Möbius disjointness for interval exchange transformations on three intervals (with A. Eskin)

We show that 3-IETs that satisfy a mild diophantine condition satisfy Sarnak's Möbius disjointness conjecture. That is, if T is such a 3-IET, for any continuous function f and x we have $\sum_{n=1}^{M} f(T^n x)\mu(n) = o(M)$, where μ is the Möbius function. We do this by showing that enough of the powers of T are pairwise disjoint. The diophantine condition is that a torus with 2 marked points related to the 3-IET is not divergent on average under the action of $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. In an appendix we show that almost every 3-IET has that all of its positive powers are disjoint.

Logarithmic laws and unique ergodicity (with R. Treviño) (submitted)

We relate different logarithm laws and unique ergodicity. Given a flat surface let $\delta_t^{EXT}(\omega)$ denote the length of the extremal length systole on $g_t\omega$. Masur showed that for almost every surface ω we have

$$\lim_{t\to\infty} \frac{-\log(\delta_t^{EXT}(\omega))}{\log(t)} = \frac{1}{2}.$$

We build an explicit example of a non-uniquely ergodic IET that satisfies this almost sure divergence rate. It is straightforward to see that Masur's paper also shows that

$$\lim_{t\to\infty} \frac{-\log(\delta_t^{F}(\omega))}{\log(t)} = \frac{1}{2},$$

where $\delta_t^F(\omega)$ is the length of the flat systole on $g_t\omega$. We show that any surface satisfying this almost sure divergence rate must be uniquely ergodic. The main techniques are using fast return times for IETs to find short simple closed curves (not just short saddle connections) and the technique of combining complexes to apply a criterion of Treviño for unique ergodicity.

Circle averages and disjointness in typical flat surfaces on every Teichmueller disc (with P. Hubert) (submitted)

We prove that for almost every surface increasing 'circles' about a fixed center equidistribute except possibly on a set of density 0. That is, for almost every ω and for any continuous function and point p on ω we have that there exists $A \subset \mathbb{R}$ with density 1 so that

$$\lim_{t\in A,\, t\to\infty} \frac{1}{2\pi}\int_0^{2\pi} f(F^t_{r_\theta\omega}(p))d\theta = \int f d\lambda^2.$$

We do this by showing that for almost every surface the flow in almost every pair of directions is disjoint. This implies that the product of these flows is typically uniquely ergodic, which allows us to derive the above result. To prove the disjointness criterion we observe that the vertical flow on $\hat{h}_s\omega = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}\omega$ and ω are isomorphic, the vertical flow on ω is isomorphic to a time change of the vertical flow on $g_t\omega$ and studying how the representation of $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$ as $\bar{h}_a g_t h_b$ changes with s and θ.

Horocycle flow orbits and lattice surface characterizations (with K. Lindsey) (submitted)

We prove new equivalences of lattice surfaces based on the horocycle flow in typical directions. The main tool is Eskin-Mirzakhani-Mohammadi's ergodic theorems for $SL(2,\mathbb{R})$ orbit closures.

A smooth mixing flow on a surface with non-degenerate fixed points (with A. Wright) (submitted)

We show that there exists a smooth mixing flow on a surface where all of the singularities are non-degenerate by constructing an example in genus 5. This answers a question of Katok-Sinai-Stepin. Ulcigrai showed that any such example is atypical (Scheglov earlier showed it is atypical in genus 2). This is connected to building a suspension flow over an IET whose roof function has symmetric logarithmic singularities.

The set of uniquely ergodic IETs is path connected (with S. Hensel) (submitted)

This paper shows that that the set of uniquely ergodic n-IETs is path connected for $n \geq 4$. This is not true for $n < 4$. The primary work is showing it for 4-IETs. The paths will use IETs corresponding to not arational foliations.

On limit sets in PMF for Teichmüller geodesics (with H. Masur and M. Wolf) (Accepted Crelle's Journal)

We show that there exist minimal and not uniquely ergodic flows on flat surfaces whose associated Teichmüller geodesics have a unique limit point in PMF. Others do not have a unique limit point. There exists non divergent Teichmüller geodesics whose limit sets are different. Also there are two divergent Teichmüller geodesics that have a shared point in their limit set. There exist ergodic (but not uniquely ergodic) vertical foliations whose limit set is not a point. Our examples come from minimal and not uniquely ergodic $\mathbb{Z}_2$ skew products of rotations introduced by W. Veech and developed by H. Masur and coauthors. C. Leinenger, A. Lenzhen and K. Rafi have some similar results.

A dichotomy for the stability of arithmetic progressions (with M. Boshernitzan) (Accepted Proceedings of the AMS)

We show that a Borel set in $[0,1]$ contains 3-term arithmetic progressions under any homeomorphism iff it contains arbitrarily long arithmetic progressions under any homeomorphism.

There exists an interval exchange with a non-ergodic generic measure (with H. Masur) (Journal of Modern Dynamics)

We show that there exists an IET with a generic measure that is not ergodic, answering a question of Boshernitzan. That is, we show there exists an IET, T, a probability measure μ and a point x so that for any $f \in C([0,1])$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=0}^{N-1} f(T^i x) = \int f d\nu$$

but ν is not an ergodic measure.

The Hausdorff dimension of non uniquely directions in $\mathcal{H}(2)$ is almost everywhere $\frac{1}{2}$ (with J. Athreya) (Geometry and Topology)

We prove that the set of not uniquely ergodic 4-IETs has Hausdorff dimension $\frac{5}{2}$, which is Hausdorff codimension $\frac{1}{2}$. We also prove that for almost every surface in $\mathcal{H}(2)$ the set of directions where the flow is not uniquely ergodic has Hausdorff dimension $\frac{1}{2}$ (which is also Hausdorff codimension $\frac{1}{2}$). We build an explicit set to show that the lower bound and the upper bound follows from work of Masur.

Every flat surface is Birkhoff and Osceledets generic in almost every direction (with A. Eskin) (Journal of Modern Dynamics)

In this paper we show that for any flat surface ω, for almost every θ under geodesic flow $r_\theta\omega$ equidistributes with respect to the unique measure supported on the closure of $SL_2(\mathbb{R})\omega$ that is not supported on a lower dimensional submanifold. Moreover, we show that the Kontsevich-Zorich cocycle has the appropriate Lyapanov spectrum for a full measure set of directions. This strengthens several known results to hold on every flat surface instead of just a full measure set of flat surfaces. The key techniques come from homogeneous dynamics and recent breakthroughs of Eskin, Mirzakhani and Mohammadi.

Every transformation is disjoint from almost every IET (Annals of Mathematics)

In this paper we show that given any transformation of Lebesgue space, almost every IET is different from it. As a corollary we show that the product of almost every pair of IETs is uniquely ergodic and so every point is recurrent. A key step in the proof is that any sequence of density 1 contains a rigidity sequence for almost every IET, strengthening a result of Veech.

Appendix C of *Right-angled billiards and volumes of moduli spaces of quadratic differentials on* $\mathbb{C}P^1$ by J. Athreya, A. Eskin and A. Zorich (Accepted Annales ENS)

This appendix shows that for almost every point in an unstable manifold for the geodesic flow on the space of quadratic differentials the circle through that point equidistributes under geodesic flow. This enables a couple of the theorems of the paper to be asymptotics rather than weak asymptotics. The argument uses Avila-Resende's result that there is a spectral gap for $SL_2(\mathbb{R})$ acting on the space of quadratic differentials and Margulis' argument for using exponential mixing to prove the equidistribution of typical circles.

Diophantine properties of IETs and general systems: Quantitative proximality and connectivity (with M. Boshernitzan) (Invetiones Mathematicae)

This paper shows a variety results on IETs, systems of linear block growth and general systems. It shows that there are no topologically mixing 3-IETs. It shows that if μ is an ergodic measure for an IET T then $T^i(x) \in B(y, \frac{\epsilon}{i})$ for any $\epsilon > 0$ and $\mu \times \mu$ a.e. (x, y). This result is proper. It also proves the following ergodic theorem: Let $f : X \to [0, \infty]$ be μ-measurable and $T : X \to X$ be μ ergodic. if $\{s_i\} \subset \mathbb{R}_+$ is non-decreasing and there exists $c > 1$ such that $\frac{s_{2n}}{s_n} > c$ for all large enough n then $\liminf_{n\to\infty} s_n f(T^n x)$ is $\in \{0, \infty\}$ for μ a.e. x. In particular it takes finite positive values on a set of measure zero. This shows that for any y we have that $\liminf_{n\to\infty} nd(T^n x, y) \in \{0, \infty\}$ for almost every x. Using the Lebesgue Density Theorem, Cassels showed earlier that for any x this was true for almost every y.

Borel-Cantelli Sequences (with M. Boshernitzan) (Journal d'Analyse Mathematique)

We classify the sequences of points $\{x_i\}_{i=1}^{\infty} \subset [0,1]$ such that for any monotone decreasing sequence of positive reals $\{r_i\}_{i=1}^{\infty}$ with $\sum_{i=1}^{\infty} r_i = \infty$ one has $\lambda(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} B(x_i, r_i)) = 1$. We extend this result to Ahlfors regular spaces.

Shrinking targets for IETs (Geometric and Functional Analysis)

In this paper we consider the following problem: Given a point $p \in [0,1)$ and a monotone decreasing sequence of positive reals $\{r_i\}_{i=1}^{\infty}$, for typical $x \in [0,1)$ and IET T will $T^i x \in B(y, r_i)$ infinitely often? The Borel-Cantelli Theorem says no if the sum of the radii converge. We show that for almost every IET if $\sum_{i=1}^{\infty} r_i = \infty$ then $T^i x \in B(y, r_i)$ for infinitely many i. The full measure set of IETs depends on the sequence. For almost every IET there exists a monotone decreasing sequence $\{r_i\}_{i=1}^{\infty}$ where $\sum_{i=1}^{\infty} r_i = \infty$ and yet for almost every x we have $T^i x \in B(y, r_i)$ for only finitely many i. However, if we restrict our sequences of radii by requiring that ir_i is monotone decreasing then one full measure set of IETs has that $T^i x \in B(y, r_i)$ infinitely often for all such $\{r_i\}_{i=1}^{\infty}$. Other related results are considered. The next paper, joint work with D. Constantine, improves this result to consider the frequency of hits to the shrinking balls.

Quantitative shrinking targets for rotations, IETs and billiards in rational polygons (with D. Constantine) (Submitted)

We consider how often the orbit of a point under an IET visits a shrinking ball about a fixed point. We show that under some assumptions it is roughly as often as one would expect.

Theorem 1. *For almost every IET T and any $\{r_i\}_{i=1}^{\infty}$ such that $\{ir_i\}$ is monotone decreasing and $\sum_{i=1}^{\infty} r_i = \infty$ we have*

$$\lim_{N\to\infty} \frac{\sum_{i=1}^{N} \chi_{B(p,a_i)}(T^i x)}{\sum_{i=1}^{N} 2a_i} \overset{a.e.x}{\to} 1.$$

We prove a variety of variants of this theorem, in particular establishing an analogous result for any fixed rational billiard. The proof follows the outline of the strong law of large numbers. A key step is establishing a quantitative version of Boshernitzan/Masur's criterion for unique ergodicity.

There exists a topologically mixing IET (Ergodic Theory and Dynamical Systems)

This paper shows that there exists an IET with the property that for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ implies that for any x we have $T^n B(x, \epsilon)$ is ϵ dense. The next paper improves on this result. In joint work with M. Boshernitzan (Diophantine properties...) we showed it is false when $n = 3$.

Topological mixing for residual sets of IETs (with J. Fickenscher) (Communications in Mathematical Physics)

This paper shows that a residual set of IETs with a permutation in a non-degenerate Rauzy class on 4 or more intervals are topologically mixing. This implies that there are topologically mixing uniquely ergodic IETs. It then proves that some billiards in L-shaped polygons are topologically mixing. It uses combinatorial methods to prove the result in the Rauzy class of (4321). It then uses the structure of Rauzy classes to extend it to the general situation.

Winning games for bounded geodesics in moduli spaces of quadratic differentials (with Y. Cheung and H. Masur) (Journal of Modern Dynamics)

We show that the set of bounded geodesics in Teichmüller space is a strong winning set for C. McMullen's variant of Schmidt's game, answering a question of McMullen. We show that the set of badly approximable interval exchanges is also strong winning. Call a direction θ in a Teichmüller disc q *bounded* if there is a compact set of Teichmüller space K_θ such that $g_t r_\theta q \in K_\theta$ for all t. We show that in every Teichmüller disc the set of bounded directions is absolutely winning for McMulllen's variant of Schmidt's game.

Every transformation is disjoint from almost every non-classical exchange (with V. Gadre) (Geometriae Dedicata)

We show that if S is an invertible measure preserving transformation of Lebesgue space and π is a 'permutation' of non-classical exchanges with at least one preserved band then for almost every $\bar{L}$ we have S is different from $T_{\pi,\bar{L}}$.

The distribution of gaps for saddle connection directions (with J. Athreya) (Geometric and Functional Analysis)

We show a variety of result on the gaps between saddle connection directions for flat surfaces. Three highlights are that for almost every surface there exists a distribution of normalized gaps, this distribution decays quadratically towards 0 at 0 and a surface has no 'small gaps' iff it is a lattice surface.

The Gap Distribution for Saddle Connection Directions on the Golden $\mathcal{L}$ (with J. Athreya and S. Leliévre) (Contemporary Mathematics)

We compute the asymptotic distribution of normalized saddle connection gaps for the golden L. This is the first example of such a computation for a flat surface that is not a torus cover.

Skew products over rotations with exotic properties (Geometriae Dedicata)

Answering questions of W. Veech and M. Boshernitzan we show that there exists a $\mathbb{Z}_2$ skew product of a badly approximable rotation that is minimal but not uniquely ergodic. We apply this to show that there is a $\mathbb{Z}$ skew product of a badly approximable rotation where the orbit of Lebesgue almost every point is dense but Lebesgue measure is not ergodic.

Omega recurrence in cocycles (with D. Ralston) (Ergodic Theory and Dynamical Systems)

Consider the skew product $\hat{R}_\alpha : [0,1) \times \mathbb{Z} \to [0,1) \times \mathbb{Z}$ by

$$\hat{R}_\alpha(x,i) = \big(x + \alpha - \lfloor x + \alpha \rfloor, i + \chi_{[0,\frac{1}{2})}(x) - \chi_{[\frac{1}{2},1)}(x)\big).$$

We show that for almost every α, x we have that $\hat{R}^i_\alpha(x,0) \in [0,1) \times \{0\}$ fairly often. We show that for some α and any x we have that $\hat{R}^i_\alpha(x,0) \in [0,1) \times \{0\}$ holds rarely.

Schrödinger operators defined by interval exchange transformations (with D. Damanik and H. Krüger) (Journal of Modern Dynamics)

Consider the dynamically defined Shrödinger operator $H_x : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ by $H_x(\bar{u})[n] = u(n+1) + u(n-1) + f(T^n x)u(n)$ where T is an IET, f is continuous and $x \in [0,1)$. We show a variety results on the spectrum of this self-adjoint operator. In particular, we use Kotani theory to show the absence of absolutely continuous spectrum in a variety of cases.

Hausdorff dimension for ergodic measures of interval exchange transformations (Journal of Modern Dynamics)

Building on an example of M. Keane this paper shows that minimal but not uniquely ergodic can have a singular ergodic measure that is carried by a Borel set of Hasudorff dimension 0. The next paper improves this.

On the Hausdorff dimensions of a singular ergodic measure for some minimal interval exchange transformations

This paper proves a variety of sharper versions of theorems in the previous paper. Additionally, in the appendix it provides the construction of an IET with a weird diophantine property: There exists a minimal and not uniquely ergodic IET T with ergodic measures μ and ν such that

$$\limsup_{n\to\infty}\frac{-\log d(T^n x,y)}{\log n}=1 \text{ for } \mu\times\nu \text{ a.e. } (x,y)$$

but

$$\limsup_{n\to\infty}\frac{-\log d(T^n x,y)}{\log n}\le\frac{1}{2} \text{ for } \nu\times\mu \text{ a.e. } (x,y).$$

Homogeneous approximation for flows on translation surfaces

Motivated by a result of L. Marchese this paper shows that for any translation surface, $\{a_i\}$ non-increasing with divergent sum we have that $|F^t_\theta x-x|<a_i$ infinitely often for almost every x and θ. This result is not implied by Marchese's result because it applies to sequences that are just non-increasing (rather than ia_i being non-increasing) and it holds on every flat surface. It does not imply Marchese's result because it does not address the difference between arbitrary pairs of discontinuities.

Ergodic homogeneous multidimensional continued fraction algorithms (with A. Nogueira)

We provide an alternate proof of Veech's result that (non-renormalized) Rauzy induction is ergodic. We extend this to show that the d-dimensional Selmer algorithm is ergodic on its absorbing set.

The densest sequence in the unit circle (with M. Boshernitzan)

We identify the densest sequence on the unit circle. Not surprisingly it is the most separated one as well.

[0,1] is not a Minimality Detector for $[0,1]^2$

This paper shows that there exists a non-minimal sequence in S^1, $(x_1,x_2,...,)$, such that for any continuous function $f:S^1\to[0,1]$ we have $(f(x_1),f(x_2),...)$ is minimal.

Arithmetic progressions in regular cantor sets

We show that the middle $\frac{1}{N}$ Cantor set contains arithmetic progressions of length $\frac{N}{50\log N}$. A draft is available upon request.