\mu_i - t_1)^2 - t_2^2 \geq 0\end{array} \tag{6.20}$$

or

$$\begin{array}{c}(2K_{min} + t_1)\ t_1 - t_2^2 \geq 0, \\ (2\mu_{min} - t_1)^2 - t_2^2 \geq 0,\end{array} \tag{6.21}$$

where

$$\begin{array}{c}K_{min} = \min\{K_1,\ K_2\}, \\ \mu_{min} = \min\{\mu_1,\ \mu_2\} = \mu_2.\end{array} \tag{6.22}$$

The limiting values $t_1^*,\ t_2^*$ of the parameters t_1 , t_2 correspond to the equalities in (6.21) (see (4.38)); they are equal to

$$t_1^* = \frac{2\mu_2^2}{K_{min} + 2\mu_2}, \tag{6.23}$$

$$t_2^* = \frac{2\mu_2(K_{min} + \mu_2)}{K_{min} + 2\mu_2}. \tag{6.24}$$

Substituting these values into the estimates (4.36) which now become the form

$$\begin{array}{c}\mathbf{Y}(\mathbf{P}^{\zeta\zeta} \cdot \mathbf{D}_0 \cdot \mathbf{P}^{\zeta\zeta}) + \mathbf{P}^{\zeta\zeta} \cdot \mathbf{T}^{\zeta\zeta} \cdot \mathbf{P}^{\zeta\zeta} \\ -(\mathbf{P}^{\zeta\zeta} \cdot \mathbf{T}^{\zeta\zeta} \cdot \mathbf{P}_\perp^{\zeta\zeta^T}) \cdot [\mathbf{P}_\perp^{\zeta\zeta} \cdot \mathbf{T}^{\zeta\zeta} \cdot \mathbf{P}_\perp^{\zeta\zeta^T}]^{-1} \cdot (\mathbf{P}_\perp^{\zeta\zeta} \cdot \mathbf{T}^{\zeta\zeta} \cdot \mathbf{P}^{\zeta\zeta^T}) =\end{array}$$

$$\begin{pmatrix} 2y(K_0) + \frac{t_2^{*2} - t_1^{*2}}{t_1^*} & 0 & 0 & 0 & 0 & 0 \\ 0 & 2y(\mu_0) + t_1^* & 0 & 0 & 0 & t_2^* \\ 0 & 0 & 2y(\mu_0) + t_1^* & 0 & -t_2^* & 0 \\ 0 & 0 & 0 & 2y(K_0) + \frac{t_2^{*2} - t_1^{*2}}{t_1^*} & 0 & 0 \\ 0 & 0 & -t_2^* & 0 & 2y(\mu_0) + t_1^* & 0 \\ 0 & t_2^* & 0 & 0 & 0 & 2y(\mu_0) + t_1^* \end{pmatrix} \geq 0 \tag{6.25}$$

we get the scalar inequality

$$2y(\mu_0) + t_1^* - t_2^* = 2y(\mu_0) - \frac{2K_{min}\mu_2}{K_{min} + 2\mu_2} \geq 0. \tag{6.26}$$

It coincides with the Hashin - Shtrikman bound on the shear modulus (2.32)–(2.37) in the well ordered case (2.31) and with the Walpole bound (2.40)–(2.41) in the badly ordered case (2.39).

6.2.2 The upper estimate.

To obtain the upper bound for the shear modulus of the mixture we use the same procedure for estimating the functional $I^{\sigma\sigma}$. In this case

$$\mathbf{e} = \{\sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_1',\ \sigma_2',\ \sigma_3'\} \tag{6.27}$$

is a six-dimensional vector consisting of the components of two stress tensors σ and σ'; the (6×6) matrices

$$\mathbf{D}_i = \mathbf{D}_i^{\sigma\sigma} = \begin{pmatrix} \mathbf{S}_i & 0 \\ 0 & \mathbf{S}_i \end{pmatrix}, \quad i = 0,\ 1,\ 2 \tag{6.28}$$

are block-diagonal. The differences $(\mathbf{D}_1 - \mathbf{D}_2)$ is nondegenerated, therefore we can use the estimate in the form (4.27).

We construct the matrix $\mathbf{T}^{\sigma\sigma}$ of a quasiafine quadratic function of the vector $\mathbf{e}$ in the same way as $\mathbf{T}^{\zeta\zeta}$ using the bilinear quasiaffine form $\mathbf{\Phi}^{\sigma\sigma}$ instead of the form $\mathbf{\Phi}^{\zeta\zeta}$

$$\mathbf{T}^{\sigma\sigma}(t_1,\ t_2) = \begin{pmatrix} -t_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_1 & 0 & 0 & 0 & t_2 \\ 0 & 0 & t_1 & 0 & -t_2 & 0 \\ 0 & 0 & 0 & -t_1 & 0 & 0 \\ 0 & 0 & -t_2 & 0 & t_1 & 0 \\ 0 & t_2 & 0 & 0 & 0 & t_1 \end{pmatrix}. \tag{6.29}$$

The restrictions (4.24), (4.25) now have the matrix form

$$\mathbf{D}_i^{\sigma\sigma} - \mathbf{T}^{\sigma\sigma}(t_1,\ t_2) =$$

$$\begin{pmatrix} \frac{1}{2K_i} + t_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2\mu_i} - t_1 & 0 & 0 & 0 & -t_2 \\ 0 & 0 & \frac{1}{2\mu_i} - t_1 & 0 & t_2 & 0 \\ 0 & 0 & 0 & \frac{1}{2K_i} + t_1 & 0 & 0 \\ 0 & 0 & t_2 & 0 & \frac{1}{2\mu_i} - t_1 & 0 \\ 0 & -t_2 & 0 & 0 & 0 & \frac{1}{2\mu_i} - t_1 \end{pmatrix} \geq 0, \tag{6.30}$$

$$i = 1,\ 2,$$

or the scalar form

$$t_1 \geq -\frac{1}{2K_{max}}, \tag{6.31}$$

$$(\frac{1}{2\mu_{max}} - t_1)^2 - t_2^2 \geq 0, \tag{6.32}$$

where

$$\begin{gathered} K_{max} = \max\{K_1,\ K_2\}, \\ \mu_{max} = \max\{\mu_1,\ \mu_2\} = \mu_1. \end{gathered} \tag{6.33}$$

The critical values $t_1^*,\ t_2^*$ of the parameters $t_1,\ t_2$ are equal

$$t_1^* = -\frac{1}{2K_{max}}, \qquad t_2^* = \frac{1}{2\mu_1} + \frac{1}{2K_{max}}. \tag{6.34}$$

The estimate (4.27) becomes the form

$$\mathbf{Y}(\mathbf{D}_0^{\sigma\sigma}) + \mathbf{T}^{\sigma\sigma}(t_1^*,\ t_2^*) =$$

$$\begin{pmatrix} y(\frac{1}{2K_0}) - t_1^* & 0 & 0 & 0 & 0 & 0 \\ 0 & y(\frac{1}{2\mu_0}) + t_1^* & 0 & 0 & 0 & t_2^* \\ 0 & 0 & y(\frac{1}{2\mu_0}) + t_1^* & 0 & -t_2^* & 0 \\ 0 & 0 & 0 & y(\frac{1}{2K_0}) - t_1^* & 0 & 0 \\ 0 & 0 & -t_2^* & 0 & y(\frac{1}{2\mu_0}) + t_1^* & 0 \\ 0 & t_2^* & 0 & 0 & 0 & y(\frac{1}{2\mu_0}) + t_1^* \end{pmatrix} \geq 0. \tag{6.35}$$

It leads to the scalar inequality

$$y(\frac{1}{2\mu_0}) + t_1^* - t_2^* = y(\frac{1}{2\mu_0}) - \frac{2}{K_{max}} - \frac{1}{\mu_1} \geq 0 \tag{6.36}$$

or

$$y(\mu_0) \leq \frac{K_{max}\mu_1}{K_{max} + 2\mu_1} \tag{6.37}$$

coinciding with the Hashin – Shtrikman bound (see (2.32)–(2.37)) in the well ordered case (2.31) and with the Walpole bound (see (2.40)–(2.41)) in the badly ordered case (2.39).

7 COUPLED BULK–SHEAR MODULI BOUNDS

Here we prove the new estimates formulated in the Theorem by using the described translation method.

7.1 The well-ordered case

7.1.1 The lower estimate.

Let us first estimate the functional $I^{\sigma\zeta\zeta}$ which allows to obtain the coupled bound, as it was mentioned above (section 4).

For this functional the vector $\mathbf{e}$ is the following 11-dimensional vector

$$\mathbf{e} = \{\sigma_1,\ \sigma_2,\ \sigma_3,\ \zeta_1,\ \zeta_2,\ \zeta_3,\ \zeta_4,\ \zeta_1',\ \zeta_2',\ \zeta_3',\ \zeta_4'\}. \tag{7.1}$$

The matrices $\mathbf{D}_i, \quad i = 0,\ 1,\ 2$ have the block-diagonal form

$$\mathbf{D}_i = \mathbf{D}_i^{\sigma\zeta\zeta} = \begin{pmatrix} \mathbf{S}_i & 0 & 0 \\ 0 & \mathbf{C}_i' & 0 \\ 0 & 0 & \mathbf{C}_i' \end{pmatrix} \tag{7.2}$$

and the projectors $\mathbf{P}$ and $\mathbf{P}_\perp$ are also block–diagonal matrices

$$\mathbf{P} = \mathbf{P}^{\sigma\zeta\zeta} = \begin{pmatrix} \mathbf{E} & 0 & 0 \\ 0 & \mathbf{P}^\zeta & 0 \\ 0 & 0 & \mathbf{P}^\zeta \end{pmatrix}, \quad \mathbf{P}_\perp = \mathbf{P}_\perp{}^{\sigma\zeta\zeta} = \begin{pmatrix} 0 & \mathbf{P}_\perp{}^\zeta & 0 \\ 0 & 0 & \mathbf{P}_\perp{}^\zeta \end{pmatrix}, \tag{7.3}$$

where $\mathbf{E}$ is the unit (3×3) matrix and $\mathbf{P}^\zeta$, $\mathbf{P}_\perp{}^\zeta$ are defined by (6.1). We use the translation matrix $\mathbf{T}^{\sigma\zeta\zeta}(t_1,\ t_2,\ t_3,\ t_4)$ with the block form

$$\mathbf{T}^{\sigma\zeta\zeta}(t_1,\ t_2,\ t_3,\ t_4) =$$

$$\begin{pmatrix} \mathbf{\Phi}^{\sigma\sigma}(-t_1,\ 0,\ 0,\ 0) & \mathbf{\Phi}^{\sigma\zeta}(0,\ -t_3,\ 0,\ 0) & \mathbf{\Phi}^{\sigma\zeta}(0,\ 0,\ -t_3,\ 0) \\ \mathbf{\Phi}^{\zeta\sigma}(0,\ -t_3,\ 0,\ 0) & \mathbf{\Phi}^{\zeta\zeta}(-t_2,\ 0,\ 0,\ 0) & \mathbf{\Phi}^{\zeta\zeta}(0,\ 0,\ 0, -t_4) \\ \mathbf{\Phi}^{\zeta\sigma}(0,\ 0, -t_3,\ 0) & \mathbf{\Phi}^{\zeta\zeta}(\ 0,\ 0,\ 0,\ -t_4) & \mathbf{\Phi}^{\zeta\zeta}(-t_2,\ 0,\ 0,\ 0) \end{pmatrix}, \tag{7.4}$$

where $\mathbf{\Phi}^{\zeta\sigma} = (\mathbf{\Phi}^{\sigma\zeta})^T$.

The restrictions (4.24), (4.25) have the form

$$\mathbf{D}_i^{\sigma\zeta\zeta} - \mathbf{T}^{\sigma\zeta\zeta}(t_1,\ t_2,\ t_3,\ t_4) =$$

$$\begin{pmatrix}
\frac{1}{2K_i}+t_1 & 0 & 0 & 0 & t_3 & 0 & 0 & 0 & 0 & t_3 & 0 \\
0 & \frac{1}{2\mu_i}-t_1 & 0 & t_3 & 0 & 0 & 0 & 0 & 0 & 0 & -t_3 \\
0 & 0 & \frac{1}{2\mu_i}-t_1 & 0 & 0 & 0 & t_3 & t_3 & 0 & 0 & 0 \\
0 & t_3 & 0 & 2K_i+t_2 & 0 & 0 & 0 & 0 & 0 & 0 & t_4 \\
t_3 & 0 & 0 & 0 & 2\mu_i-t_2 & 0 & 0 & 0 & 0 & t_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 2\mu_i-t_2 & 0 & 0 & -t_4 & 0 & 0 \\
0 & 0 & t_3 & 0 & 0 & 0 & t_2 & -t_4 & 0 & 0 & 0 \\
0 & 0 & t_3 & 0 & 0 & 0 & -t_4 & 2K_i+t_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -t_4 & 0 & 0 & 2\mu_i-t_2 & 0 & 0 \\
t_3 & 0 & 0 & 0 & t_4 & 0 & 0 & 0 & 0 & 2\mu_i-t_2 & 0 \\
0 & -t_3 & 0 & t_4 & 0 & 0 & 0 & 0 & 0 & 0 & t_2
\end{pmatrix} \geq 0,$$

$$i = 1,\ 2. \tag{7.5}$$

Let us find the critical values of the parameters $t_1, ...,\ t_4$. We mention first, that each of the matrices $[\ \mathbf{D}_i^{\sigma\zeta\zeta} - \mathbf{T}^{\sigma\zeta\zeta}],\quad i = 1, 2$ is divided into the direct sum of four blocks

$$\mathbf{D}_i^{\sigma\zeta\zeta} - \mathbf{T}^{\sigma\zeta\zeta} = A_i^{1,\ 5,\ 10} \oplus A_i^{2,\ 4,\ 11} \oplus A_i^{3,\ 7,\ 8} \oplus A_i^{6,\ 9}. \tag{7.6}$$

The matrix $A_i^{a,b,c}$ here denotes a diagonal minor of the matrix $[\ \mathbf{D}_i^{\sigma\zeta\zeta} - \mathbf{T}^{\sigma\zeta\zeta}\ ]$; it consists from the elements standing on the intersection of lines and colums with numbers $a,\ b,\ c$.

The matrices in the right hand side of (7.6) have the following form:

$$A_i^{1,\ 5,\ 10} = \begin{pmatrix} \frac{1}{2K_i}+t_1 & t_3 & t_3 \\ t_3 & 2\mu_i - t_2 & t_4 \\ t_3 & t_4 & 2\mu_i - t_2 \end{pmatrix}, \tag{7.7}$$

$$A_i^{2,\ 4,\ 11} = \begin{pmatrix} \frac{1}{2\mu_i}-t_1 & t_3 & -t_3 \\ t_3 & 2K_i + t_2 & t_4 \\ -t_3 & t_4 & t_2 \end{pmatrix}, \tag{7.8}$$

$$A_i^{3,\ 7,\ 8} = \begin{pmatrix} \frac{1}{2\mu_i} - t_1 & t_3 & t_3 \\ t_3 & t_2 & -t_4 \\ t_3 & -t_4 & 2K_i + t_2 \end{pmatrix}, \tag{7.9}$$

$$A_i^{6,\ 9} = \begin{pmatrix} 2\mu_i - t_2 & -t_4 \\ -t_4 & 2\mu_i - t_2 \end{pmatrix}. \tag{7.10}$$

The critical values t_i^* of the parameters t_i are determined from the following system

$$det A_1^{1,\ 5,\ 10} = 0, \tag{7.11}$$

$$det A_2^{1,\ 5,\ 10} = 0, \tag{7.12}$$

$$det A_2^{2,\ 4,\ 11} = 0, \tag{7.13}$$

$$det A_2^{6,\ 9} = 0. \tag{7.14}$$

They are equal

$$t_1^* = -\frac{1}{2K_1} + \frac{(K_1-K_2)\mu_2^2(K_1+\mu_1)(K_1+\mu_2)}{K_1\mu_1(K_1\mu_1+K_1\mu_2+2\mu_1\mu_2)(K_2+2\mu_2)(K_1+\mu_2)-K_1\mu_2(K_2\mu_1+K_2\mu_2+2\mu_1\mu_2)(K_1+2\mu_2)(K_1+\mu_1)}, \tag{7.15}$$

$$t_2^* = \mu_1 + \mu_2 - \frac{K_1(\mu_1-\mu_2)(K_1\mu_1+K_1\mu_2+2\mu_1\mu_2)(K_2\mu_1+K_2\mu_2+2\mu_1\mu_2)}{\mu_1(K_1\mu_1+K_1\mu_2+2\mu_1\mu_2)(K_2+2\mu_2)(K_1+\mu_2)-\mu_2(K_2\mu_1+K_2\mu_2+2\mu_1\mu_2)(K_1+2\mu_2)(K_1+\mu_1)}, \tag{7.16}$$

$$t_3^* = \sqrt{(\frac{1}{2\mu_1} - t_1^*)(t_2^* - \frac{\mu_2^2}{K_1 + 2\mu_2})}, \tag{7.17}$$

$$t_4^* = \mu_2 - t_2^*. \tag{7.18}$$

One can check also that such choice of the parameters $t_i, \ \ i = 1, ..., 4$ provides the minimal sum of the ranks of the matrices $[\mathbf{D}_1^{\sigma\zeta\zeta} - \mathbf{T}^{\sigma\zeta\zeta}(t_1^*,\ t_2^*,\ t_3^*,\ t_4^*)$ $]$ and $[\mathbf{D}_2^{\sigma\zeta\zeta} - \mathbf{T}^{\sigma\zeta\zeta}(t_1^*,\ t_2^*,\ t_3^*,\ t_4^*)]$ and the nonnegativity of these matrices.

The estimate (4.37) becomes the form

$$det[\mathbf{Y}(\mathbf{P}^{\sigma\zeta\zeta} \cdot \mathbf{D}_0^{\sigma\zeta\zeta} \cdot (\mathbf{P}^{\sigma\zeta\zeta})^T) + \mathbf{P}^{\sigma\zeta\zeta} \cdot \mathbf{T}^{\sigma\zeta\zeta} \cdot (\mathbf{P}^{\sigma\zeta\zeta})^T$$

$$-(\mathbf{P}^{\sigma\zeta\zeta} \cdot \mathbf{T}^{\sigma\zeta\zeta} \cdot (\mathbf{P}_\perp^{\sigma\zeta\zeta})^T) \cdot [\mathbf{P}_\perp^{\sigma\zeta\zeta} \cdot \mathbf{T}^{\sigma\zeta\zeta} \cdot (\mathbf{P}_\perp^{\sigma\zeta\zeta})^T]^{-1} \cdot (\mathbf{P}_\perp^{\sigma\zeta\zeta} \cdot \mathbf{T}^{\sigma\zeta\zeta} \cdot (\mathbf{P}^{\sigma\zeta\zeta})^T)] \geq 0, \tag{7.19}$$

where for the isotropic tensors D_0

$$\mathbf{Y}(\mathbf{P}^{\sigma\zeta\zeta}\cdot\mathbf{D}_0^{\sigma\zeta\zeta}\cdot(\mathbf{P}^{\sigma\zeta\zeta})^T)+\mathbf{P}^{\sigma\zeta\zeta}\cdot\mathbf{T}^{\sigma\zeta\zeta}\cdot(\mathbf{P}^{\sigma\zeta\zeta})^T$$

$$-(\mathbf{P}^{\sigma\zeta\zeta}\cdot\mathbf{T}^{\sigma\zeta\zeta}\cdot(\mathbf{P}_\perp^{\sigma\zeta\zeta})^T)\cdot[\mathbf{P}_\perp^{\sigma\zeta\zeta}\cdot\mathbf{T}^{\sigma\zeta\zeta}\cdot(\mathbf{P}_\perp^{\sigma\zeta\zeta})^T]^{-1}\cdot(\mathbf{P}_\perp^{\sigma\zeta\zeta}\cdot\mathbf{T}^{\sigma\zeta\zeta}\cdot(\mathbf{P}^{\sigma\zeta\zeta})^T)=$$

$$\begin{pmatrix}
\frac{1}{2K_0}-t_1^* & 0 & 0 & 0 & -t_3^* & 0 & 0 & 0 & -t_3^* \\
0 & \frac{1}{2\mu_0}+\frac{t_1^*t_2^*+t_3^{*2}}{t_2^*} & 0 & -\frac{t_3^*(t_2^*+t_4^*)}{t_2^*} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2\mu_0}+\frac{t_1^*t_2^*+t_3^{*2}}{t_2^*} & 0 & 0 & 0 & -\frac{t_3^*(t_2^*+t_4^*)}{t_2^*} & 0 & 0 \\
0 & -\frac{t_3^*(t_2^*+t_4^*)}{t_2^*} & 0 & 2K_0+\frac{t_4^{*2}-t_2^{*2}}{t_2^*} & 0 & 0 & 0 & 0 & 0 \\
-t_3^* & 0 & 0 & 0 & 2\mu_0+t_2^* & 0 & 0 & 0 & -t_4^* \\
0 & 0 & 0 & 0 & 0 & 2\mu_0+t_2^* & 0 & t_4^* & 0 \\
0 & 0 & -\frac{t_3^*(t_2^*+t_4^*)}{t_2^*} & 0 & 0 & 0 & 2K_0+\frac{t_4^{*2}-t_2^{*2}}{t_2^*} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & t_4^* & 0 & 2\mu_0+t_2^* & 0 \\
-t_3^* & 0 & 0 & 0 & -t_4^* & 0 & 0 & 0 & 2\mu_0+t_2^*
\end{pmatrix}\geq 0, \tag{7.20}$$

The matrix in the left hand side of (7.20) also can be presented as a direct sum of four blocks of the type (7.7)–(7.10)

$$\mathbf{Y}(\mathbf{P}^{\sigma\zeta\zeta}\cdot\mathbf{D}_0^{\sigma\zeta\zeta}\cdot(\mathbf{P}^{\sigma\zeta\zeta})^T)+\mathbf{P}^{\sigma\zeta\zeta}\cdot\mathbf{T}^{\sigma\zeta\zeta}\cdot(\mathbf{P}^{\sigma\zeta\zeta})^T$$

$$-(\mathbf{P}^{\sigma\zeta\zeta}\cdot\mathbf{T}^{\sigma\zeta\zeta}\cdot(\mathbf{P}_\perp^{\sigma\zeta\zeta})^T)\cdot[\mathbf{P}_\perp^{\sigma\zeta\zeta}\cdot\mathbf{T}^{\sigma\zeta\zeta}\cdot(\mathbf{P}_\perp^{\sigma\zeta\zeta})^T]^{-1}\cdot(\mathbf{P}_\perp^{\sigma\zeta\zeta}\cdot\mathbf{T}^{\sigma\zeta\zeta}\cdot(\mathbf{P}^{\sigma\zeta\zeta})^T)$$

$$=A_0^{1,\ 5,\ 9}\oplus A_0^{2,\ 4}\oplus A_0^{3,\ 7}\oplus A_0^{6,\ 8}, \tag{7.21}$$

where

$$A_0^{1,\ 5,\ 9}=\begin{pmatrix} y(\frac{1}{2K_0})-t_1^* & -t_3^* & -t_3^* \\ -t_3^* & y(2\mu_0)+t_2^* & -t_4^* \\ -t_3^* & -t_4^* & y(2\mu_0)+t_2^* \end{pmatrix}, \tag{7.22}$$

$$A_0^{2,\ 4}=A_0^{3,\ 7}=\begin{pmatrix} y(2\mu_0)+\frac{t_1^*t_2^*+t_3^{*2}}{t_2^*} & -\frac{t_3^*(t_2^*+t_4^*}{t_2^*} \\ -\frac{t_3^*(t_2^*+t_4^*}{t_2^*} & y(2K_0)+\frac{t_4^{*2}-t_2^{*2}}{t_2^*} \end{pmatrix}, \tag{7.23}$$

$$A_0^{6,\ 8}=\begin{pmatrix} y(2\mu_0)+t_2^* & t_4^* \\ t_4^* & y(2\mu_0)+t_2^* \end{pmatrix}. \tag{7.24}$$

One can check that the most restrictive estimates on the set of the pairs $(y(K_0),\ y(\mu_0))$ come from the inequality

$$det(A_0^{1,\ 5,\ 9}) \geq 0. \tag{7.25}$$

The last inequality gives us the lower estimate (3.2), (3.4) of the Theorem. The boundary of the set of possible pairs $(y(K_0), y(\mu_0))$ which satisfy this estimate is defined by the fraction linear curve $y(\mu_0) = F_1(y(K_0))$. The direct calculation shows that it passes through the points (3.20).

7.1.2 The upper estimate.

To study this case we deal with the functional $I^{\zeta\sigma\sigma}$. The proof of the estimate is the same as in previous case. Here the vector $\mathbf{e}$ is equal

$$\mathbf{e} = \{\zeta_1,\ \zeta_2,\ \zeta_3,\ \zeta_4, \sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_1',\ \sigma_2',\ \sigma_3'\}. \tag{7.26}$$

The matrices $\mathbf{D}_i,\quad i = 1,\ 2$ have the block diagonal form

$$\mathbf{D}_i = \mathbf{D}_i^{\zeta\sigma\sigma} = \begin{pmatrix} \mathbf{C}_i' & 0 & 0 \\ 0 & \mathbf{S}_i & 0 \\ 0 & 0 & \mathbf{S}_i \end{pmatrix}, \qquad i = 1,\ 2, \tag{7.27}$$

projectors P and $\mathbf{P}_\perp$ have the block forms

$$\mathbf{P} = \mathbf{P}^{\zeta\sigma\sigma} = \begin{pmatrix} \mathbf{P}^\zeta & 0 & 0 \\ 0 & \mathbf{E} & 0 \\ 0 & 0 & \mathbf{E} \end{pmatrix}, \quad \mathbf{P}_\perp = \mathbf{P}_\perp{}^{\zeta\sigma\sigma} = (\, \mathbf{P}_\perp{}^\zeta \quad 0 \quad 0 \,), \tag{7.28}$$

the matrix $\mathbf{T}^{\zeta\sigma\sigma}(t_1,\ t_2,\ t_3,\ t_4)$ is chosen in the following block form

$$\mathbf{T}^{\zeta\sigma\sigma}(t_1,\ t_2,\ t_3,\ t_4) =$$

$$\begin{pmatrix} \mathbf{\Phi}^{\zeta\zeta}(-t_1,\ 0,\ 0,\ 0) & \mathbf{\Phi}^{\zeta\sigma}(0,\ -t_3,\ 0,\ 0) & \mathbf{\Phi}^{\zeta\sigma}(0,\ 0,\ -t_3,\ 0) \\ \mathbf{\Phi}^{\sigma\zeta}(0,\ -t_3,\ 0,\ 0) & \mathbf{\Phi}^{\sigma\sigma}(-t_2,\ 0,\ 0,\ 0) & \mathbf{\Phi}^{\sigma\sigma}(0,\ 0,\ 0, t_4) \\ \mathbf{\Phi}^{\sigma\zeta}(0,\ 0, -t_3,\ 0) & \mathbf{\Phi}^{\sigma\sigma}(0,\ 0,\ 0,\ t_4) & \mathbf{\Phi}^{\sigma\sigma}(-t_2,\ 0,\ 0,\ 0) \end{pmatrix}. \tag{7.29}$$

As in the previous case each of the (10×10) matrices $\mathbf{D}_i^{\zeta\sigma\sigma} - \mathbf{T}^{\zeta\sigma\sigma}$

$$\mathbf{D}_i^{\zeta\sigma\sigma} - \mathbf{T}^{\zeta\sigma\sigma}(t_1,\ t_2,\ t_3,\ t_4) =$$

$$\begin{pmatrix}
2K_i + t_1 & 0 & 0 & 0 & 0 & t_3 & 0 & 0 & 0 & t_3 \\
0 & 2\mu_i - t_1 & 0 & 0 & t_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2\mu_i - t_1 & 0 & 0 & 0 & 0 & t_3 & 0 & 0 \\
0 & 0 & 0 & t_1 & 0 & 0 & t_3 & 0 & -t_3 & 0 \\
0 & t_3 & 0 & 0 & \frac{1}{2K_i} + t_2 & 0 & 0 & 0 & 0 & 0 \\
t_3 & 0 & 0 & 0 & 0 & \frac{1}{2\mu_i} - t_2 & 0 & 0 & 0 & t_4 \\
0 & 0 & 0 & t_3 & 0 & 0 & \frac{1}{2\mu_i} - t_2 & 0 & -t_4 & 0 \\
0 & 0 & t_3 & 0 & 0 & 0 & 0 & \frac{1}{2K_i} + t_2 & 0 & 0 \\
0 & 0 & 0 & -t_3 & 0 & 0 & -t_4 & 0 & \frac{1}{2\mu_i} - t_2 & 0 \\
t_3 & 0 & 0 & 0 & 0 & t_4 & 0 & 0 & 0 & \frac{1}{2\mu_i} - t_2
\end{pmatrix} \tag{7.30}$$

can be represented as a direct sum of four blocks

$$\mathbf{D}_i^{\sigma\zeta\zeta} - \mathbf{T}^{\sigma\zeta\zeta}(t_1,\ t_2,\ t_3,\ t_4) = A_i^{1,\ 6,\ 10} \oplus A_i^{2,\ 5} \oplus A_i^{3,\ 8} \oplus A_i^{4,\ 7,\ 9}, \tag{7.31}$$

where

$$A_i^{1,\ 6,\ 10} = \begin{pmatrix} 2K_i + t_1 & t_3 & t_3 \\ t_3 & \frac{1}{2\mu_i} - t_2 & t_4 \\ t_3 & t_4 & \frac{1}{2\mu_i} - t_2 \end{pmatrix}, \tag{7.32}$$

$$A_i^{2,\ 5} = A_i^{3,\ 8} = \begin{pmatrix} 2\mu_i - t_1 & t_3 \\ t_3 & \frac{1}{2K_i} + t_2 \end{pmatrix}, \tag{7.33}$$

$$A_i^{4,\ 7,\ 9} = \begin{pmatrix} t_1 & t_3 & -t_3 \\ t_3 & \frac{1}{2\mu_i} - t_2 & -t_4 \\ -t_3 & -t_4 & \frac{1}{2\mu_i} - t_2 \end{pmatrix}. \tag{7.34}$$

The critical values t_i^* of four parameters t_i of the translation matrix are determined from the system of two equations

$$det A_1^{2,\ 5} = 0, \tag{7.35}$$

$$det A_2^{2,\ 5} = 0, \tag{7.36}$$

and two scalar equations following from the equality

$$rank A_1^{4,\ 7,\ 9} = 1. \tag{7.37}$$

They are equal

$$t_1^* = \frac{2\mu_1^2\mu_2(K_1 - K_2)}{K_1\mu_1(K_2+\mu_1) - K_2\mu_2(K_1+\mu_1)}, \tag{7.38}$$

$$t_2^* = \frac{\mu_2(K_1+\mu_1) - \mu_1(K_2+\mu_1)}{2K_1\mu_1(K_2+\mu_1) - 2K_2\mu_2(K_1+\mu_1)}, \tag{7.39}$$

$$t_3^* = \sqrt{t_1^*(\frac{1}{2\mu_1} - t_2^*)}, \tag{7.40}$$

$$t_4^* = \frac{1}{2\mu_1} - t_2^*. \tag{7.41}$$

Such a choice of the parameters provides the nonnegativity of the matrices $\mathbf{D}_1^{\sigma\zeta\zeta} - \mathbf{T}^{\sigma\zeta\zeta}(t_1^*,\ t_2^*,\ t_3^*,\ t_4^*)$ and $\mathbf{D}_2^{\sigma\zeta\zeta} - \mathbf{T}^{\sigma\zeta\zeta}(t_1^*,\ t_2^*,\ t_3^*,\ t_4^*)$.

We obtain the upper bound from the inequality

$$det[\mathbf{Y}(\mathbf{P}^{\sigma\zeta\zeta}\cdot\mathbf{D}_0^{\sigma\zeta\zeta}\cdot(\mathbf{P}^{\sigma\zeta\zeta})^T) + \mathbf{P}^{\sigma\zeta\zeta}\cdot\mathbf{T}^{\sigma\zeta\zeta}\cdot(\mathbf{P}^{\sigma\zeta\zeta})^T$$

$$-(\mathbf{P}^{\sigma\zeta\zeta}\cdot\mathbf{T}^{\sigma\zeta\zeta}\cdot(\mathbf{P}_\perp^{\sigma\zeta\zeta})^T)\cdot[\mathbf{P}_\perp^{\sigma\zeta\zeta}\cdot\mathbf{T}^{\sigma\zeta\zeta}\cdot(\mathbf{P}_\perp^{\sigma\zeta\zeta})^T]^{-1}\cdot(\mathbf{P}_\perp^{\sigma\zeta\zeta}\cdot\mathbf{T}^{\sigma\zeta\zeta}\cdot(\mathbf{P}^{\sigma\zeta\zeta})^T)] \geq 0. \tag{7.42}$$

The matrix in the square brackets is divided into four uncoupled blocks for isotropic tensors $\mathbf{D}_0$; the most restrictive estimate comes from one of them

$$det A_0^{1,\ 6,\ 9} \geq 0. \tag{7.43}$$

This inequality can be rewritten in the form (3.2), (3.8); calculation shows that the boundary fraction linear curve passes through the points (3.21).

7.2 The badly ordered case.

The procedure doesn't change in this case, the only difference is in the expressions for parameters in the translation method.

7.2.1 The lower estimate.

We estimate the functional $I^{\zeta\zeta\zeta}$. The 12-dimensional vector $\mathbf{e}$ is equal

$$\mathbf{e} = \{\zeta_1,\ \zeta_2,\ \zeta_3,\ \zeta_4,\ \zeta_1',\ \zeta_2',\ \zeta_3',\ \zeta_4',\ \zeta_1'',\ \zeta_2'',\ \zeta_3'',\ \zeta_4''\}. \tag{7.44}$$

The matrices $\mathbf{D}_i,\quad i = 1,\ 2$ have the form

$$\mathbf{D}_i = \mathbf{D}_i^{\zeta\zeta\zeta} = \begin{pmatrix} \mathbf{C}_i' & 0 & 0 \\ 0 & \mathbf{C}_i' & 0 \\ 0 & 0 & \mathbf{C}_i' \end{pmatrix}, \qquad i = 1,\ 2. \tag{7.45}$$

The projectors P and $\mathbf{P}_\perp$ are the block matrices

$$\mathbf{P} = \mathbf{P}^{\zeta\zeta\zeta} = \begin{pmatrix} \mathbf{P}^\zeta & 0 & 0 \\ 0 & \mathbf{P}^\zeta & 0 \\ 0 & 0 & \mathbf{P}^\zeta \end{pmatrix}, \quad \mathbf{P}_\perp = \mathbf{P}_\perp{}^{\zeta\zeta\zeta} = \begin{pmatrix} \mathbf{P}_\perp{}^\zeta & 0 & 0 \\ 0 & \mathbf{P}_\perp{}^\zeta & 0 \\ 0 & 0 & \mathbf{P}_\perp{}^\zeta \end{pmatrix}, \tag{7.46}$$

and the matrix T has the block form

$$\mathbf{T}^{\zeta\zeta\zeta}(t_1,\ t_2,\ t_3,\ t_4) =$$

$$\begin{pmatrix} \mathbf{\Phi}^{\zeta\zeta}(-t_1,\ 0,\ 0,\ 0) & \mathbf{\Phi}^{\zeta\zeta}(0,\ -t_3,\ 0,\ 0) & \mathbf{\Phi}^{\zeta\zeta}(0,\ 0, -t_3,\ 0) \\ \mathbf{\Phi}^{\zeta\zeta}(0, -t_3\ 0,\ 0) & \mathbf{\Phi}^{\zeta\zeta}(-t_2,\ 0,\ 0,\ 0) & \mathbf{\Phi}^{\zeta\zeta}(0,\ 0,\ 0, -t_4) \\ \mathbf{\Phi}^{\zeta\zeta}(0,\ 0, -t_3,\ 0) & \mathbf{\Phi}^{\zeta\zeta}(0,\ 0,\ 0, -t_4) & \mathbf{\Phi}^{\zeta\zeta}(-t_2,\ 0,\ 0,\ 0) \end{pmatrix}. \tag{7.47}$$

Each of the (12×12) matrices $\mathbf{D}_i^{\zeta\zeta\zeta} - \mathbf{T}^{\zeta\zeta\zeta},\quad i = 1,\ 2$ is a direct sum of the diagonal minors

$$\mathbf{D}_i^{\zeta\zeta\zeta} - \mathbf{T}^{\zeta\zeta\zeta} = A_i^{1,\ 6,\ 11} \oplus A_i^{2,\ 5,\ 12} \oplus A_i^{3,\ 8,\ 9} \oplus A_i^{4,\ 7,\ 10}, \tag{7.48}$$

where

$$A_i^{1,\ 6,\ 11} = \begin{pmatrix} 2K_i + t_1 & t_3 & t_3 \\ t_3 & 2\mu_i - t_2 & t_4 \\ t_3 & t_4 & 2\mu_i - t_2 \end{pmatrix}, \tag{7.49}$$

$$A_i^{2,\ 5,\ 12} = \begin{pmatrix} 2\mu_i - t_1 & -t_3 & t_3 \\ -t_3 & 2K_i + t_2 & t_4 \\ t_3 & t_4 & t_2 \end{pmatrix}, \tag{7.50}$$

$$A_i^{3,\ 8,\ 9} = \begin{pmatrix} 2\mu_i - t_1 & -t_3 & -t_3 \\ -t_3 & t_2 & -t_4 \\ -t_3 & -t_4 & 2K_i + t_2 \end{pmatrix}, \tag{7.51}$$

$$A_i^{1,\ 6,\ 11} = \begin{pmatrix} t_1 & t_3 & -t_3 \\ t_3 & 2\mu_i - t_2 & -t_4 \\ -t_3 & -t_4 & 2\mu_i - t_2 \end{pmatrix}. \tag{7.52}$$

The critical values t_i^* of the parameters t_i of the translation matrix $\mathbf{T}^{\zeta\zeta\zeta}$ are determined by the conditions

$$detA_1^{2,\ 5,\ 12} = 0, \tag{7.53}$$

$$detA_2^{2,\ 5,\ 12} = 0, \tag{7.54}$$

$$rankA_2^{4,\ 7,\ 10} = 1, \tag{7.55}$$

and are equal

$$t_1^* = \frac{2\mu_1\mu_2^2(K_2 - K_1)}{\mu_1(K_2+\mu_2)(K_1+2\mu_2) - \mu_2(K_1+\mu_2)(K_2+2\mu_2)}, \tag{7.56}$$

$$t_2^* = \frac{2\mu_2^2(\mu_1(K_2+\mu_2) - \mu_2(K_1+\mu_2))}{\mu_1(K_2+\mu_2)(K_1+2\mu_2) - \mu_2(K_1+\mu_2)(K_2+2\mu_2)}, \tag{7.57}$$

$$t_3^* = \sqrt{t_1^*(2\mu_2 - t_2^*)}, \tag{7.58}$$

$$t_4^* = 2\mu_2 - t_2^*. \tag{7.59}$$

As in the previous case they provide the minimal sum of the ranks of the matrices $\mathbf{D}_1^{\zeta\zeta\zeta} - \mathbf{T}^{\zeta\zeta\zeta}(t_1^*,\ t_2^*,\ t_3^*,\ t_4^*)$ and $\mathbf{D}_2^{\zeta\zeta\zeta} - \mathbf{T}^{\zeta\zeta\zeta}(t_1^*,\ t_2^*,\ t_3^*,\ t_4^*)$ and non-negativity of these mastrices.

The matrix [$\mathbf{Y}(\mathbf{P}^{\zeta\zeta\zeta}\cdot\mathbf{D}_0^{\zeta\zeta\zeta}\cdot(\mathbf{P}^{\zeta\zeta\zeta})^T) + \mathbf{P}^{\zeta\zeta\zeta}\cdot\mathbf{T}^{\zeta\zeta\zeta}\cdot(\mathbf{P}^{\zeta\zeta\zeta})^T - (\mathbf{P}^{\zeta\zeta\zeta}\cdot\mathbf{T}^{\zeta\zeta\zeta}\cdot(\mathbf{P}_\perp^{\zeta\zeta\zeta})^T)\cdot[\mathbf{P}_\perp^{\zeta\zeta\zeta}\cdot\mathbf{T}^{\zeta\zeta\zeta}\cdot(\mathbf{P}_\perp^{\zeta\zeta\zeta})^T]^{-1}\cdot(\mathbf{P}_\perp^{\zeta\zeta\zeta}\cdot\mathbf{T}^{\zeta\zeta\zeta}\cdot(\mathbf{P}^{\zeta\zeta\zeta})^T)$] for the isotropic tensors D_0 also has a block diagonal structure. One could check that the most restrictive estimates correspond to vanishing of the determinant of the block $A_0^{1,\ 4,\ 9}$

$$detA_0^{1,\ 4,\ 9} \geq 0. \tag{7.60}$$

This condition leads to the inequality (3.3), (3.12) of the Theorem; the boundary curve passes through the points (3.22).

7.2.2 The upper estimate.

For this estimate we consider the functional $I^{\sigma\sigma\sigma}$, which is a quadratic form of the vector $\mathbf{e}$

$$e = \{\sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_1',\ \sigma_2',\ \sigma_3',\ \sigma_1'',\ \sigma_2'',\ \sigma_3''\}. \tag{7.61}$$

The matrices $\mathbf{D}_i,\quad i=1,\ 2$ are defined by the expression

$$\mathbf{D}_i = \mathbf{D}_i^{\sigma\sigma\sigma} = \begin{pmatrix} \mathbf{S}_i & 0 & 0 \\ 0 & \mathbf{S}_i & 0 \\ 0 & 0 & \mathbf{S}_i \end{pmatrix}, \qquad i=1,\ 2, \tag{7.62}$$

the matrix T is equal

$$\mathbf{T}^{\sigma\sigma\sigma}(t_1,\ t_2,\ t_3,\ t_4) =$$

$$\begin{pmatrix} \mathbf{\Phi}^{\sigma\sigma}(-t_1,\ 0,\ 0,\ 0) & \mathbf{\Phi}^{\sigma\sigma}(0,-t_3,\ 0,\ 0,\ 0) & \mathbf{\Phi}^{\sigma\sigma}(0,\ 0,\ 0,-t_3) \\ \mathbf{\Phi}^{\sigma\sigma}(0,-t_3,\ 0,\ 0) & \mathbf{\Phi}^{\sigma\sigma}(-t_2,\ 0,\ 0,\ 0) & \mathbf{\Phi}^{\sigma\sigma}(0,\ 0,-t_4,\ 0) \\ \mathbf{\Phi}^{\sigma\sigma}(0,\ 0,\ 0,-t_3) & \mathbf{\Phi}^{\sigma\sigma}(0,\ 0,-t_4,\ 0) & \mathbf{\Phi}^{\sigma\sigma}(-t_2,\ 0,\ 0,\ 0) \end{pmatrix}. \tag{7.63}$$

The differences $(\mathbf{D}_1 - \mathbf{D}_2)$ is nondegenerated, therefore we can use the estimates in the form (4.27). Each of the (9×9) matrices $[\mathbf{D}_i^{\sigma\sigma\sigma} - \mathbf{T}^{\sigma\sigma\sigma}(t_1,\ t_2,\ t_3,\ t_4)]$ i=1,2 is represented as a direct sum of four blocks

$$\mathbf{D}_i^{\zeta\zeta\zeta} - \mathbf{T}^{\zeta\zeta\zeta} = A_i^{1,\ 5,\ 9} \oplus A_i^{2,\ 4} \oplus A_i^{3,\ 7} \oplus A_i^{6,\ 8}, \tag{7.64}$$

where

$$A_i^{1,\ 5,\ 9} = \begin{pmatrix} \frac{1}{2K_i} + t_1 & t_3 & t_3 \\ t_3 & \frac{1}{2\mu_i} - t_2 & t_4 \\ t_3 & t_4 & \frac{1}{2\mu_i} - t_2 \end{pmatrix}, \tag{7.65}$$

$$A_i^{2,\ 4} = \begin{pmatrix} \frac{1}{2\mu_i} - t_1 & -t_3 \\ -t_3 & \frac{1}{2K_i} + t_2 \end{pmatrix}, \tag{7.66}$$

$$A_i^{3,\ 7} = \begin{pmatrix} \frac{1}{2\mu_i} - t_1 & -t_3 \\ -t_3 & \frac{1}{2K_i} + t_2 \end{pmatrix}, \tag{7.67}$$

$$A_i^{6,\ 8} = \begin{pmatrix} \frac{1}{2\mu_i} - t_2 & -t_4 \\ -t_4 & \frac{1}{2\mu_i} - t_2 \\ . & \end{pmatrix} \tag{7.68}$$

The critical values t_i^* of the parameters t_i of the translation matrix $\mathbf{T}^{\sigma\sigma\sigma}$ are obtained from the equations

$$det A_2^{1,\ 5,\ 9} = 0, \tag{7.69}$$

$$det A_1^{2,\ 4} = 0, \tag{7.70}$$

$$det A_2^{2,\ 4} = 0, \tag{7.71}$$

$$det A_1^{6,\ 8} = 0, \tag{7.72}$$

and are equal

$$t_1^* = -\frac{1}{2K_2} + \frac{\mu_1\mu_2(K_2-K_1)(K_2+\mu_2)(K_2+\mu_1)}{K_2^2\mu_1(K_1\mu_1+K_1\mu_2+2\mu_1\mu_2)(K_2+\mu_2)-K_1K_2\mu_2(K_2\mu_1+K_2\mu_2+2\mu_1\mu_2)(K_2+\mu_1)}, \tag{7.73}$$

$$t_2^* = \frac{1}{4\mu_1} + \frac{1}{4\mu_2} - \frac{K_2(\mu_1-\mu_2)(K_1\mu_1+K_1\mu_2+2\mu_1\mu_2)(K_2\mu_1+K_2\mu_2+2\mu_1\mu_2)}{K_2\mu_1^2\mu_2(K_1\mu_1+K_1\mu_2+2\mu_1\mu_2)(K_2+\mu_2)-K_1\mu_1\mu_2^2(K_2\mu_1+K_2\mu_2+2\mu_1\mu_2)(K_2+\mu_1)}, \tag{7.74}$$

$$t_3^* = \sqrt{(\frac{1}{2\mu_1} - t_1^*)(\frac{1}{2K_1} + t_2^*)}, \tag{7.75}$$

$$t_4^* = 1/(2\mu_1) - t_2^*. \tag{7.76}$$

The estimates come from the inequality

$$det[\mathbf{Y}(\mathbf{D}_0^{\sigma\sigma\sigma}) + \mathbf{T}^{\sigma\sigma\sigma}(t_1^*,\ t_2^*,\ t_3^*,\ t_4^*)] \geq 0. \tag{7.77}$$

the most restrictive ones correspond to vanishing of the determinant of the block $A_0^{1,\ 5,\ 9}$ of the matrix $[\mathbf{Y}(\mathbf{D}_0^{\sigma\sigma\sigma}) + \mathbf{T}^{\sigma\sigma\sigma}(t_1^*,\ t_2^*,\ t_3^*,\ t_4^*)]$

$$det A_0^{1,\ 5,\ 9} \geq 0. \tag{7.78}$$

This condition leads to the inequality (3.3), (3.16) of the Theorem. One can check that boundary curve passes through the points (3.23).

This statement concludes the prove of the Theorem .

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8 REFERENCES

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A APPENDIX

Here we show transformation of the inequality (4.23). Consider first the case of the nondegenerate matrix $(D_1 - D_2)$. We literally follow here the paper [37] by Milton and begin with the inequality (4.23). Observe that for any matrix $\mathbf{X}$ and a constant $m_1,\ m_2,\ m_1+m_2=1$ we have

$$m_1\mathbf{X}+m_2\mathbf{E}-m_1m_2(\mathbf{X}-\mathbf{E})\cdot(m_1\mathbf{E}+m_2\mathbf{X})^{-1}\cdot(\mathbf{X}-\mathbf{E})$$

$$=[(m_1\mathbf{X}+m_2\mathbf{E})\cdot(m_1\mathbf{E}+m_2\mathbf{X})-m_1m_2(\mathbf{X}-\mathbf{E})\cdot(\mathbf{X}-\mathbf{E})]\cdot(m_1\mathbf{E}+m_2\mathbf{X})^{-1}$$

$$=(m_1+m_2)^2\mathbf{X}\cdot(m_1\mathbf{E}+m_2\mathbf{X})^{-1}=(m_1\mathbf{X}^{-1}+m_2\mathbf{E})^{-1}. \tag{A.1}$$

Setting $\mathbf{X}=(\mathbf{D}_1-\mathbf{T})\cdot(\mathbf{D}_2-\mathbf{T})^{-1}$ and multiplying on the right by $(\mathbf{D}_2-\mathbf{T})$ produces

$$(m_1(\mathbf{D}_1-\mathbf{T})^{-1}+m_2(\mathbf{D}_2-\mathbf{T})^{-1})^{-1}$$

$$=m_1\mathbf{D}_1+m_2\mathbf{D}_2-\mathbf{T}-m_1m_2(\mathbf{D}_1-\mathbf{D}_2)\cdot(m_2\mathbf{D}_1+m_1\mathbf{D}_2-\mathbf{T})^{-1}\cdot(\mathbf{D}_1-\mathbf{D}_2). \tag{A.2}$$

Combining this with (4.23) we obtain

$$\mathbf{D}_0\geq m_1\mathbf{D}_1+m_2\mathbf{D}_2-m_1m_2(\mathbf{D}_1-\mathbf{D}_2)\cdot(m_2\mathbf{D}_1+m_1\mathbf{D}_2-\mathbf{T})^{-1}\cdot(\mathbf{D}_1-\mathbf{D}_2). \tag{A.3}$$

The tensor transformation Y (4.26) can be rewritten in the form

$$\mathbf{D}_0=m_1\mathbf{D}_1+m_2\mathbf{D}_2-m_1m_2(\mathbf{D}_1-\mathbf{D}_2)\cdot(m_2\mathbf{D}_1+m_1\mathbf{D}_2+\mathbf{Y})^{-1}\cdot(\mathbf{D}_1-\mathbf{D}_2). \tag{A.4}$$

Comparing (A.3) and (A.4) and assuming $(\mathbf{D}_1-\mathbf{D}_2)$ is nondegenerated gives the bound (4.27).

Let assume now that the matrices of the initial materials have equal eigendirections and eigenvalues. To get the estimate for this case let us project the ineguality (A.3) by means of $\mathbf{P}$. We have

$$\mathbf{P}\cdot\mathbf{D}_0\cdot\mathbf{P}^T\geq\mathbf{P}\cdot(m_1\mathbf{D}_1+m_2\mathbf{D}_2)\cdot\mathbf{P}^T$$
$$-m_1m_2\mathbf{P}\cdot(\mathbf{D}_1-\mathbf{D}_2)\cdot(m_2\mathbf{D}_1+m_1\mathbf{D}_2-\mathbf{T})^{-1}\cdot(\mathbf{D}_1-\mathbf{D}_2)\cdot\mathbf{P}^T. \tag{A.5}$$

Now we represent the matrix $(m_2\mathbf{D}_1+m_1\mathbf{D}_2-\mathbf{T})$ in the block form

$$(m_2\mathbf{D}_1+m_1\mathbf{D}_2-\mathbf{T})=\begin{pmatrix}\mathbf{K} & \mathbf{L}\\ \mathbf{L}^T & \mathbf{M}\end{pmatrix}, \tag{A.6}$$

where

$$\mathbf{K}=\mathbf{P}\cdot(m_2\mathbf{D}_1+m_1\mathbf{D}_2-\mathbf{T})\cdot\mathbf{P}^T, \tag{A.7}$$

$$\mathbf{L} = \mathbf{P} \cdot (m_2\mathbf{D}_1 + m_1\mathbf{D}_2 - \mathbf{T}) \cdot {\mathbf{P}_\perp}^T, \tag{A.8}$$

$$\mathbf{M} = \mathbf{P}_\perp \cdot (m_2\mathbf{D}_1 + m_1\mathbf{D}_2 - \mathbf{T}) \cdot {\mathbf{P}_\perp}^T, \tag{A.9}$$

and suppose that the matrix $\mathbf{M}$ is nondegenerated. One can check that the inverse matrix being written in the similar block form has the upper left block $\mathbf{K'}$

$$\mathbf{K'} = (\mathbf{K} - \mathbf{L} \cdot \mathbf{M^{-1}L^T})^{-1}. \tag{A.10}$$

Substituting this into the (A.5) and keeping in mind that $\mathbf{P} \cdot \mathbf{D}_1 \cdot {\mathbf{P}_\perp}^T = \mathbf{P} \cdot \mathbf{D}_2 \cdot {\mathbf{P}_\perp}^T = 0$ and $\mathbf{P}_\perp \cdot (\mathbf{D}_1 - \mathbf{D}_2) \cdot {\mathbf{P}_\perp}^T = 0$ (see (4.28)) we come to the inequality

$$\begin{gathered}
\mathbf{P} \cdot \mathbf{D}_0 \cdot \mathbf{P}^T \geq \mathbf{P} \cdot (m_1\mathbf{D}_1 + m_2\mathbf{D}_2) \cdot \mathbf{P}^T \\
-m_1 m_2 [\mathbf{P} \cdot (\mathbf{D}_1 - \mathbf{D}_2) \cdot \mathbf{P}^T] \cdot [\mathbf{P} \cdot (m_2\mathbf{D}_1 + m_1\mathbf{D}_2 - \mathbf{T}) \cdot \mathbf{P}^T \\
-(\mathbf{P} \cdot \mathbf{T} \cdot {\mathbf{P}_\perp}^T) \cdot (\mathbf{P}_\perp \cdot (m_2\mathbf{D}_1 + m_1\mathbf{D}_2 \\
-\mathbf{T}) \cdot {\mathbf{P}_\perp}^T) \cdot (\mathbf{P}_\perp \cdot \mathbf{T} \cdot {\mathbf{P}_\perp}^T)]^{-1} \cdot [\mathbf{P}_\perp \cdot (\mathbf{D}_1 - \mathbf{D}_2) \cdot \mathbf{P}^T].
\end{gathered} \tag{A.11}$$

Comparing (A.11) and the following form of the relation (4.35)

$$\begin{gathered}
\mathbf{P} \cdot \mathbf{D}_0 \cdot \mathbf{P}^T = \mathbf{P} \cdot (m_1\mathbf{D}_1 \cdot + m_2\mathbf{D}_2) \cdot \mathbf{P}^T \\
-m_1 m_2 [\mathbf{P} \cdot (\mathbf{D}_1 - \mathbf{D}_2) \cdot \mathbf{P}^T] \cdot [\mathbf{P} \cdot (m_2\mathbf{D}_1 + m_1\mathbf{D}_2) \cdot \mathbf{P}^T \\
+\mathbf{Y}(\mathbf{P} \cdot \mathbf{D_0} \cdot \mathbf{P^T})]^{-1} \cdot [\mathbf{P} \cdot (\mathbf{D}_1 - \mathbf{D}_2) \cdot \mathbf{P}^T]
\end{gathered} \tag{A.12}$$

and assuming the nondegenerasy of the difference $[\mathbf{P} \cdot (\mathbf{D}_1 - \mathbf{D}_2) \cdot \mathbf{P}^T]$ we end up with the expression (4.36).