Caroline Series' The modular surface and continued fractions

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April 21, 2020

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Regular Continued Fractions

Way to represent $x > 0$ as

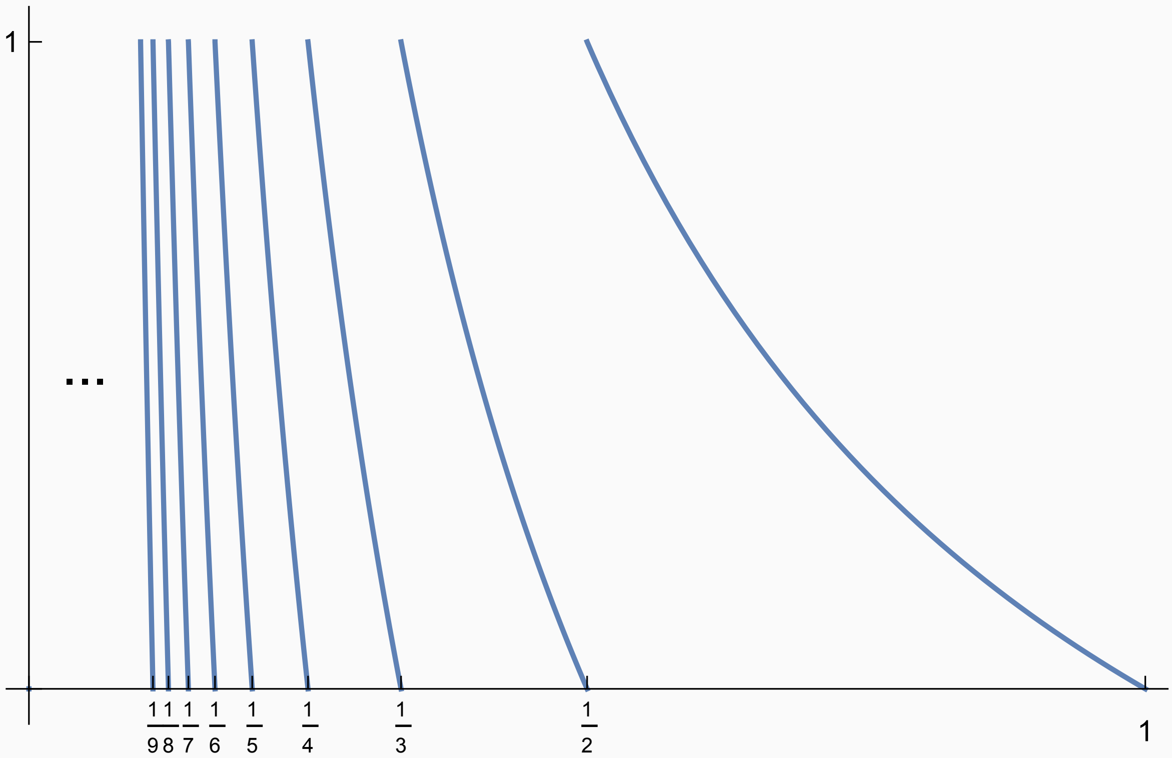
- $x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} = [a_0; a_1, a_2, \ldots].$
- Expansion terminates if and only if x is rational.
- Every irrational number has a unique continued fraction expansion.
- $\frac{1}{[a_0; a_1, a_2, \ldots]} = \dfrac{1}{a_0 + [0; a_1, a_2, \ldots]} = [0; a_0, a_1, a_2, \ldots].$
- $\frac{1}{[0; a_1, a_2, \ldots]} = a_1 + \dfrac{1}{a_2 + [0; a_3, \ldots]} = [a_1; a_2, a_3, \ldots].$

Gauss map

Define $T : [0, 1] \to [0, 1]$ by

$$T(x) = \begin{cases} \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} = \begin{cases} \frac{1}{x} - k & \text{for } x \in \left(\frac{1}{k+1}, \frac{1}{k} \right] \\ 0 & \text{if } x = 0 \end{cases}.$$

Gauss map



$$T([0; a_1, a_2, a_3, \ldots]) = [0; a_2, a_3, \ldots] = [a_2, a_3, \ldots]$$

Natural Extension

Define $\bar{T} : [0,1)^2 \to [0,1)^2$ by

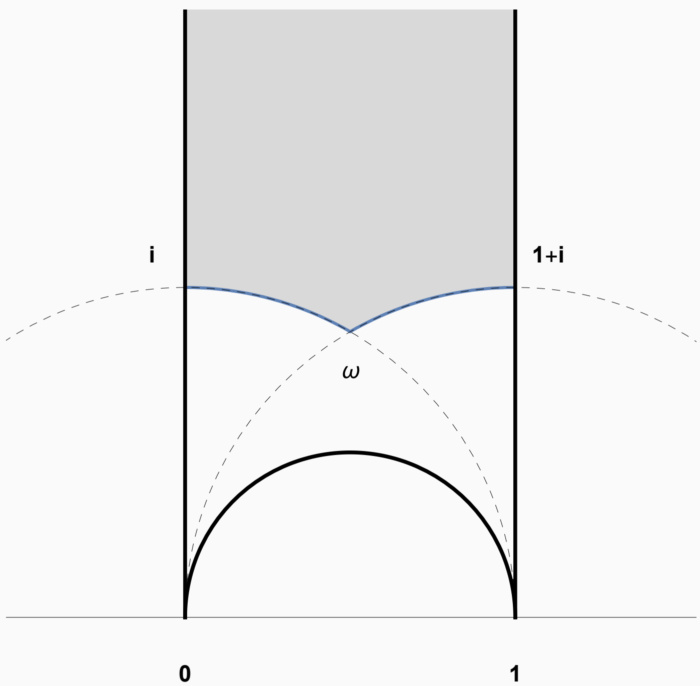
$$\bar{T}(x,y) = \begin{cases} \left(\frac{1}{x} - k, \frac{1}{y+k}\right) & \text{for } x \in \left(\frac{1}{k+1}, \frac{1}{k}\right] \\ (0,y) & \text{if } x = 0 \end{cases}.$$

$$\bar{T}(([a_1, a_2, \ldots], [a_0, a_{-1}, \ldots])) = ([a_2, a_3, \ldots], [a_1, a_0, a_{-1}, \ldots]).$$

Farey Tessellation

$\mathbb{H} := \{x + iy \mid y > 0\}$

Act by $PSL(2,\mathbb{Z})$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$



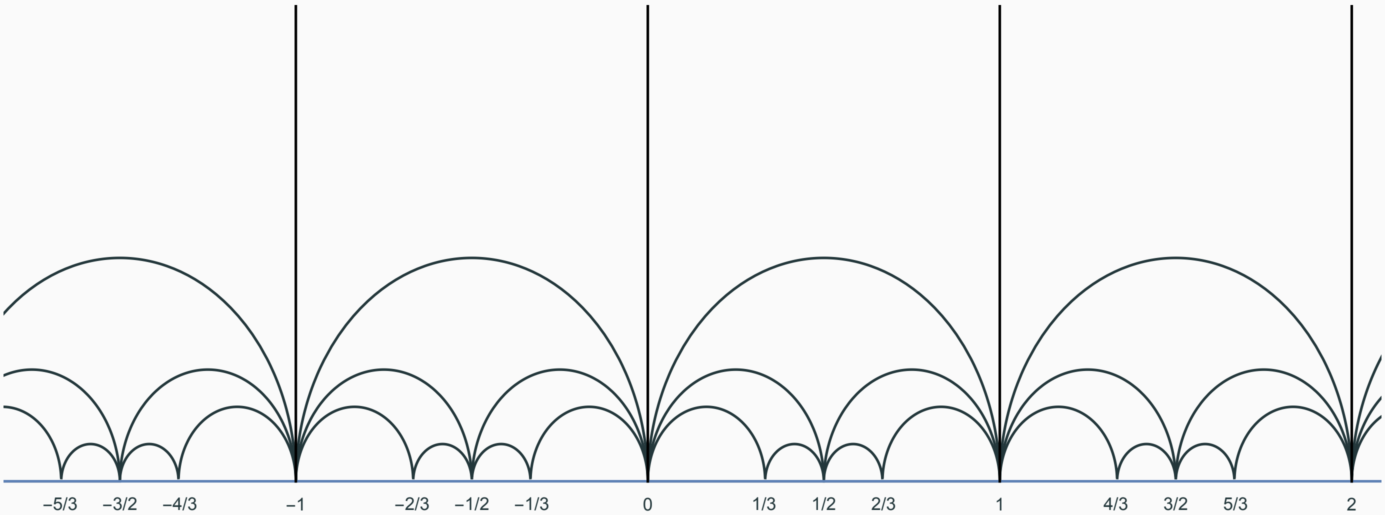
Farey Tessellation

Rest of this talk base on Caroline Series, *The modular surface and continued fractions*, J. London Math. Soc. 2 (1985).

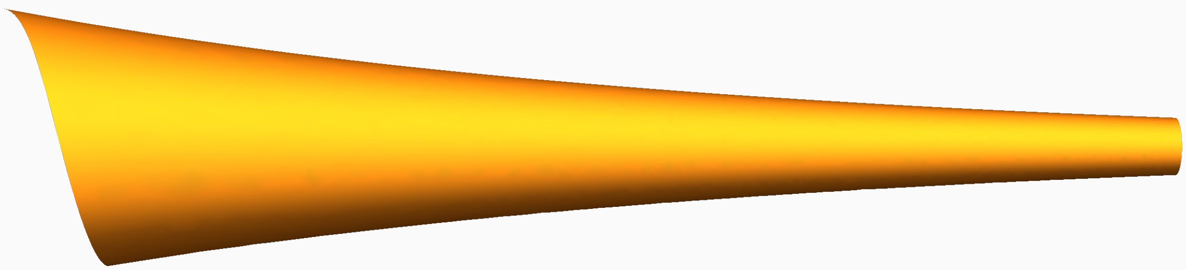
$\mathbb{H} := \{x + iy \mid y > 0\}$

Connect two rational numbers $\frac{p}{q}, \frac{p'}{q'}$ iff $pq' - p'q = \pm 1$.

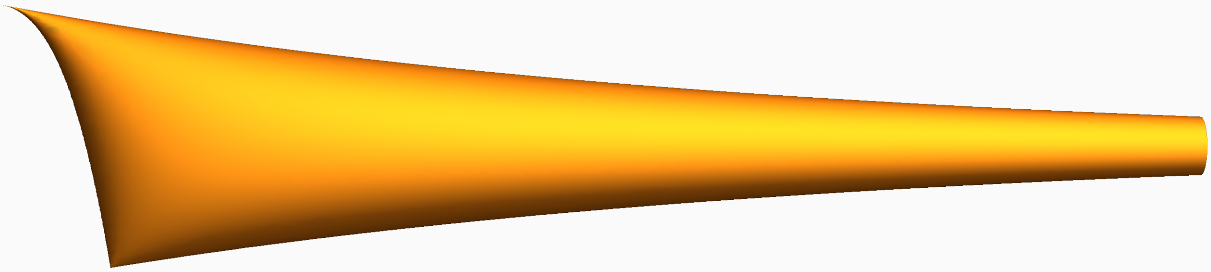
Images of the imaginary axis under $PSL(2, \mathbb{Z})$.



Modular Surface



Modular Surface



Translating back to the upper half plane

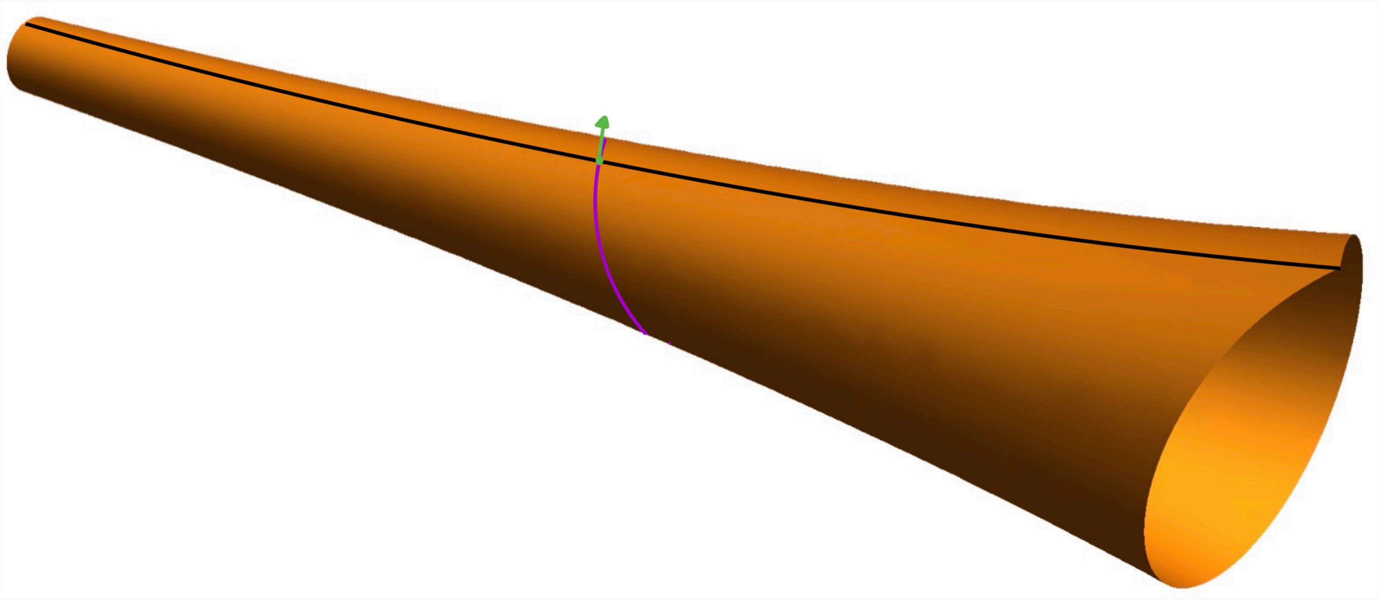
1. Hyperbolic geodesics are unique. Identify $(\gamma_\infty, \gamma_{-\infty}) \in \mathbb{R}^2$ with the geodesic γ from $\gamma_{-\infty}$ to γ_∞.

$$\mathcal{S} = \{(\gamma_\infty, \gamma_{-\infty}) \in \mathbb{R}^2 : 0 < |\gamma_{-\infty}| \leq 1 \leq |\gamma_\infty|, \\ -\operatorname{sign}(\gamma_{-\infty}) = -\operatorname{sign}(\gamma_\infty)\}$$

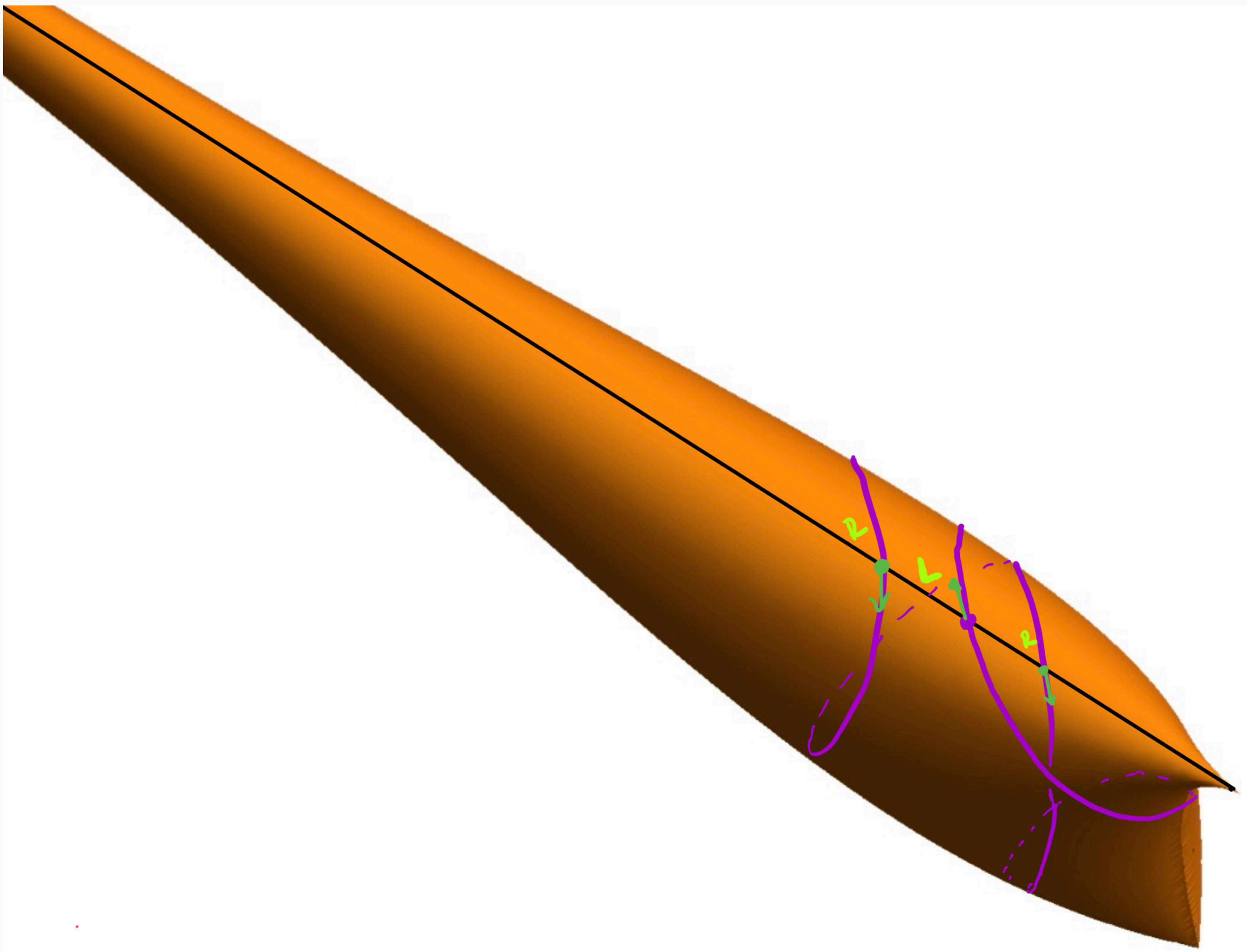
$$\mathcal{A} = \{\gamma \in \mathbb{H} : (\gamma_\infty, \gamma_{-\infty}) \in \mathcal{S}\}$$

2. Unit tangent vectors define geodesics. Let ξ_γ be where γ intersects $i\mathbb{R}$. Identify $(u_\gamma, \xi_\gamma) \in T^1\mathcal{M} = T^1(PSL(2,\mathbb{Z})\backslash\mathbb{H})$ with $(\gamma_\infty, \gamma_{-\infty})$
3. A map on $\mathcal{S}$ induces a map on $T^1\mathcal{M}$.

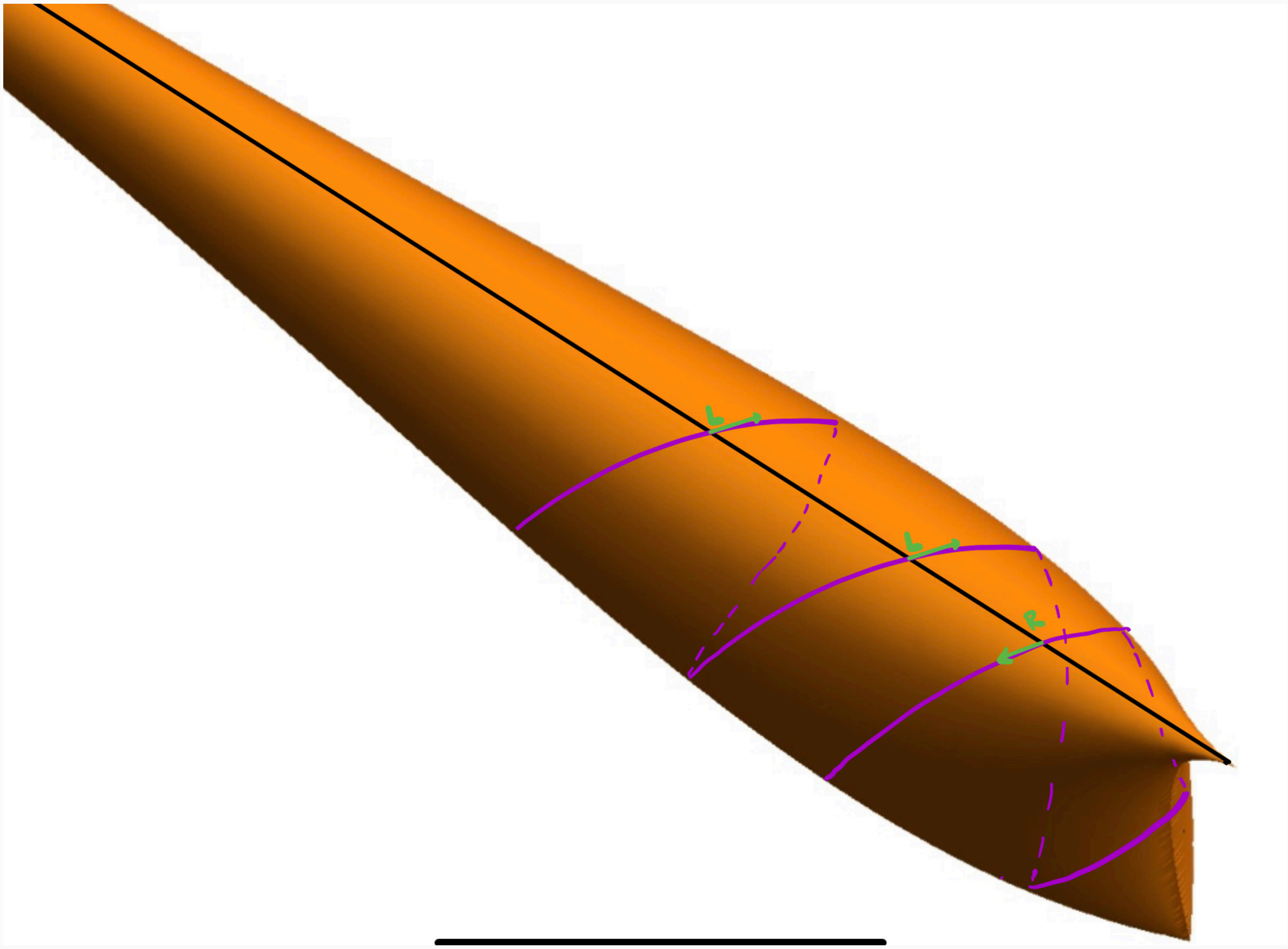
Modular Surface



Modular Surface



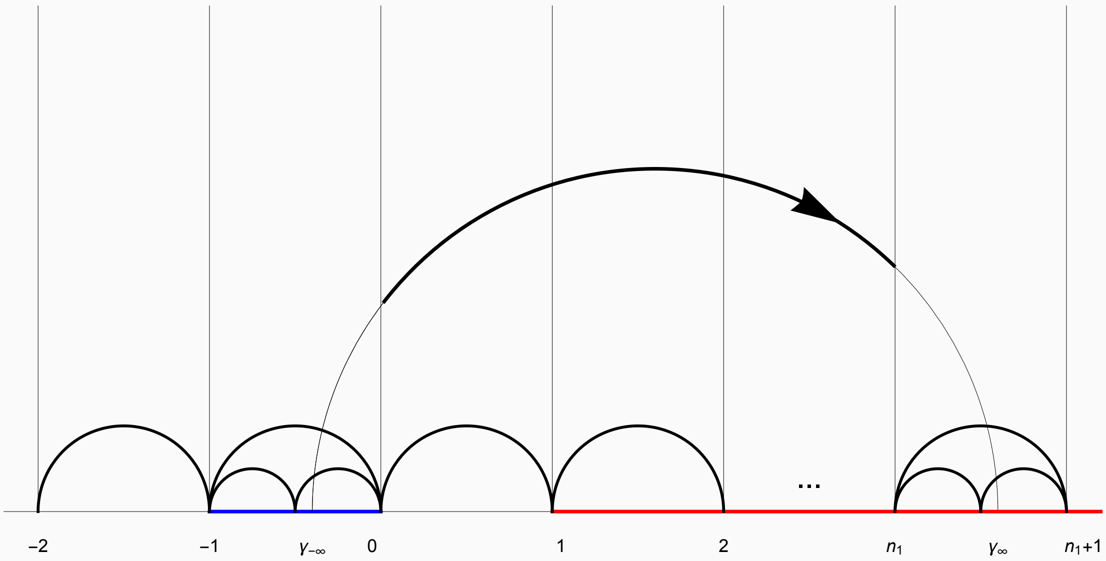
Modular Surface

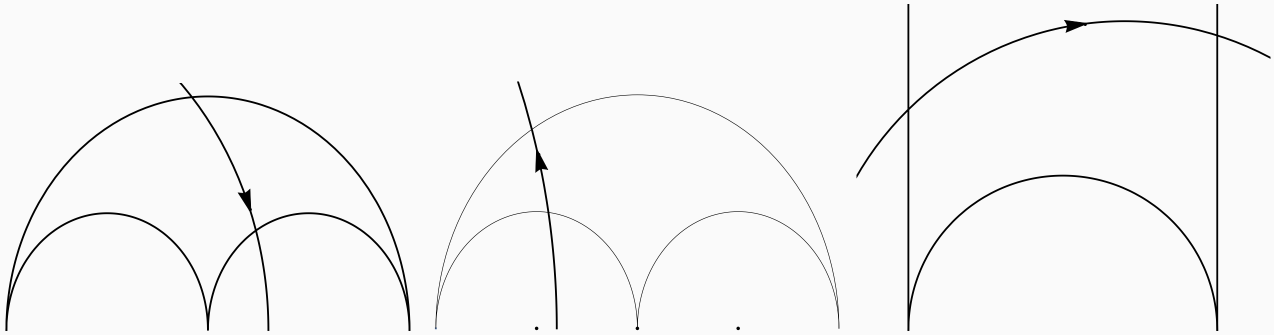


Geodesics

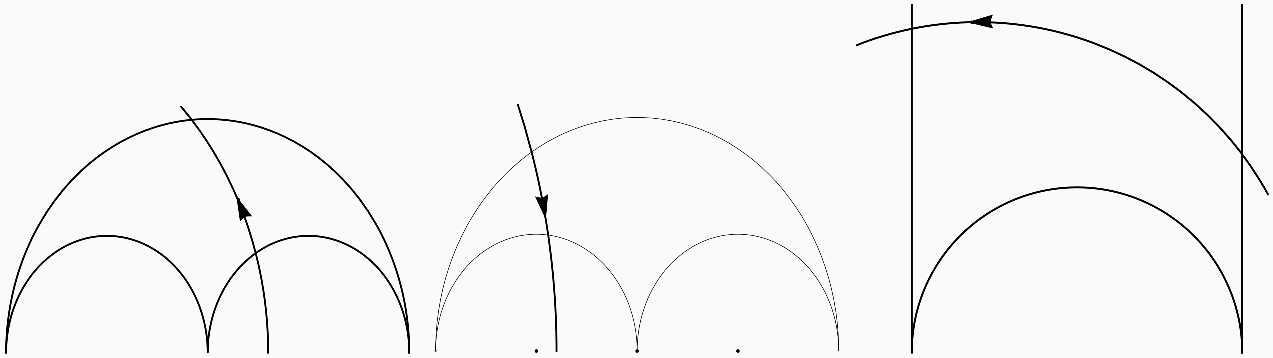
Let $\mathcal{S}$ be the set of geodesics γ with endpoints

- $\gamma_{-\infty} \in (-1, 0), \gamma_{\infty} \geq 1$
- $\gamma_{-\infty} \in (0, 1), \gamma_{\infty} \leq -1$



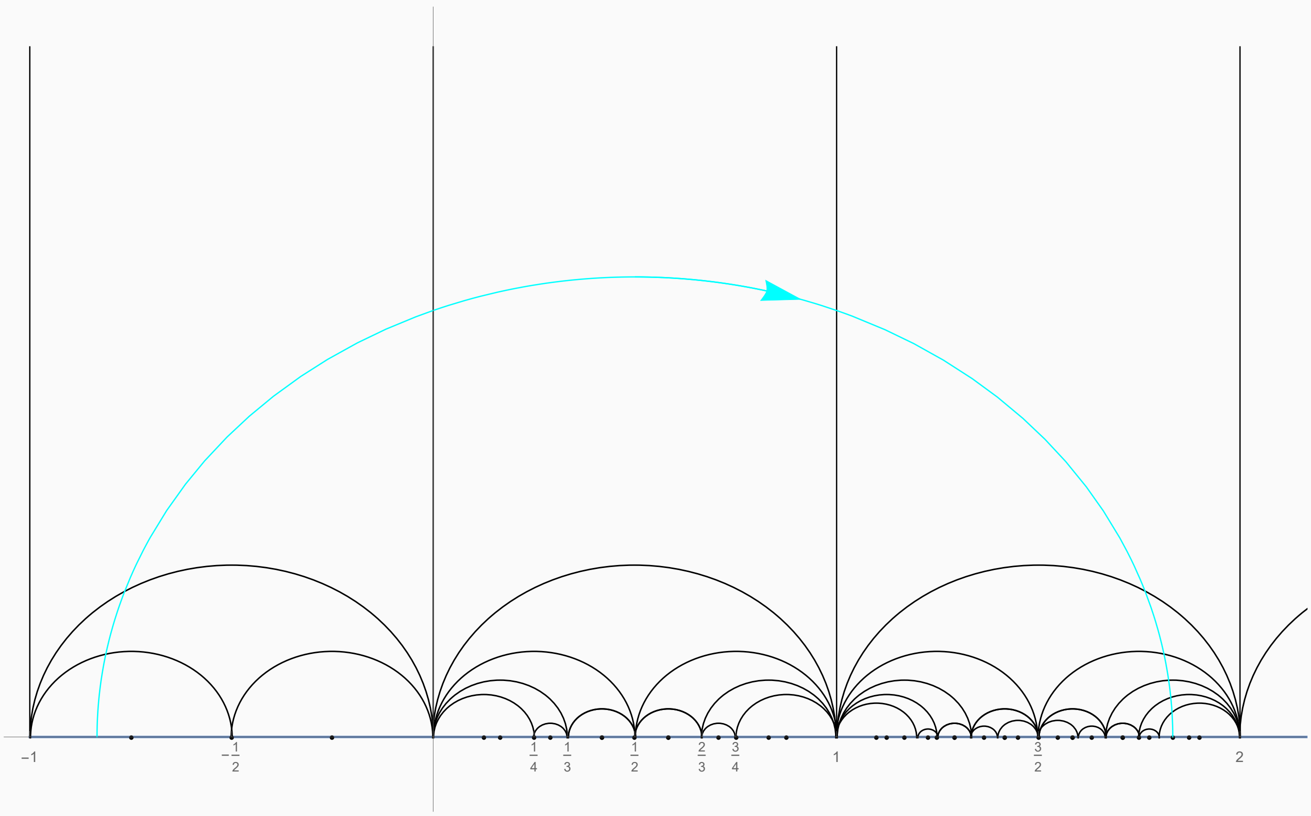


Some segments of type *L*

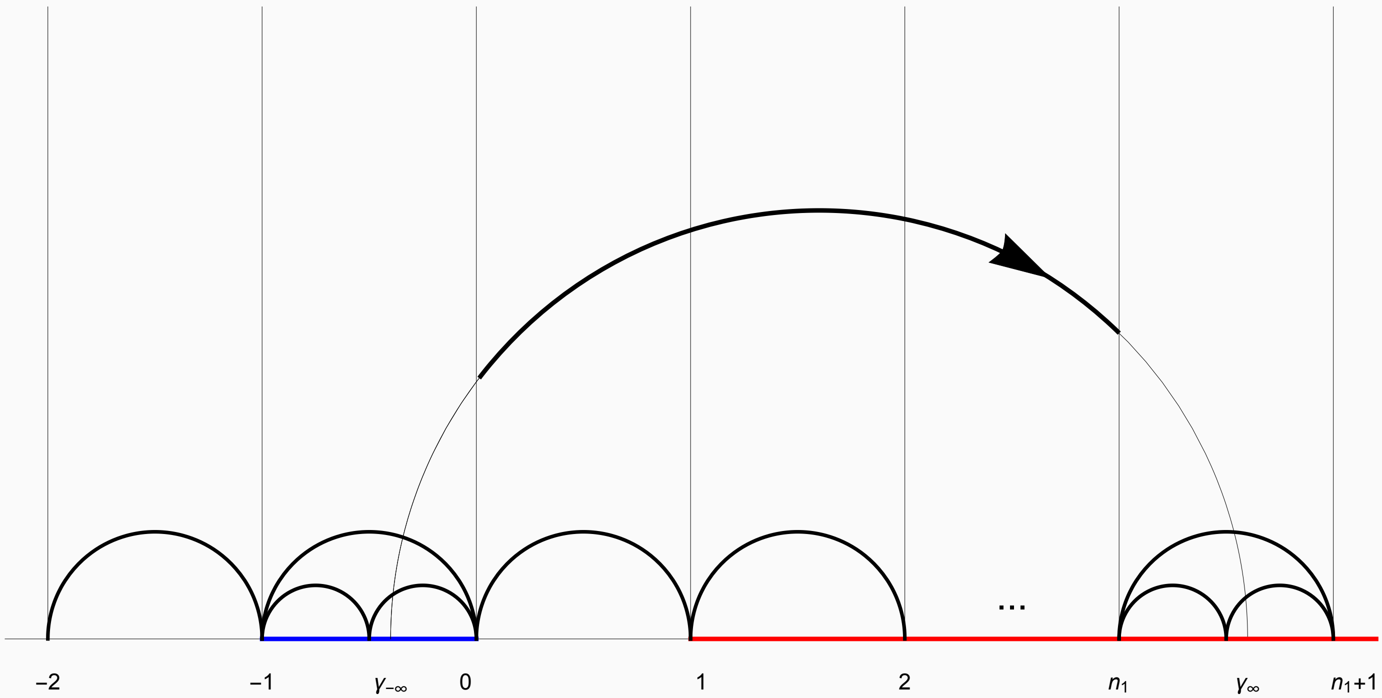


Some segments of type *R*

Example



Example



Cutting sequence $\ldots RR\xi_\gamma L^{n_1} R^1 L \ldots$

Let $X = \{(u_\gamma, \xi_\gamma) \in T^1\mathcal{M} :$ cutting sequence change type at $\xi_\gamma\}$.

Theorem (Series Theorem A, '84)

The map $i : \mathcal{A} \to X, i(\gamma) = \pi((u_\gamma, \xi_\gamma))$ is surjective, continuous, and open. It is injective except for the two oppositely oriented geodesics joining $+1$ to -1 have the same image.

A geodesic from $\gamma_{-\infty}$ to γ_∞ has two options:

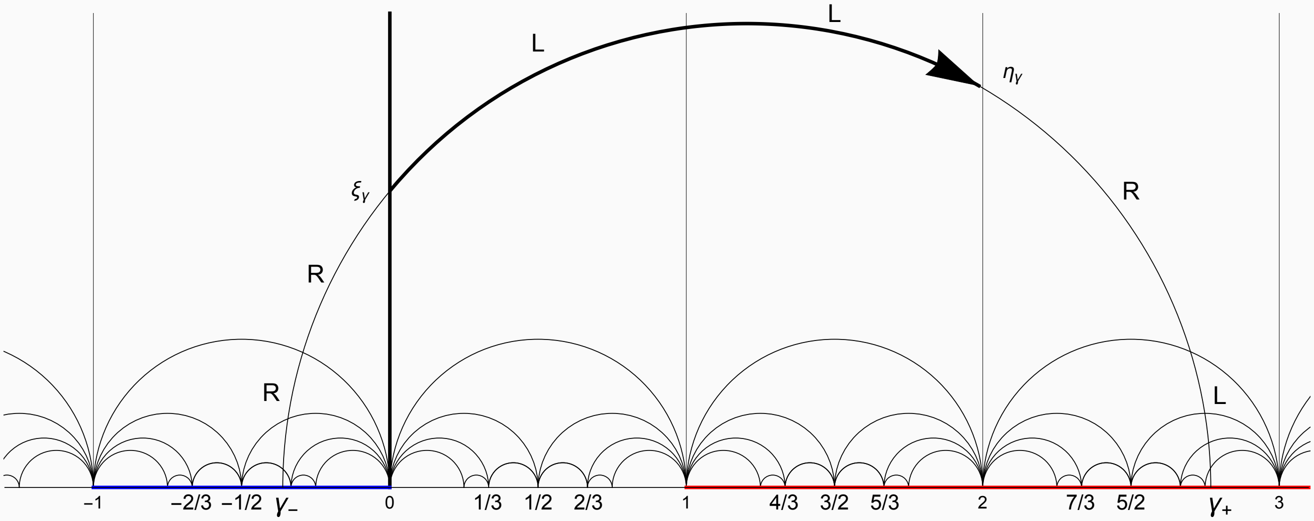
- $\gamma_{-\infty} \in (-1, 0),\ \gamma_\infty \in (1, \infty)$. *This geodesic has the coding* $\dots L^{n_{-2}} R^{n_{-1}} \xi_\gamma L^{n_0} R^{n_1} L^{n_2} \dots$

$$\gamma_{-\infty} = -[n_{-1}, n_{-2}, \dots] \text{ and } \gamma_\infty = n_0 + [n_1, n_2, \dots]$$

- $\gamma_{-\infty} \in (0, 1),\ \gamma_\infty \in (-\infty, -1)$.

$$\gamma_{-\infty} = [n_{-1}, n_{-2}, \dots] \text{ and } \gamma_\infty = -\left(n_0 + [n_1, n_2, \dots]\right).$$

Example

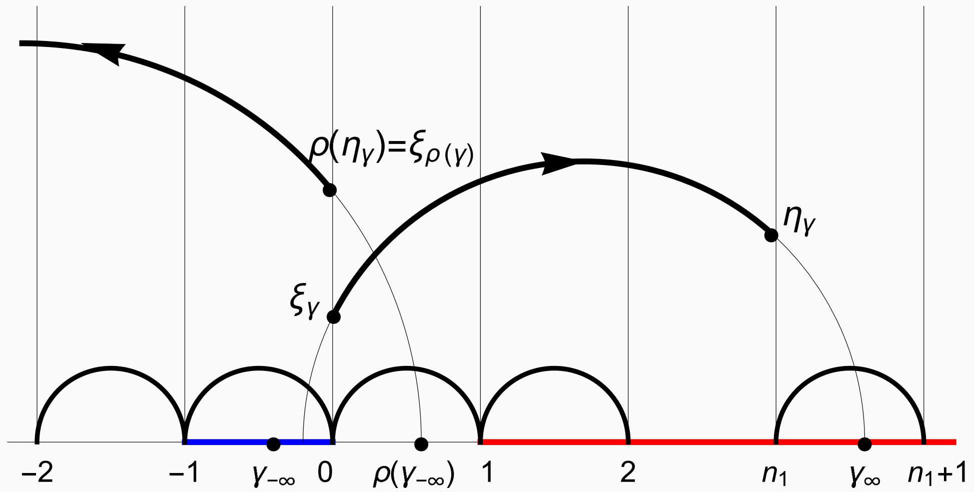


$\ldots LR^2\xi_\gamma L^2RL^3 \ldots$ corresponds to $-[0;3,1,\ldots]$ and $[2;1,3,\ldots]$

Action on Upper Half Plane

Case 1, $\gamma_\infty > 1$.

Define ρ on $\mathcal{S}$ by $(x, y) \mapsto (\frac{1}{a_1 - x}, \frac{1}{a_1 - y})$.



$$\ldots L^{n_{-1}} R^{n_0} \xi_\gamma L^{n_1} R^{n_2} \cdots \mapsto L^{n_{-1}} R^{n_0} L^{n_1} \xi_{\rho(\gamma)} R^{n_2} \ldots$$

Case 2, $\gamma_\infty < -1$, $(x, y) \mapsto (\frac{1}{-a_1 - x}, \frac{1}{-a_1 - y})$.

Section

Proposition (Corollary to Series' Theorems B & C)

Let X be the set of unit tangent vectors $u_\gamma \in T^1\mathcal{M}$ based at $\pi(\xi_\gamma)$ pointing along $\pi(\gamma)$, and $i(\gamma) = u_\gamma$.

The map $\bar{\rho} : X \to X$ given by $\bar{\rho}(u_\gamma) = i(\rho(\gamma))$ is invertible, and the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad\bar{\rho}\quad} & X \\ {\scriptstyle J\circ i^{-1}}\downarrow & & \downarrow{\scriptstyle J\circ i^{-1}} \\ (0,1]^2 & \xrightarrow{\quad\bar{T}\quad} & (0,1]^2. \end{array}$$

commutes, where $J : \mathcal{S} \to (0,1]^2$ is the invertible map defined by

$$J(x,y) := \mathrm{sign}(x)(1/x, -y)$$

Invariant Measure

The invariant measure for the geodesic flow on $T^1\mathbb{H}$ is

$$\frac{d\alpha d\beta d\theta}{(\alpha-\beta)^2}$$

Using the map J and projecting, we get

$$d\bar{\mu} = \frac{1}{\log 2}\frac{dxdy}{(xy+1)^2}$$

$$d\mu = \frac{1}{\log 2}\frac{dx}{x+1}$$

We find that T and $\bar{T}$ are ergodic:

T is *ergodic* if for every μ-measurable set A such that $T^{-1}A = A$, either $\mu(A) = 0$ or $\mu(X\backslash A) = 0$.

Applications

- $\alpha > 1$ has a purely periodic continued fraction expansion if and only if α is a quadratic irrational with $\alpha = \overline{[n_1; n_2, \ldots, n_{2r}]}, -\bar{\alpha} = \overline{[0; n_{2r}, n_{2r-1}, \ldots, n_1]}$
- The tail of the expansion of α is periodic if and only if α is a quadratic irrational.
- $d(\xi_\gamma, \eta_\gamma) = \frac{1}{2}\log(\gamma_\infty \gamma_{-\infty} \rho(\gamma_\infty)\rho(\gamma_{-\infty}))$.
- Length of closed geodesics on $\mathcal{M}$ is $\frac{1}{2}\log\frac{(\rho^{2r})'(\gamma_\infty)}{(\rho^{2r})'(\gamma_{-\infty})}$.