Factors

Let $X = (X, \mathcal{M}, \mu, T)$, $Y = (Y, \mathcal{A}, \nu, S)$ be two ergodic probability measure preserving systems.

We say $Y$ is a factor of $X$ if there exists $\pi : X \to Y$ so that $\mu(\pi^{-1}(A)) = \nu(A)$ for all $A \in \mathcal{A}$ and $\pi \circ T = S \circ \pi$. 

\[
\begin{array}{ccc}
X & \xrightarrow{T} & X \\
\downarrow \pi & & \downarrow \pi \\
Y & \xrightarrow{S} & Y
\end{array}
\]
Examples

- The one point system is a factor of every system.
- \((X_1 \times X_2, T_1 \times T_2, \mu_1 \otimes \mu_2)\) has \((X_1, T_1, \mu_1)\) as a factor, with factor map \(\pi_1\), projection onto the first coordinate.
  - \(T_1 \circ \pi_1\) is also a factor map.
- Every system is a factor of itself.
  - \(\text{Id}\) is a factor map
  - \(T^k\) is a factor map.
- \(R_\alpha : [0,1) \rightarrow [0,1)\) by \(R(x) = x + \alpha\) mod 1 has \(R_{k\alpha}\) as a factor for all \(k \in \mathbb{N}\).
  - The factor map is \(x \rightarrow kx\) mod 1
  - Because \(k(x + \alpha) = kx + k\alpha\)
- In fact if \(X\) is not weakly mixing than any eigenfunction is a factor map.
$X$ is *prime* if the only factor maps of $X$ are projection to the one point system or isomorphisms.

**Example:** Let $X = \mathbb{Z}/p\mathbb{Z}$, $\mu$ be counting measure and $T = +1 \mod p$. 
Objects on $X$ related to factors of $X$

What about $X$ can tell us about its factors?

Well $\bigcup_{x \in X} \left( \pi^{-1}(\pi x) \times \pi^{-1}(\pi x) \right) \subset X \times X$ is one object.

Moreover there is a version of this for measures.
Disintegration of measures: If \( Y \) is a factor of \( X \) with factor map \( \pi \) as above then:
For \( \nu \) a.e. \( y \in Y \) there exists a probability measure \( \mu_y \) (on \( X \)) so that \( \mu_y(\pi^{-1}(y)) = 1 \) and

\[
\mu = \int_Y \mu_y d\nu(y).
\]

That is \( \mu(A) = \int_Y \mu_y(A) d\nu(y) \).

**Example 1:** \( X = (X_1 \times X_2, \mu_1 \otimes \mu_2, T_1 \times T_2), \pi((x_1, x_2)) = x_1. \)

\((\mu_1 \otimes \mu_2)_{x_1} \) “is the copy of \( \mu_2 \) supported on \( \{x_1\} \times X_2.\)”

That is, \( (\mu_1 \otimes \mu_2)_{x_1}(A) = \mu_2(\{z \in X_2 : (x_1, z) \in A\}). \)
Example 2:

\[ \mathbf{X} = ([0, 1), \text{Leb}, R_\alpha), \quad \pi(x) = kx \mod 1. \]

\[ \mu_x = \frac{1}{k} \left( \sum_{i=0}^{k-1} \delta_{x + \frac{i}{k}} \right) \] where \( x + \frac{i}{k} \) is taken mod 1.
Relatively independent joining

Let $Y$ be a factor of $X$ with factor map $\pi$ and $\mu_y$ be the measures coming from the disintegration of measures as above.

Want an analogue of: $\bigcup_{x \in X} \left( \pi^{-1}(\pi x) \times \pi^{-1}(\pi x) \right) \subset X \times X$.

The relatively independent joining over $Y$ is

$$\sigma_\pi = \int_Y \mu_y \otimes \mu_y \, d\nu(y),$$

a measure on $X \times X$.

**Example 1:** $X = (X_1 \times X_2, \mu_1 \otimes \mu_2, T_1 \times T_2), \pi((x_1, x_2)) = x_1$.

$$\sigma_\pi(A) = \int_{X_1} \mu_2 \otimes \mu_2(\{(a, b) : (x_1, a, x_1, b) \in A\}) \, d\mu_1(x_1).$$

**Example 2:** $X = ([0, 1), \text{Leb}, R_\alpha), \pi(x) = 6x \mod 1$.

$$\sigma_\pi(A) = \int_{[0,1]} \frac{1}{6} |\{i \in \{0, \ldots, 5\} : (x, x + \frac{i}{6}) \in A\}| \, d\text{Leb}(x).$$
Properties of $\sigma_{\pi}$

$\sigma_{\pi}$ is a measure on $X \times X$ that is

- $T \times T$ invariant
- and projects to $\mu$ in both coordinates.

A map with these two properties is called a self-joining of $\mu$.

Other examples of self joinings:
- $\mu \times \mu$ (which is also the relatively independent joining over the map to the 1 point system).
- $\sigma$ where $\sigma(A) = \mu(\{x : (x, T^k x) \in A\})$.

We denote this measure $\Delta_{T^k}(\mu)$.
Rudolph’s criterion

\( X \) has minimal self-joinings if any ergodic self-joining is either the product measure or \( \Delta_{T_k}(\mu) \) for some \( k \).

**Theorem**

(Rudolph) Weakly mixing systems with minimal self-joinings are prime.
Veech’s criterion

$\mathbf{X}$ has property $S$ if any ergodic self-joining is either the product measure $\mu \times \mu$ or is 1-1 on almost every fiber.

Equivalently, if $C(T)$ denotes the centralizer of $T$. That is, the set of $F : X \to X$ so that $F$ preserves $\mu$ and commutes with $T$. If $\sigma$ is an ergodic self-joining of $\mathbf{X}$ other than $\mu \times \mu$ then there exists $F \in C(T)$ so that $\sigma(A) = \mu(\{x : (x, Fx) \in A\})$. (Disintegration of measures applied to projection onto the first coordinate.)

We denote this measure,

$$\Delta_F(\mu).$$

Theorem

(Veech) If $\mathbf{X}$ has property $S$ than any non-trivial factor comes from modding out by a compact subgroup of the centralizer.
If $X = ([0, 1), Leb, R_\alpha)$ with $\alpha$ irrational, then $C(R_\alpha) = S^1$. The compact subgroups are given by the set of rotations by the $k^{th}$ roots of unity. Modding out by one these is identifying the fibers of the times $k$ mod 1 map.
Ergodic decomposition

Let \(J(X)\) be the set of self-joinings of \(X\).

- It is convex.
- It is weak-* compact.
- It is the convex hull of its extreme points, \(J^e(X)\), the ergodic self-joinings of \(X\).

**Ergodic decomposition:** Let \(Y\) be a factor of \(X\) with factor map \(\pi\). Let \(\sigma_\pi\) be the measure as before. There exists a unique Borel probability measure on \(J^e(X)\), \(P\), so that

\[
\sigma_\pi(A) = \int_{J^e(X)} \tau(A) dP(\tau)
\]

(*)

for all measurable \(A \subset X \times X\).
More generally

This is a special example of a more general theorem. Let $X' = (X', B', \mu', T')$ be a (not necessarily ergodic) probability measure preserving system. Then there exists a unique measure $P'$ giving full measure to the $T'$ ergodic and invariant probability measures so that

$$\mu' = \int \tau dP' (\tau).$$

Our previous theorem is the case $X' = (X \times X, \sigma_\pi, T \times T)$.

**Example:** Let $T' : [0, 1) \times [0, 1) \to [0, 1) \times [0, 1)$ by $T'(x, y) = (x, R_x(y))$ and $\mu' = \text{Leb}^2$. Observe that $\mu$ is not ergodic, but for almost every $x$, $\text{Leb}_x(A) = \text{Leb}(\{y : (x, y) \in A\})$ is.

We have $\mu' = \int \text{Leb}_x d\text{Leb}(x)$. 

Recall, there exists a unique Borel probability measure on $J^e(X)$, $\mathbb{P}$, so that

$$\sigma_\pi(A) = \int_{J^e(X)} \tau(A) d\mathbb{P}(\tau) \quad (*)$$

for all measurable $A \subset X \times X$.

**Example:** If $X = ([0,1), \text{Leb}, R_\alpha)$ with $\alpha$ irrational, $\pi$ is times $k$ mod 1.

Let $R_{\frac{i}{k}}(x) = x + \frac{i}{k}$ mod 1.

$$\sigma_\pi = \sum_{i=0}^{k-1} \Delta R_{\frac{i}{k}}(\mu)$$

Note that $\Delta R_\beta(\mu)$ is $R_\alpha \times R_\alpha$ ergodic for all $\beta$. 
**Exercise:** If $\pi$ is not the map to the one point system, 
$\mathbb{P}(\mu \times \mu) = 0$.

- That is, $\mathbb{P}\left(\bigcup_{F \in C(T)} \Delta F\right) = 1$.

So we get a measure $\hat{\mathbb{P}}$ on $C(T)$.

To complete the theorem, it suffices to show there exists $K \subset C(T)$ so that $\hat{\mathbb{P}}(K) = 1$ and $F_*\hat{\mathbb{P}} = \hat{\mathbb{P}}$ for all $F \in K$.

Indeed, we have a topological group, $K$, with a probability measure on it that it is invariant under the group action. So, $K$ is compact. Indeed, because $C(T)$ and thus $K$ is separable, $\hat{\mathbb{P}}(U) > 0$ for every non-empty open set $U$. If $K$ were not compact, there would be a non-empty neighborhood of $Id$, $V$ and $k_1, \ldots \in K$ so $k_i V \cap k_j V = \emptyset$ for all $i \neq j$. As $\hat{\mathbb{P}}(k_i V) = \hat{\mathbb{P}}(V) > 0$ this would contradict that $\hat{\mathbb{P}}$ is a probability measure.
Preserving $\hat{\mathbb{P}}$

If $\pi \circ F = \pi$ almost everywhere then $(id \times F)_*\mathbb{P} = \mathbb{P}$.
Indeed,

$$\int_{J^e(X)} (id \times F)_*\tau d\mathbb{P}(\tau) = (id \times F)_*\sigma_\pi$$

$$= \int_X (id \times F)_*(\mu_{\pi(x)} \otimes \mu_{\pi(x)}) d\pi_\ast \mu(x)$$

$$= \sigma_\pi$$

Since $\mathbb{P}$ is the unique such measure, $(id \times F)_*\mathbb{P} = \mathbb{P}$.
So $F$ preserves $\hat{\mathbb{P}}$. 
Generalizing $\pi \circ F = \pi$ to $J(X)$

Let,

$$I(\pi) = \{ \tau \in J(X) : (\pi \times \pi)_* \tau = (\pi \times \pi)_* \Delta_{Id(\mu)} = \Delta_{Id(\nu)} \}.$$ 

Properties: $I(\pi)$ is

- Convex
- Compact
- Contains $\sigma_\pi$.
- Extremal

Because $\sigma_\pi \in I(\pi)$, and $I(\pi)$ is extremal, convex and compact $\mathbb{P}$ is supported on $I(\pi) \cap J^e(X)$. 
$I(\pi) \cap J^e(X) = \{ \Delta_F(\mu) : F \in C(T) \text{ and } \pi \circ F = \pi \}.$

So $\{ F \in C(T) : \pi \circ F = F \}$ preserves $\hat{P}$ and has full $\hat{P}$ measure.
References


Disintegration of measures and ergodic decomposition: Michael Hochman’s notes
http://math.huji.ac.il/~m hochman/courses/ergodic-theory-2012/notes.final.pdf

Haar measure: Ryan Vinroot’s notes
http://www.math.wm.edu/~vinroot/PadicGroups/haar.pdf

More on joinings: Theirry de la Rue
https://hal.archives-ouvertes.fr/hal-02469083/document