Let $\mathcal{A}$ denote a finite set. $\mathcal{A}^{\mathbb{N}}$ with product topology is a compact metric space.

$$d(\mathbf{x}, \mathbf{y}) = 2^{-\inf\{i : x_i \neq y_i\}}$$

gives the topology.
The left shift $L : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ by $L(\mathbf{x})_i = x_{i+1}$ is a continuous map.

We consider $X \subset \mathcal{A}^{\mathbb{N}}$ closed and so that $X = L(X)$.
By the Krylov-Bogolyubov Theorem for any such X there is an L invariant Borel probability measure.

Examples:

- $X = \mathcal{A}^{\mathbb{N}}$.
- $X = \{\mathbf{x} \in \{0,1\}^{\mathbb{N}} :\ \text{if } x_i = 1 \text{ then } x_{i+1} \neq 1\}$.
- $X = \{(0, 0, \dots)\}$
- Given $\mathbf{x} \in \mathcal{A}^{\mathbb{N}}$, let $Y = \overline{\{L^i \mathbf{x}\}_{i=0}^{\infty}}$ and $X = \cap_{i=1}^{\infty} L^i Y$, the ω-limit set of $\mathbf{x}$.

Let

$$B_n(X) = \{(a_1, ..., a_n) \in \mathcal{A}^n : \exists \mathbf{x} \in X \text{ with } x_i = a_i \text{ for all } 1 \leq i \leq n\}.$$

An important function is $n \to |B_n(X)|$.

- $\lim_{n\to\infty} \frac{1}{n} \log(|B_n(X)|)$ exists and is the topological entropy of X.
- If there exists C so that $|B_n(X)| \leq Cn$ for all n we say X has linear block growth.

Theorem

(Hedlund-Morse) If there exists n so that $|B_n(X)| = |B_{n+1}(X)|$ then $|B_{n+k}| = |B_n|$ for all $k \geq 0$. Moreover every element of X is (eventually) periodic.

Proof.

- For each $(a_1, ..., a_n) \in B_n(X)$ there exists unique $u \in \mathcal{A}$ so that $(a_1, ..., a_n, u) \in B_{n+1}(X)$.
- Applying this to $(a_2, ..., a_n, u)$ there exists a unique v so that $(a_2, ..., a_n, u, v) \in B_{n+1}$ and so $(a_1, ..., a_n, u, v)$ is the unique element of $B_{n+2}(X)$ starting $a_1, ..., a_n$.
- Iterating this, for all $k \geq 0$ there exists a unique element of B_{n+k} starting with $a_1, ..., a_n$.

□

The bound from the Hedlund-Morse Theorem is optimal. There exists (X, L), a shift dynamical system so that $|B_n(X)| = n + 1$ for all n.

Let $0 < \alpha < 1$ so that $\alpha \notin \mathbb{Q}$ and $R : [0, 1) \to [0, 1)$ by $R(x) = x + \alpha - \lfloor x + \alpha \rfloor$. Let $\tau : [0, 1) \to \{0, 1\}$ by

$$\tau(x) = \begin{cases} 0 & \text{if } x \in [0, \alpha) \\ 1 & \text{else.} \end{cases}$$

Let $\mathfrak{c} : [0, 1) \to \{0, 1\}^{\mathbb{N}}$ by $\mathfrak{c}(x)_i = \tau(R^i x)$.
$X = \overline{c([0, 1))}$ satisfies $|B_n(X)| = n + 1$ for all n.

1. Show $\mathfrak{c}([0,1))$ is not closed in $\{0,1\}^{\mathbb{N}}$.
2. Describe $\overline{c\big([0,1)\big)} \setminus c\big([0,1)\big)$.
3. Show $|B_n(X)| = n+1$.

The following notions may be helpful for the exercises:

A word $(a_1, ..., a_n) \in B_n(x)$ is called *right special* if there exists (at least) two different symbols $u, v \in \mathcal{A}$ so that $(a_1, ..., a_n, u), (a_1, ..., a_n, v) \in B_{n+1}(X)$.

A word $(a_1, ..., a_n) \in B_n(x)$ is called *left special* if there exists (at least) two different symbols $u, v \in \mathcal{A}$ so that $(u, a_1, ..., a_n), (v, a_1, ..., a_n) \in B_{n+1}(X)$.

We say a word is *special* if it is either left or right special.

Further reading:
Substitutions in Dynamics, Arithmetics and Combinatorics by Pytheas Fogg.
-See: https://www.irif.fr/$\sim$berthe/Fogg.html

An introduction to symbolic dynamics and coding by Douglas Lind and Brian Marcus.