2.2.1 Regular continued fractions

The regular continued fraction expansion of $x > 0$ is

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}} = [a_0; a_1, a_2, \dots],$$

where $a_i \in \mathbb{Z},\ a_0 \geq 0$ and $a_i > 0$ for $i \geq 1$. When $x < 0$, the continued fraction expansion is $x = -|x| = -[a_0; a_1, a_2, \dots]$.

The dynamical system associated to the regular continued fractions is $([0,1], \mathcal{B}_{[0,1]}, T, \mu)$ where $\mathcal{B}_{[0,1]}$ is the Borel σ-algebra on $[0,1]$, the *Gauss map* $T : [0,1] \to [0,1]$ (Figure 2.2.1) is defined by

$$T(x) = \begin{cases} \frac{1}{x} - k & \text{if } x \in (\frac{1}{k+1}, \frac{1}{k}], \\ 0 & \text{if } x = 0, \end{cases} \tag{2.2.1}$$

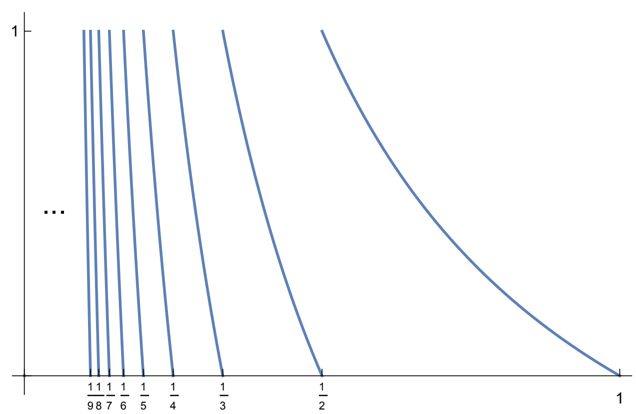

Figure 2.2.1: Gauss Map T

and μ is the Gauss probability measure $\mu(A) = \dfrac{1}{\log 2} \displaystyle\int_A \dfrac{dx}{1+x}$, $A \in \mathcal{B}_{[0,1]}$. The first digit of the continued fraction expansion of x is k. If we look at how T acts on the continued fraction expansion of $x = [0; a_1, a_2, \dots]$, we get

$$T([0; a_1, a_2, \dots]) = [0; a_2, a_3, \dots].$$

So, T deletes the first digit of the continued fraction expansion. In fact, we can write the regular continued fraction expansion of x by repeatedly applying T and recording which integer we subtract. The fact that μ is T-invariant was discovered by Gauss (see [21, Lemma 3.5]).

Since T is an infinite-to-one map, it is often easier to consider its invertible natural extension $\bar{T}$ that keeps track of all possible "futures" and "pasts" of x. We define the "futures" of x as $t_n = [0; a_{n+1}, a_{n+2}, \dots] = T^n(x)$ and the "past" of x as $v_n = [0; a_n, a_{n-1}, \dots, a_2, a_1]$. That is, $\bar{T}$ acts as

$$\left(\cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}, \cfrac{1}{a_0 + \cfrac{1}{a_{-1} + \dots}} \right) \mapsto \left(\cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}, \cfrac{1}{a_1 + \cfrac{1}{a_0 + \dots}} \right).$$

Formally, $\bar{T} : [0,1]^2 \to [0,1]^2$ is defined by

$$\bar{T}(x,y) = \begin{cases} (\frac{1}{x} - k, \frac{1}{k+y}) & \text{if } x \in (\frac{1}{k+1}, \frac{1}{k}], \\ (0, y) & \text{if } x = 0. \end{cases}$$

Thus, $\bar{T}$ acts as a skew-shift over the map T. The first coordinate of $\bar{T}$ acts as the Gauss map, and the second as the inverse branch of the Gauss map for a given k. In this way, $\bar{T}$ keeps track of both the future of T and the past. Then $\bar{T}^n(x,0) = (t_n, v_n)$. In the regular continued fraction case, we know that $\bar{T}$ is defined on $[0,1]^2$, but the natural extension domain can be more complicated for other types of continued fractions. Series [54] connected the action of $\bar{T}$ on the bi-infinite sequence of digits to a coding of geodesics on $\mathcal{M}$. The outlines of her argument are presented in Chapter 3.

2.2.2 Generalizing the natural extension construction and dual continued fraction expansions

There are many ways to define a continued fraction expansion. Throughout this thesis, I require that the numerators are ± 1 and impose various restrictions on the denominators. For the odd, grotesque, extended odd, and α-odd continued fractions, the denominators must be odd. For the even and extended even continued fractions, the denominators must be even. For the Lehner and Farey expansions discussed in Chapter 5, the denominator must be 1 or 2 based on whether the next (for Lehner) or preceding (for Farey) numerator is $+1$ or -1.

For the purposes of this subsection, I am going to start with a generic continued fraction