Arnoux's coding of the geodesic flow on the modular surface

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- GOALS of paper: In an elementary manner, give explicit coordinates for the unit tangent bundle of the modular surface and thereby derive explicit expressions for the geodesic flow. Relate this to continued fractions.

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- Earlier work: Ford, Artin, Adler-Flatto, Series. For more on coding, see various works of S. Katok.

Möbius action induced identifications

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- $T^1(\mathbb{H}) \longleftrightarrow PSL_2(\mathbb{R})$,
- and, $T^1(\mathcal{M}) \longleftrightarrow SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$.

Lattices of covolume one and $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$

We can also identify $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$ with 'area one' lattices:

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- A lattice $\Gamma \subset \mathbb{R}^2$ uniquely corresponds to the set of its positively oriented bases.
- A positively oriented basis of a lattice uniquely corresponds to the rows of a 2×2 real matrix of positive determinant.
- A lattice of covolume one has its positively oriented bases forming elements of $SL_2(\mathbb{R})$. Oriented changes of basis for such a Γ are effectuated exactly by left multiplication by elements of $SL_2(\mathbb{Z})$.

Theorem 1, Canonical bases

Theorem (Arnoux's coset representatives) The set $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$ is in 1-1 correspondence with the set of matrices $A = \begin{pmatrix} a & c \\ -b & d \end{pmatrix}$ such that $(a, b, c, d) \in \Omega_0 \cup \Omega_1 \cup \Omega_2$ where

$$\Omega_0 = \{(a, b, c, d) \in \mathbb{R}^4 \mid ad + bc = 1,\ 0 < b < 1 \leq a,\ 0 \leq c < d\},$$
$$\Omega_1 = \{(a, b, c, d) \in \mathbb{R}^4 \mid ad + bc = 1,\ 0 \leq a < 1 \leq b,\ 0 \leq d < c\},$$
$$\Omega_2 = \{(a, b, 0, d) \in \mathbb{R}^4 \mid ad = 1,\ 0 < b < a < 1\}.$$

Geodesic flow acts (locally) as $(a, b, c, d) \mapsto (ae^{t/2}, be^{t/2}, ce^{-t/2}, de^{-t/2})$.

Theorem 2: Zippered rectangles give (interiors of) Ω_0, Ω_1

Theorem (Arnoux's construction) To each lattice Γ, meeting the coordinate axes only at the origin, the *algorithm below* uniquely associates a positively oriented basis and a fundamental domain that is the union of two rectangles whose bases are aligned and such that one rectangle has width less than 1 and the other greater than one, with the wider taller than the narrower rectangle.

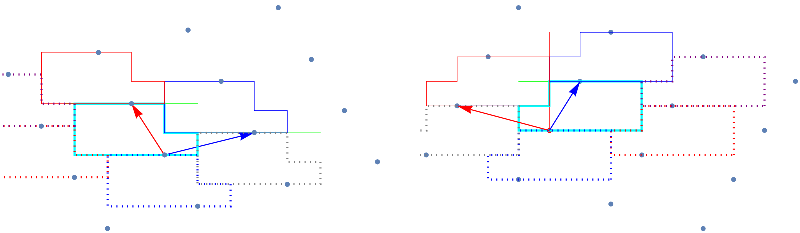

Figure: Cases: Ω_0 and Ω_1.

Theorem 3, Gauss map as factor

Theorem (Arnoux's extension) Let $\Sigma_0 \subset \Omega_0$ be the subset where $a = 1$, and $\Sigma_1 \subset \Omega_1$ be the subset where $b = 1$. Then
(1) the geodesic flow maps the 2D boundary face Σ_0 to a boundary face of Ω_0 that can naturally be identified with Σ_1;
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Theorem 4, verifying Lévy's constant

- **Theorem (Lévy 1936)** For almost every real $x \in [0,1]$ the series of its regular continued fraction denominators $(q_n)_{n\geq 0}$ satisfies
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- Let τ_n be the n-th return time of (thus, the flow value bringing) $(x,0) \in \Sigma_0$ to $\Sigma_0 \cup \Sigma_1$. He shows that q_n/n and $\tau_n/2n$ limit to the same value and argues by Birkhoff sums using the ergodicity of the geodesic flow to get the result.

Algorithm

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- The upwards vertical ray emanating from x meets a unique first H_y. (Red in figure.) **Assume this intersection is to the left of y.** This is the case of Ω_1.

Figure: Construction begins

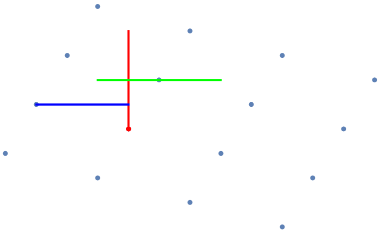

Figure: Beginning of construction, case of intersection left of y.

Algorithm, cont'd

- Let a be the (horizontal) distance from the intersection to y and c be the (vertical) distance from it to x. Thus, $y = x + a + ic$ using complex notation.

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- Let V_x be the open vertical segment from x up to H_y. Of those $z \in \Gamma$ such that H_z extends to intersect V_x, there is a unique z minimizing the distance from z to V_x. (See blue in figure.) Choose this z, label that (horizontal) distance b and label the (vertical) distance from x up to the point of intersection as d.

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- Note that $\boxed{0 < a < 1 \leq b; \quad 0 < d < c}$.

The reversed ‘L’

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- Above the right portion of the bottom, erect the rectangle of height c.
- Form the matrix $A = \begin{pmatrix} a & c \\ -b & d \end{pmatrix}$.

The fundamental domain $\mathcal{F}$; basis: rows of A

Let $\mathcal{F}$ be the reversed ‘L’ so formed. One shows that $\mathcal{F}$ is a fundamental domain for Γ and also the rows of A are a positively oriented basis of Γ. (See figure, next slide.) The theorem thus holds in this case.

Figure: tiling by images of $\mathcal{F}$; identifications give torus

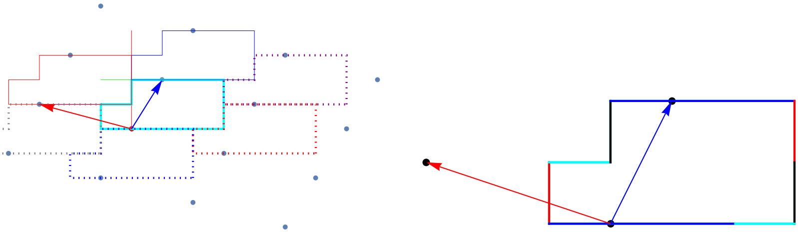
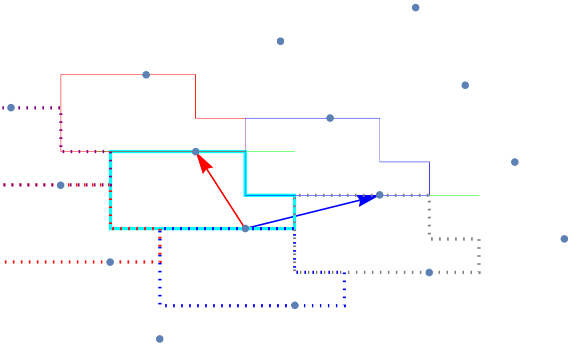


Figure: Case of intersection left of y. Left: $\mathcal{F}$ and its translates. Right: the fundamental domain identifies to a torus. Note that it is the suspension of an IET. Recall $A = \begin{pmatrix} a & c \\ -b & d \end{pmatrix}$.

Ω_0, case of 'L' shape

When the first intersection above is to the **right** of y, we label differently so as to still find an oriented basis corresponding to $A = \begin{pmatrix} a & c \\ -b & d \end{pmatrix}$. Now, $\boxed{0 < b < 1 \leq a; \quad 0 < c < d}$.



Accounting for *all* Γ, 1

- Try to apply the algorithm to a general Γ. It could be Γ includes a 'short' vertical vector and we find that y lies directly above x. If the algorithm succeeds other than this, we choose to include this in the case of Ω_1. That is, we include the possibility of $a = 0$ in the definition of Ω_1.

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- This was an innocuous choice, as the $(b = 0)$-boundary of Ω_0 as a subset of $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$ is identified with the $(a = 0)$-boundary of Ω_1:

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 0 & a^{-1} \\ -a & a^{-1} - c \end{pmatrix}.$$

Accounting for *all* Γ, 2

- The algorithm in the restricted setting remains unchanged if we include x in V_x. Now a general Γ could have a 'short' horizontal vector so x and z are horizontally aligned. We hence include the possibility of $c = 0$ in the Ω_0 and $d = 0$ in the Ω_1.

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- Similarly, if Γ has a 'very short' horizontal vector, it could be that there are two choices for y. Choose the option whose intersection point is to the right; thus, $0 < b < 1$. Let the horizontal distance between those two choices be a. Then we find a basis given by $(a, 0), (-b, d)$. The case of $a \geq 1$ is already in Ω_0. Otherwise, we have $0 < b < a < 1$, $c = 0$ and define Ω_2 to address exactly this case.

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- With this, all covolume one Γ are accounted for with the union

$$\Omega_0 = \{(a, b, c, d) \in \mathbb{R}^4 \mid ad + bc = 1,\ 0 < b < 1 \leq a,\ 0 \leq c < d\},$$

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Geodesic flow

- Under the identification of $T^1\mathcal{M}$ with $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$, the geodesic flow is effectuated by right multiplication by diagonal matrices of the form $g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$.

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- This flow acts on the various $\mathcal{F}$, stretching horizontally while contracting vertically. Similarly for the image of our positively oriented bases.
- In particular, $(a, b, c, d) \mapsto (ae^{t/2}, be^{t/2}, ce^{-t/2}, de^{-t/2})$.

Entrance and exit; 3-D

- With mild abuse,
  $\Omega_0 = \{(a, b, c) \in \mathbb{R}^3 \mid 0 < b < 1 \leq a,\ 0 \leq c < 1/(a+b)\}$.
  The flow enters on the $a = 1$-face of boundary and exits on $b = 1$-face of boundary. (See next figure, then come back.)

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- Cut-and-stack of our rectangles (see figure) shows that exiting face is identified with the entering face, also given by $b = 1$, of $\Omega_1 = \{(a, b, d) \in \mathbb{R}^4 \mid 0 \leq a < 1 \leq b,\ 0 \leq d < 1/(a+b)\}$, as

$$\begin{pmatrix} 1 & \lfloor a \rfloor \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & c \\ -1 & d \end{pmatrix} = \begin{pmatrix} \{a\} & c + \lfloor a \rfloor d \\ -1 & d \end{pmatrix},$$

  since $0 \leq \{a\} = a - \lfloor a \rfloor < 1$ and
  $ad = (1 - c) \leq 1 < a/(1 + \{a\})$ (since the 'exit' has $a > 1$).

Flow figures

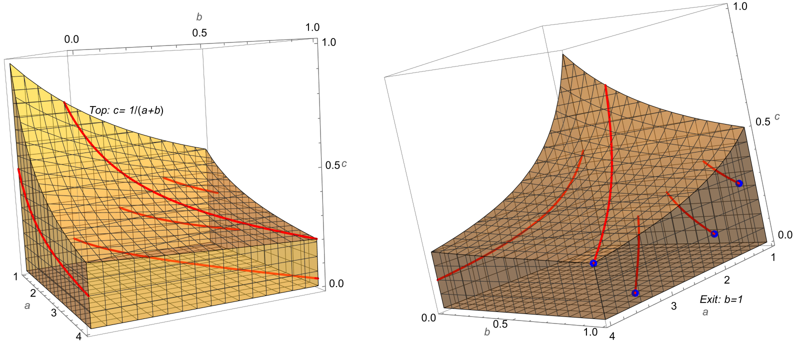

Figure: Two views of flow through Ω_0 from $a = 1$ to $b = 1$ faces.

Cut-and-stack figure

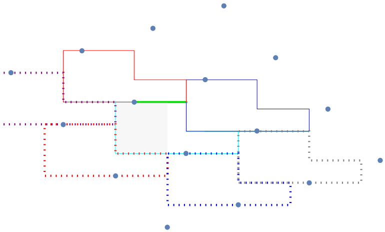

Figure: Here $b = 1$, and (for my ease) $\lfloor a \rfloor = 1$, thus one grey box is cut and then stacked above on the right. In particular, $(-b, d)$ is unchanged.

One 2D map

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- The $b = 1$-face of Ω_1 can be parametrized by (a, d) with $0 \leq a < 1$, $0 \leq d < 1/(a + 1)$. The map is then

$$(x, y)_0 \mapsto (1/x - \lfloor 1/x \rfloor, x - x^2 y)_1.$$

Second 2D map, Theorem 2 is proven.

- The $b = 1$ face of Ω_1 flows to its $a = 1$ boundary face. (Except those points with $a = 0$.) This gives $(x, y)_1 = (a, d) - - > (a', b', c', d') = (\,1, a^{-1}, a(1 - ad), ad\,)$.

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- Identify to $a = 1$-face of Ω_0

$$\begin{pmatrix} 1 & 0 \\ \lfloor b' \rfloor & 1 \end{pmatrix} \begin{pmatrix} 1 & c' \\ -b' & d' \end{pmatrix} = \begin{pmatrix} 1 & c' \\ -\{b'\} & d' + \lfloor b' \rfloor c' \end{pmatrix},$$

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- Using the (new) (b, c), the map is then

$$(x, y)_1 \mapsto (\,1/x - \lfloor 1/x \rfloor, x - x^2 y\,)_0.$$

Theorem 3, Gauss map as factor. Repeated!

Theorem (Arnoux's extension) Let $\Sigma_0 \subset \Omega_0$ be the subset where $a = 1$, and $\Sigma_1 \subset \Omega_1$ be the subset where $b = 1$. Then
(1) the geodesic flow maps the 2D boundary face Σ_0 to a boundary face of Ω_0 that can naturally be identified with Σ_1;
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The natural extension.

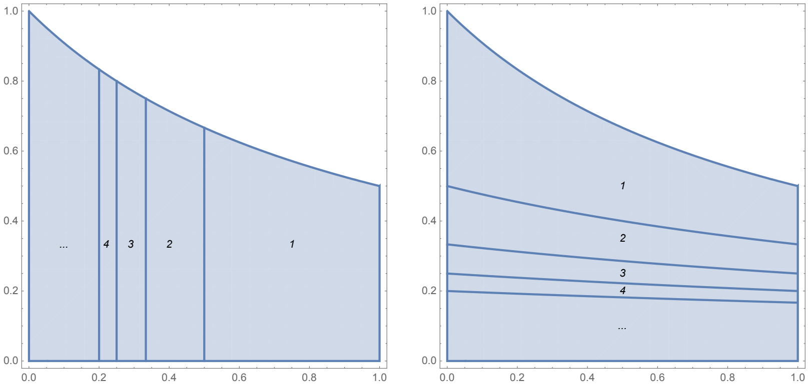

Figure: The natural extension map.

Thanks!