Factors

Let $\mathbf{X} = (X, \mathcal{M}, \mu, T)$, $\mathbf{Y} = (Y, \mathcal{A}, \nu, S)$ be two ergodic probability measure preserving systems.

We say $\mathbf{Y}$ is a *factor* of $\mathbf{X}$ if there exists $\pi : X \to Y$ so that $\mu(\pi^{-1}(A)) = \nu(A)$ for all $A \in \mathcal{A}$ and $\pi \circ T = S \circ \pi$.

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\pi"] & X \arrow[d, "\pi"] \\ Y \arrow[r, "S"] & Y \end{tikzcd}$$

Examples

- The one point system is a factor of every system.
- $(X_1 \times X_2, T_1 \times T_2, \mu_1 \otimes \mu_2)$ has (X_1, T_1, μ_1) as a factor, with factor map π_1, projection onto the first coordinate.
  - $T_1 \circ \pi_1$ is also a factor map.
- Every system is a factor of itself.
  - Id is a factor map
  - T^k is a factor map.
- $R_\alpha : [0,1) \to [0,1)$ by $R(x) = x + \alpha$ mod 1 has $R_{k\alpha}$ as a factor for all $k \in \mathbb{N}$.
  - The factor map is $x \to kx$ mod 1
  - Because $k(x+\alpha) = kx + k\alpha$
- In fact if **X** is not weakly mixing than any eigenfunction is a factor map.

Prime

X is *prime* if the only factor maps of **X** are projection to the one point system or isomorphisms.

Example: Let $X = \mathbb{Z}/p\mathbb{Z}$, μ be counting measure and $T = +1$ mod p.

Objects on $\mathbf{X}$ related to factors of $\mathbf{X}$

What about $\mathbf{X}$ can tell us about its factors?

Well $\cup_{x\in\mathbf{X}}\Big(\pi^{-1}(\pi x)\times\pi^{-1}(\pi x)\Big)\subset X\times X$ is one object.

Moreover there is a version of this for measures.

Disintegration of measures: If $\mathbf{Y}$ is a factor of $\mathbf{X}$ with factor map π as above then:
For ν a.e. $y \in Y$ there exists a probability measure μ_y (on X) so that $\mu_y(\pi^{-1}(y)) = 1$ and

$$\mu = \int_Y \mu_y d\nu(y).$$

That is $\mu(A) = \int_Y \mu_y(A) d\nu(y)$.

Example 1: $\mathbf{X} = (X_1 \times X_2, \mu_1 \otimes \mu_2, T_1 \times T_2)$, $\pi((x_1, x_2)) = x_1$.
$(\mu_1 \otimes \mu_2)_{x_1}$ "is the copy of μ_2 supported on $\{x_1\} \times X_2$."
That is, $(\mu_1 \otimes \mu_2)_{x_1}(A) = \mu_2(\{z \in X_2 : (x_1, z) \in A\})$.

Example 2:
$\mathbf{X} = ([0,1), Leb, R_\alpha)$, $\pi(x) = kx \mod 1$.
$\mu_x = \frac{1}{k}(\sum_{i=0}^{k-1} \delta_{x+\frac{i}{k}})$ where $x + \frac{i}{k}$ is taken mod 1.

Relatively independent joining

Let $\mathbf{Y}$ be a factor of $\mathbf{X}$ with factor map π and μ_y be the measures coming from the disintegration of measures as above.

Want an analogue of: $\cup_{x \in \mathbf{X}} \Big(\pi^{-1}(\pi x) \times \pi^{-1}(\pi x) \Big) \subset X \times X$.

The *relatively independent joining over* $\mathbf{Y}$ is

$$\sigma_\pi = \int_Y \mu_y \otimes \mu_y d\nu(y),$$

a measure on $X \times X$.

Example 1: $\mathbf{X} = (X_1 \times X_2, \mu_1 \otimes \mu_2, T_1 \times T_2)$, $\pi((x_1, x_2)) = x_1$.

$$\sigma_\pi(A) = \int_{X_1} \mu_2 \otimes \mu_2(\{(a, b) : (x_1, a, x_1, b) \in A\}) d\mu_1(x_1).$$

Example 2: $\mathbf{X} = ([0, 1), Leb, R_\alpha)$, $\pi(x) = 6x \mod 1$.

$$\sigma_\pi(A) = \int_{[0,1)} \frac{1}{6} |\{i \in \{0, ..., 5\} : (x, x + \frac{i}{6}) \in A\}| dLeb(x).$$

Properties of σ_π

σ_π is a measure on $X \times X$ that is

- $T \times T$ invariant
- and projects to μ in both coordinates.

A map with these two properties is called a *self-joining* of μ.

Other examples of self joinings:
•$\mu \times \mu$ (which is also the relatively independent joining over the map to the 1 point system).
• σ where $\sigma(A) = \mu(\{x : (x, T^k x) \in A\})$.
We denote this measure $\Delta_{T^k}(\mu)$

Rudolph's criterion

X has *minimal self-joinings* if any ergodic self-joining is either the product measure or $\Delta_{T^k}(\mu)$ for some k.

Theorem

(Rudolph) Weakly mixing systems with minimal self-joinings are prime.

Veech's criterion

X has *property S* if any ergodic self-joining is either the product measure $\mu \times \mu$ or is 1-1 on almost every fiber.
Equivalently, if $C(T)$ denotes the centralizer of T. That is, the set of $F : X \to X$ so that F preserves μ and commutes with T. If σ is an ergodic self-joining of **X** other than $\mu \times \mu$ then there exists $F \in C(T)$ so that $\sigma(A) = \mu(\{x : (x, Fx) \in A\})$. (Disintegration of measures applied to projection onto the first coordinate.)
We denote this measure,

$$\Delta_F(\mu).$$

Theorem

(Veech) If **X** *has property S than any non-trivial factor comes from modding out by a compact subgroup of the centralizer.*

An example

If $\mathbf{X} = ([0,1), Leb, R_\alpha)$ with α irrational, then $C(R_\alpha) = S^1$.
The compact subgroups are given by the set of rotations by the k^{th} roots of unity.
Modding out by one these is identifying the fibers of the times k mod 1 map.

Ergodic decomposition

Let $J(\mathbf{X})$ be the set of self-joinings of $\mathbf{X}$.

- It is convex.
- It is weak-* compact.
- It is the convex hull of its extreme points, $J^e(\mathbf{X})$, the ergodic self-joinings of $\mathbf{X}$.

Ergodic decomposition: Let $\mathbf{Y}$ be a factor of $\mathbf{X}$ with factor map π. Let σ_π be the measure as before. There exists a unique Borel probability measure on $J^e(\mathbf{X})$, $\mathbb{P}$, so that

$$\sigma_\pi(A) = \int_{J^e(\mathbf{X})} \tau(A) d\mathbb{P}(\tau) \qquad (*)$$

for all measurable $A \subset X \times X$.

More generally

This is a special example of a more general theorem.
Let $\mathbf{X}' = (X', \mathcal{B}', \mu', T')$ be a (not necessarily ergodic) probability measure preserving system. Then there exists a unique measure $\mathbb{P}'$ giving full measure to the T' ergodic and invariant probability measures so that

$$\mu' = \int \tau d\mathbb{P}'(\tau).$$

Our previous theorem is the case $\mathbf{X}' = (X \times X, \sigma_\pi, T \times T)$.
Example: Let $T' : [0,1) \times [0,1) \to [0,1) \times [0,1)$ by $T'(x,y) = (x, R_x(y))$ and $\mu' = Leb^2$. Observe that μ is not ergodic, but for almost every x, $Leb_x(A) = Leb(\{y : (x,y) \in A\})$ is.
We have $\mu' = \int Leb_x dLeb(x)$.

Back to the relatively independent joining over a factor

Recall, there exists a unique Borel probability measure on $J^e(\mathbf{X})$, $\mathbb{P}$, so that

$$\sigma_\pi(A) = \int_{J^e(\mathbf{X})} \tau(A) d\mathbb{P}(\tau) \qquad (*)$$

for all measurable $A \subset X \times X$.

Example: If $\mathbf{X} = ([0,1), Leb, R_\alpha)$ with α irrational, π is times k mod 1.

Let $R_{\frac{j}{k}}(x) = x + \frac{j}{k}$ mod 1.

$$\sigma_\pi = \sum_{i=0}^{k-1} \Delta_{R_{\frac{i}{k}}}(\mu)$$

Note that $\Delta_{R_\beta}(\mu)$ is $R_\alpha \times R_\alpha$ ergodic for all β.

Exercise: If π is not the map to the one point system, $\mathbb{P}(\mu \times \mu) = 0$.

- That is, $\mathbb{P}(\cup_{F \in C(T)} \Delta_F) = 1$.

So we get a measure $\hat{\mathbb{P}}$ on $C(T)$.

To complete the theorem, it suffices to show there exists $K \subset C(T)$ so that $\hat{\mathbb{P}}(K) = 1$ and $F_* \hat{\mathbb{P}} = \hat{\mathbb{P}}$ for all $F \in K$.

Indeed, we have a topological group, K, with a probability measure on it that it is invariant under the group action. So, K is compact. Indeed, because $C(T)$ and thus K is separable, $\hat{P}(U) > 0$ for every non-empty open set U. If K were not compact, there would be a non-empty neighborhood of Id, V and $k_1, \dots \in K$ so $k_i V \cap k_j V = \emptyset$ for all $i \neq j$. As $\hat{P}(k_i V) = \hat{P}(V) > 0$ this would contradict that $\hat{P}$ is a probability measure.

Preserving $\hat{P}$

If $\pi \circ F = \pi$ almost everywhere then $(id \times F)_* \mathbb{P} = \mathbb{P}$.
Indeed,

$$\begin{aligned}
\int_{J^e(\mathbf{X})} (id \times F)_* \tau \, d\mathbb{P}(\tau) &= (id \times F)_* \sigma_\pi \\
&= \int_X (id \times F)_* (\mu_{\pi(x)} \otimes \mu_{\pi(x)}) d\pi_* \mu(x) \\
&= \sigma_\pi
\end{aligned}$$

Since $\mathbb{P}$ is the unique such measure, $(id \times F)_* \mathbb{P} = \mathbb{P}$.
So F preserves $\hat{P}$.

Generalizing $\pi \circ F = \pi$ to $J(\mathbf{X})$

Let,

$$I(\pi) = \{\tau \in J(\mathbf{X}) : (\pi \times \pi)_* \tau = (\pi \times \pi)_* \Delta_{Id}(\mu) = \Delta_{Id}(\nu)\}.$$

Properties: $I(\pi)$ is

- Convex
- Compact
- Contains σ_π.
- Extremal

Because $\sigma_\pi \in I(\pi)$, and $I(\pi)$ is extremal, convex and compact $\mathbb{P}$ is supported on $I(\pi) \cap J^e(\mathbf{X})$.

$I(\pi) \cap J^e(\mathbf{X}) = \{\Delta_F(\mu) : F \in C(T) \text{ and } \pi \circ F = \pi\}$.

So $\{F \in C(T) : \pi \circ F = F\}$ preserves $\hat{\mathbb{P}}$ and has full $\hat{\mathbb{P}}$ measure.

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