Accessibility

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Learning Seminar - Nov 2020

Ergodic diffeomorphisms

Let $f : M \to M$ be a diffeomorphism of a compact manifold and μ be a Borel probability measure.

Question: When is μ ergodic for f?

Let μ be a smooth volume form on the manifold M and $\mathrm{Diff}^r_\mu(M)$ the set of C^r diffeomorphisms where $r \geq 1$ that preserve volume.

Question: When is $f \in \mathrm{Diff}_\mu(M)$ ergodic for μ? When is there an open set $\mathscr{U} \subset \mathrm{Diff}^r_\mu(M)$ such that each $f \in \mathscr{U}$ is ergodic for μ?

Hopf (1939) was able to answer this in part using an argument that is now called the Hopf argument. Anosov used this to prove ergodicity for volume preserving C^2Anosov diffeomorphisms.

Hopf argument

Facts:

1. f is ergodic for μ if and only if $g \in L^1$ is f-invariant implies g is constant almost everywhere.
2. $B_+(g)(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g(f^j(x))$ exists almost everywhere
3. $B_-(g)(x) \lim_{n\to-\infty} \frac{1}{|n|} \sum_{j=n+1}^{0} g(f^j(x))$ exists almost everywhere

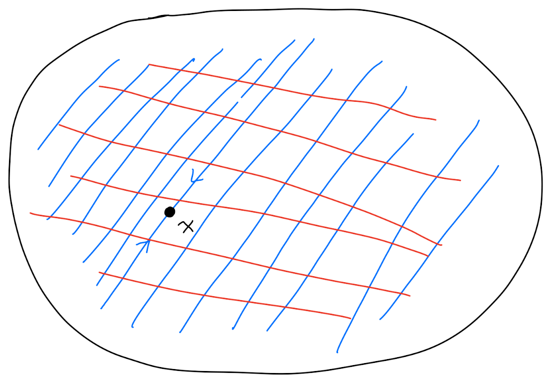
Let f be Anosov (uniformly hyperbolic on the entire manifold). Then for each $x \in M$ there exist stable and unstable manifolds $W^s(x)$ and $W^u(x)$

Claim: For g continuous we have $y \in W^s(x) \implies B_+(g)(x) = B_+(g)(y)$ if it exists. (A similar statement holds for unstable manifolds and $B_-(g)(x) = B_-(g)(y)$.)

Proof: Given $\epsilon > 0$ there exists a $\delta > 0$ such that if $d(x, y) < \delta$ then $|g(x) - g(y)| < \epsilon$. After a finite number of iterates we know points on the stable manifold become delta close. Then the difference in the averages is $< 2\epsilon$. Since ϵ is arbitrary we are done. $\square$

Hopf argument - part 2

If g is continuous, then $B_+(g)$ and $B_-(g)$ are constant along stable and unstable manifolds when these exist.



This seems to imply it is constant everywhere. There is a technical point (absolute continuity) needed in the proof and this is why Anosov needs C^2 diffeomorphisms.

Partial hyperbolicity

In the 1990s there was progress in extending Anosov's result to a larger class of systems.

Definition: A diffeomorphism f is partially hyperbolic if there exists an f-invariant splitting $E^s \oplus E^c \oplus E^u$ such that

- each vector in E^s is uniformly contracted,
- each vector in E^u is uniformly expanded, and
- each vector in E^c does not contract as much as those in E^s or expand as much as those in E^u.

For each $x \in M$ there exist stable and unstable manifolds ($W^s(x)$ and $W^u(x)$) tangent to the distributions E^s and E^u. There may be center leaves tangent to E^c denoted W^c, but this isn't guaranteed.

Idea: Extend the Hopf argument. Problem is that the stable and unstable manifolds don't "fill up" the manifold.

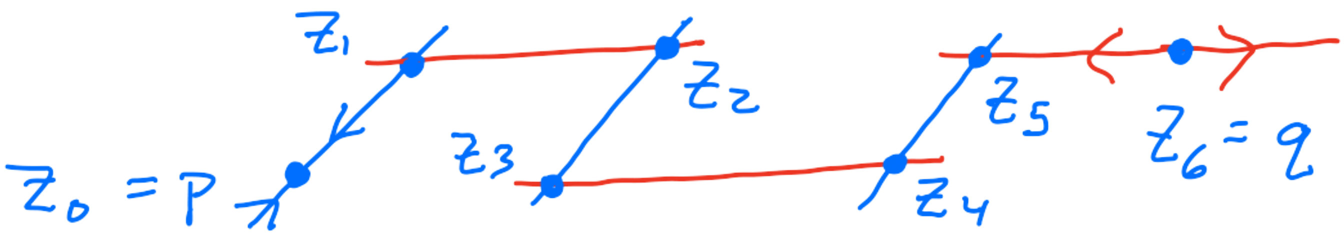
How to overcome this?

Using a notion called accessibility this can then be carried over the entire manifold.

Accessibility

Definition: Two points p and q are accessible (or in the same accessibility class) if there are points z_i with $z_0 = p$, $z_\ell = q$ and $z_i \in W^\alpha(z_{i-1})$ for $i = 1,\ldots,\ell$ and $\alpha = s$ or u.

Accessibility is an equivalence relation and the flow is accessible if the entire manifold is an accessibility class.



Transitive flows

Theorem: (Brin 75) Let f be a partially hyperbolic diffeomorphism of a compact Riemannian manifold M. Assume that f preserves a smooth measure on M and has the accessibility property. Then for almost every $x \in M$ the orbit of x is dense in M. In particular, f is transitive.

Idea of Proof: Let U and V be open sets. We want to show for almost every point in U the orbit intersects V. Let $\{z_0, \ldots, z_k\}$ be a us-path with z_0 in U and z_k in V. Then there is a neighborhood around z_{k-1} where almost every orbits intersects V. This follows by the Poincaré Recurrence Theorem - if x is recurrent and $y \in W^s(x) \cup W^u(x)$ then y enters V.

Then there is a neighborhood around z_{k-2} whose orbit intersects the neighborhood around z_{k-1}. Now continue to z_0. $\square$

Main results

Theorem: (Dolgopyat-Wilkinson 03, Avila-Crovisier-Wilkinson, preprint) If M is a smooth compact manifold and $r \geq 1$, then stable accessibility is C^1 dense among

- all
- volume-preserving
- symplectic

partially hyperbolic C^r diffeomorphisms on M.

Theorem: (F, Hasselblatt, preprint) For any smooth compact manifold M and $r \geq 1$, C^1-stable accessibility is C^1 dense among

- all
- volume-preserving
- symplectic

partially hyperbolic C^r flows on M.

C^1

In the C^1 setting there are perturbation techniques that help with obtaining accessibility.

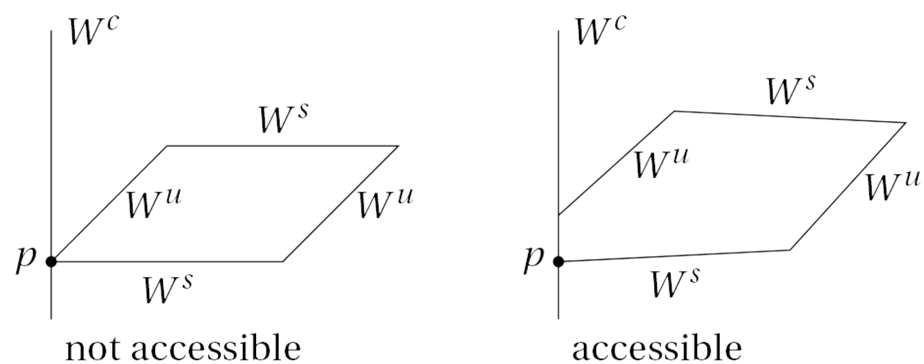
1. Franks' Lemma: We can locally linearize a diffeomorphism (or flow) in a neighborhood of a periodic orbit and the linear map can be the one we prescribe. More generally, this allows one to freely perturb the derivative of a diffeomorphism or flow along a finite piece of orbit, and then the linear map is pasted in (using the exponential map).

2. Closing Lemmas: Such as Pugh's Closing Lemma allow us to perturb to create periodic points near nonwandering points. This can also be done for chain recurrent points from results of Bonatti and Crovisier.

Remark: For the C^1 setting if we want to make an ϵ-small C^1 perturbation we can do this is a neighborhood of size ϵ.

These results don't exist for higher regularity. The problem is in making an ϵ-small C^2 perturbations requires a neighborhood of scale $1/\epsilon$ and this gets worse with higher regularity

Brin quadrilaterals



A Brin quadrilateral is a 4-legged us-path if there is a center leaf for p then it starts and ends on a point in the leaf. If there isn't a center leaf one can use disks that are nearly tangent to the center direction.

These play a central role in accessibility. If the stable and unstable manifolds are jointly integrable, then there is no displacement in the center direction. If the quadrilateral does not close up then there is displacement in the center direction, and if the center is 1-dimensional this implies accessibility.

Local Perturbations

1. To produce accessibility one wants to perturb the map so the invariant subbundles (stable and unstable) are changed. The problem is that with recurrence you could undo the accessibility in another location.

2. However, the changes done in one location are quickly reduced further along the orbit. To use this one first wants to make perturbations along orbits with long return times - so the change is dampened by the time it returns.

3. These kind of local perturbations require the C^1 topology. This is the only place where C^1 perturbations are necessary in the proof.

2-step approach

Step 1: Find small disks "in the center direction" that have

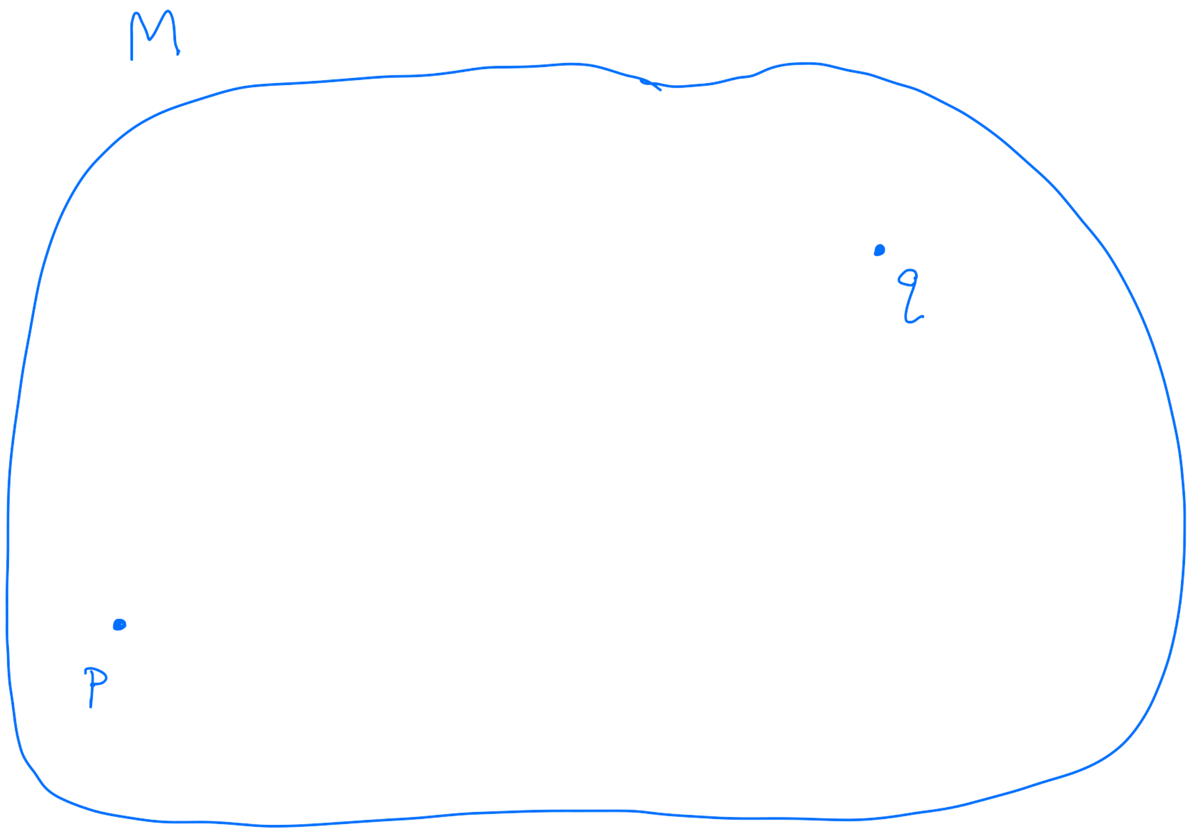
- long return times,
- are sufficiently dense, and
- pairwise disjoint.

Given any disk we want to ensure that given any point p there is a us-path from p to some point on the disk. This implies accessibility "modulo" disks.

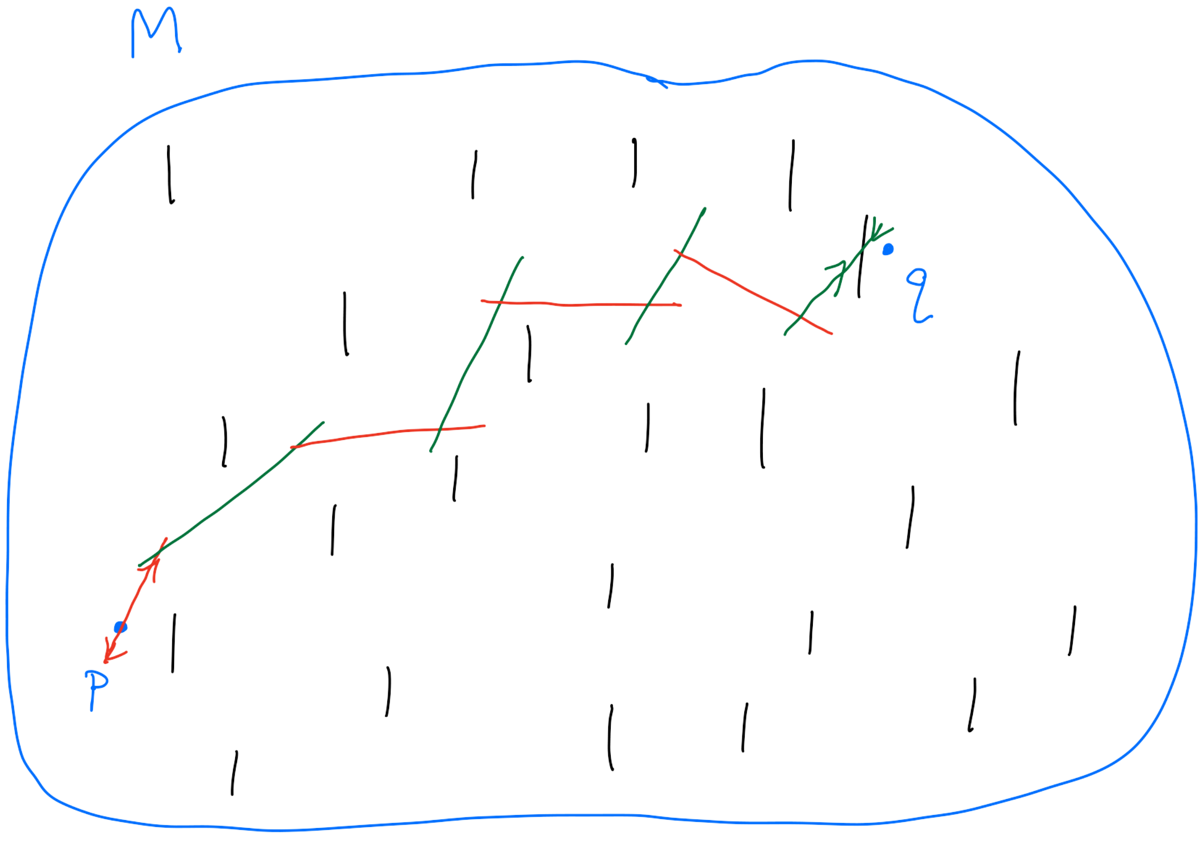
Lastly, we want to ensure these properties are robust.

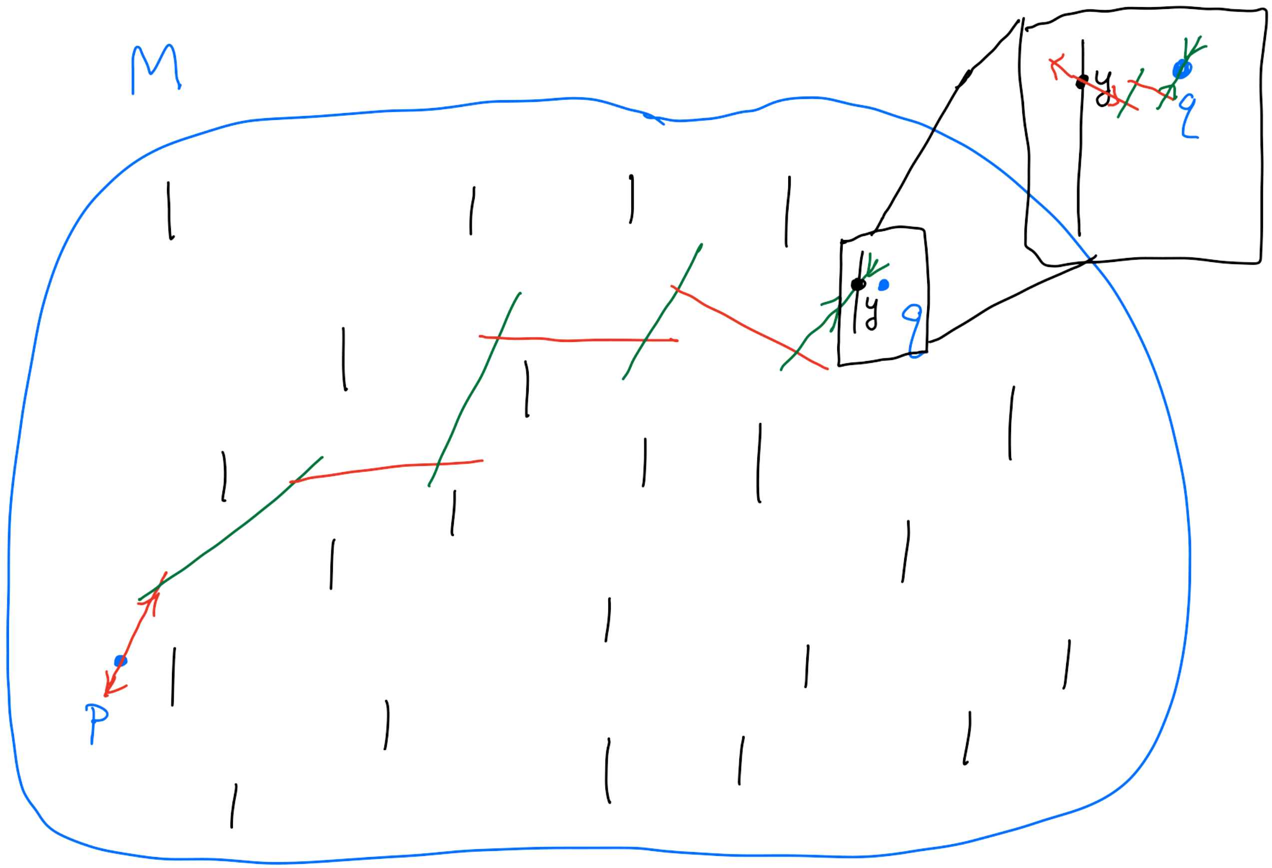
Step 2: Perturb the flow locally near the disks to ensure that there is a neighborhood of the disk so that any 2 points in the neighborhood are accessible by us-paths that stay close to the disk.

Idea: So given points p and q the first step ensures there is a us-path to a disk close to q and the second step ensures we can extend the us-path from the point on the disk to q.



M
q
p

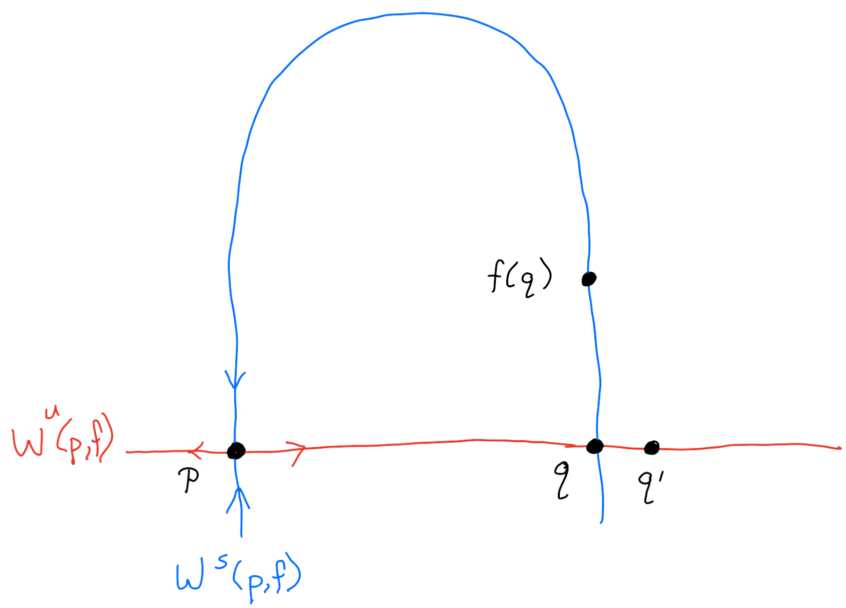

M
p
q



M
P
y
q
y
q

Local perturbations

To locally change the dynamics a natural way is to post-compose the diffeomorphism with a map close to the identity and equal to the identity outside the desired neighborhood. The new map moves points to the desired new location.



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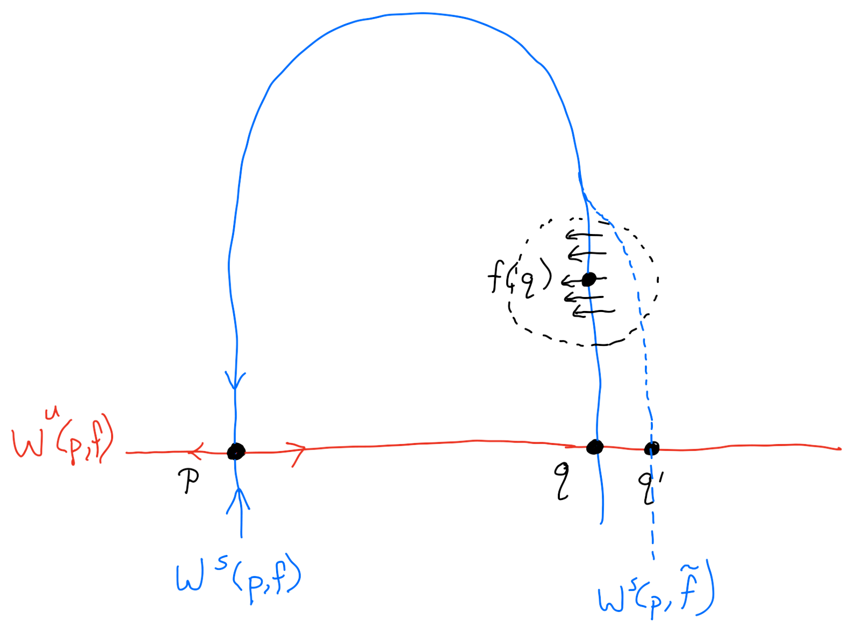


Adapted charts and admissible disks

Adapted charts: The first step is ensuring there are local coordinates that are well-adapted for the dynamics. These charts are chosen so the dynamics are nearly integrable for local Brin quadrilaterals.

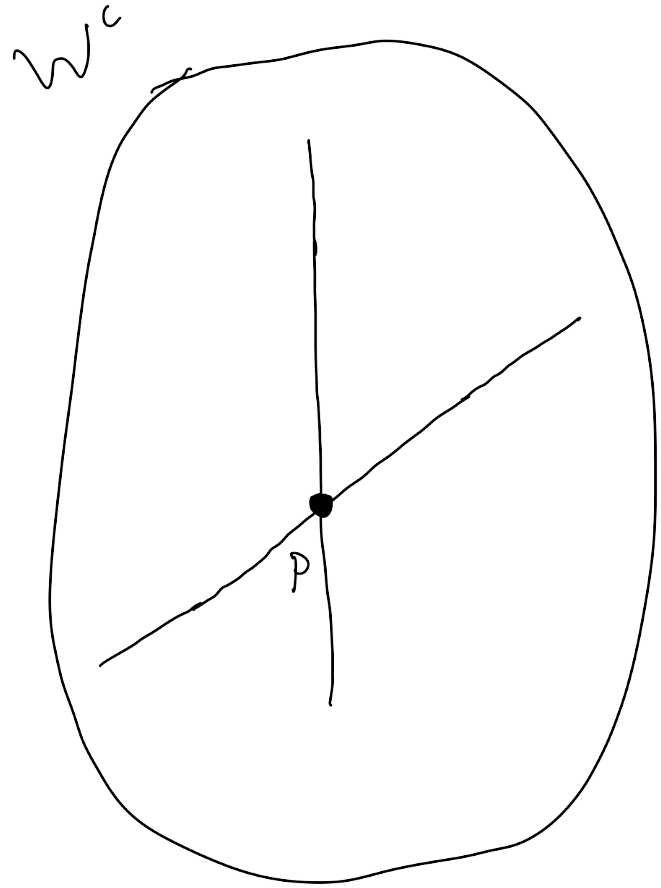
Proposition: For a partially hyperbolic set Λ for a diffeomorphism f there exists a $\delta > 0$ such that if $T > 0$, then there exists a family $\mathscr{D}$ of center disks such that each disk has a return time $> T$ has diameter $< T^{-1}$ and a map g that is δ close to f and g is stably accessible on $\mathscr{D}$.

Accessibility near disks

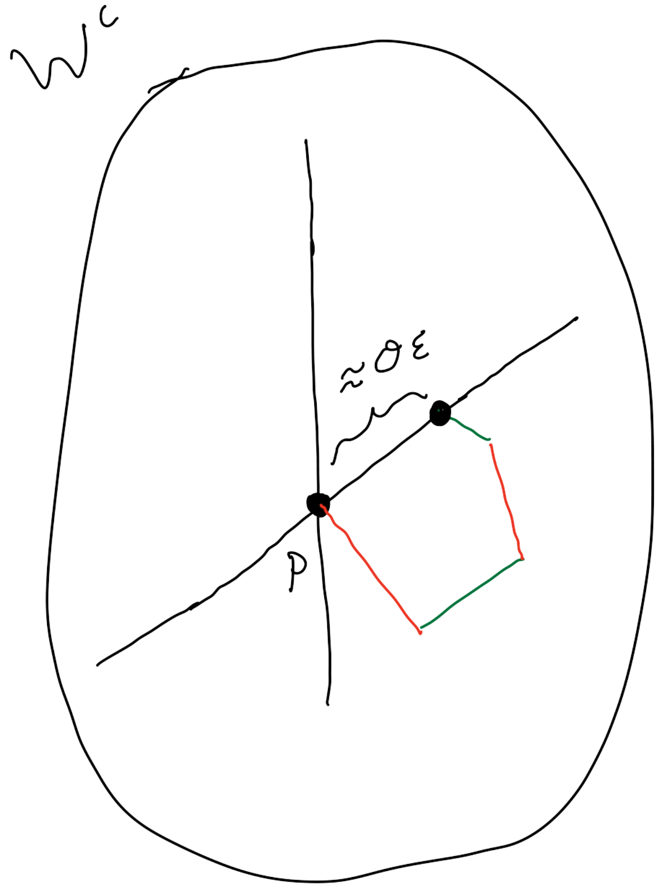
Definition: For a center direction of dimension j the map is θ-accessible in a small ball of radius $\approx\epsilon$ if there are j Brin quadrilaterals in the ball whose endpoints are distance $\approx \pm\,\theta\epsilon$ from the initial point in each of the j directions.

- If the map is θ-accessible near a point p in a center disk, then there is a neighborhood of p in the center disk that is accessible to p. (Result just relies on Brouwer's Fixed Point Theorem)

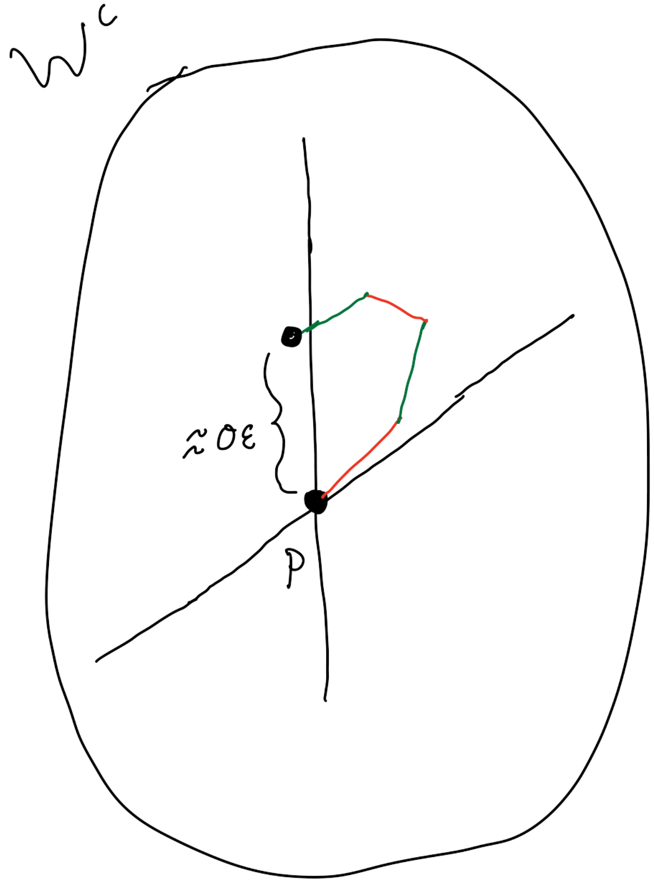
- Theorem can then be proved if we can perturb the map so that it is θ-accessible near all the center disks we have chosen simultaneously.

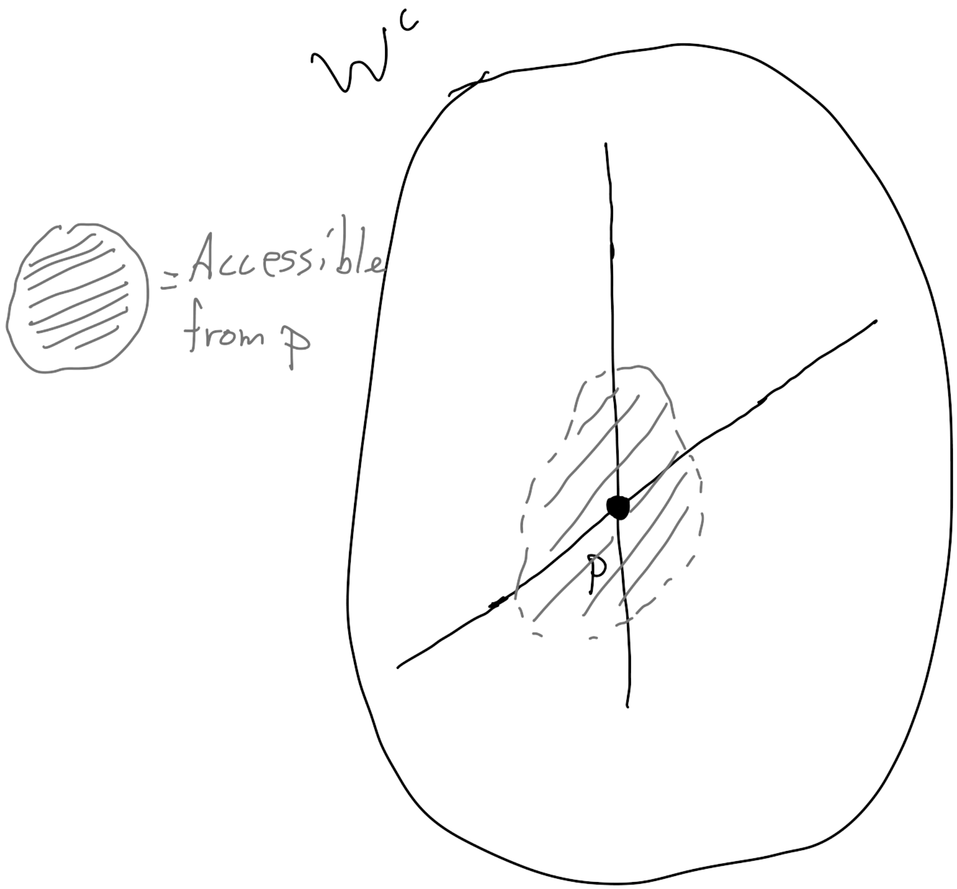


W^c
p



W^c
$\approx \theta\varepsilon$
p

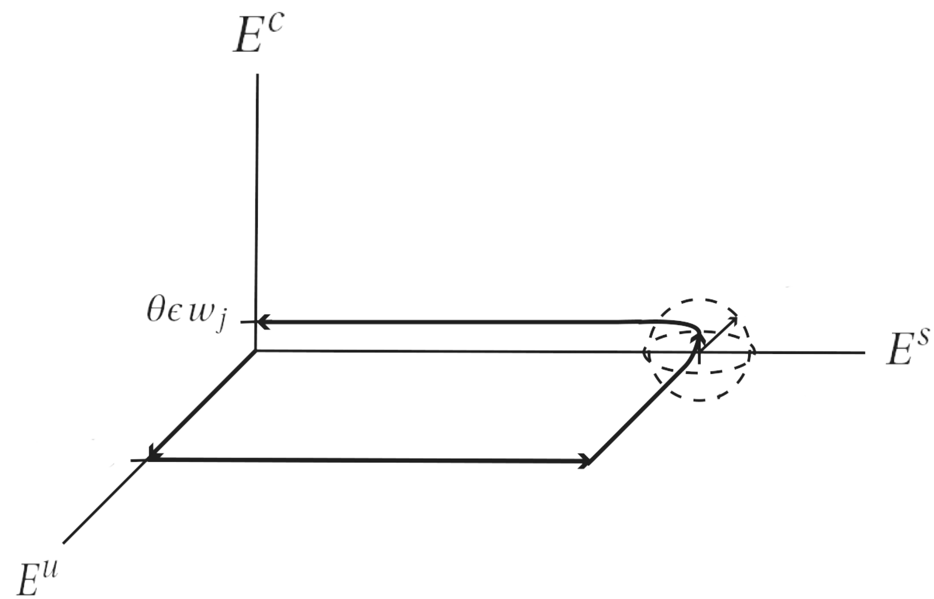

W^c
≈ Oε
p



W^c
= Accessible from p
p

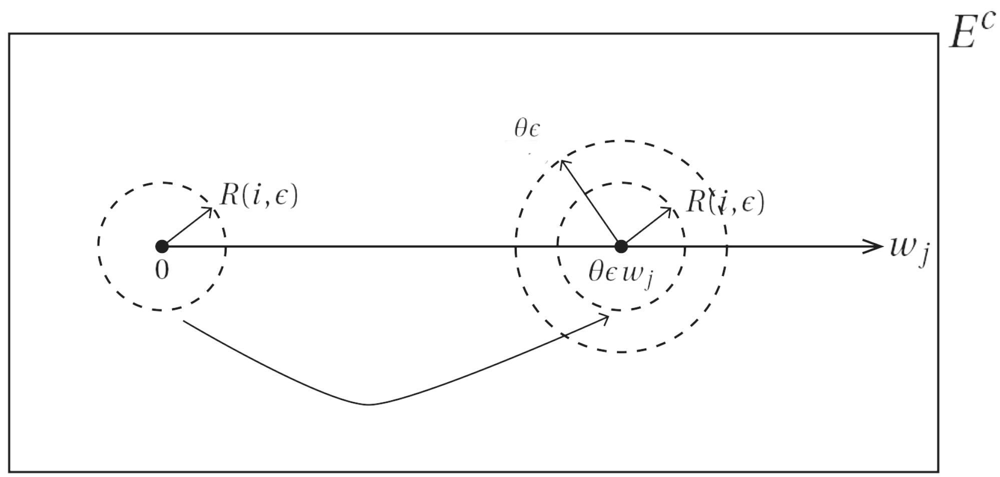
Perturbations

Start with a quadrilateral that ends near the initial point. Now in a neighborhood of the third endpoint perturb the map in the w_j direction for the center.



Creating θ-accessibility

For the original flow we suppose we return within a ball of size $R(i, \epsilon)$. The key is that $R(i, \epsilon) \in o(\epsilon)$. Now the perturbation described on the previous slide says we can end near $\theta \epsilon w_j$ as desired.



Conclusion

Corollary 1: (Dolgopyat Wilkinson 03) For $r \geq 1$, there is a C^1 open and dense set $\mathscr{U}$ among volume preserving partially hyperbolic C^r diffeomorphisms such that each diffeomorphism in $\mathscr{U}$ is topologically transitive.

Corollary 2: (Dolgopyat Wilkinson 03) Let M be a symplectic manifold with $\dim(M) \leq 4$. The C^1-closure of the stably transitive diffeomorphisms in $\mathrm{Diff}^r_\omega(M)$ coincides with the C^1-closure of the partially hyperbolic ones.

Corollary 3: (Dolgopyat Wilkinson 03) If $r \geq 1$, and M has a symplectic form ω, then there is a C^1 open and dense set of transitive diffeomorphisms among C^r partially hyperbolic diffeomorphisms preserving ω.

Open questions

Question 1: For $r \geq 1$ is accessibility C^ropen and dense among partially hyperbolic diffeomorphisms? What about among those preserving a volume form or symplectic form?

A diffeomorphism has a dominated splitting $TM = E \oplus F$ if any expansion by a vector in E is less then the expansion in F and any contraction in F is dominated by stronger contraction in E. This is weaker then partial hyperbolicity.

Conjecture: (Dolgopyat, Wilkinson, 03) In the space of volume preserving diffeomorphisms the C^1-closure of the stably transitive diffeomorphisms coincides with the closure of the diffeomorphisms admitting a dominated splitting.

Thank you!