NO IET IS MIXING

from

"IET and some special flows are not mixing" A. Katok

IET: interval exchange transf.

$I = [0, b]$ and $f : [0, b] \to [0, b]$

we want $f$ to be 1-1 and continuous except at finitely many points
$f$ preserves the Lebesgue measure

Formally: $n > 0 \in \mathbb{N}$

$\Delta = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ s.t. $\sum \lambda_i = b-a$

$T_{\Delta, \sigma} = T : [0, b]$
for $1 \leq i \leq n$ \hspace{1em} a_i = \sum_{1 \leq j \leq i} \lambda_j^i

b_i = \sum_{1 \leq j \leq \sigma(i)} \lambda_{\sigma^{-1}(j)}^i

for $x \in \mathcal{I}$ we define

$\tau(x) = x + b_i - a_i$ for $x \in \left[ a_i, a_i + \lambda_i \right]

\textbf{def:} If $m$ is the minimal positive integer such $\mathcal{I}(T)$ has a representation as above we shall say that $f$ is a IET of $m$ intervals.

& n
\[ f : [0, 1] \to \lambda = (1/2, 1/2) \]
\[ \sigma = (12) \]

\[ \begin{array}{c}
\circlearrowleft \\
0 \quad 1/2 \quad 1
\end{array} \]

Remark

In principle an ISM there may be other invariant measures different from the Lebesgue one

\((X, \mu, f)\) dynamical system

Define (mixing) let \((X, \mathcal{B}, \mu, T)\) be a dynamical system. Then T is mixing if \(\forall A, B \in \mathcal{B}\) we have
\[ \lim_{n \to +\infty} \mu(T^n A \cap B) = \mu(A) \mu(B) \]

Theorem (A. Keleti)

\[ f: \mathbb{R} \to \mathbb{R}^2 \] is a TET

\[ \mu \text{ is any Borel measure on } \mathbb{R} \]

which is \( f \)-invariant.

\[ f \text{ is not mixing} \]

Basic idea: if \( f \) were mixing

Then for \( A = B \) one would have
\[
\lim_{n \to \infty} \mu(f^{-n}A \cap A) = \mu(A)^2
\]

The idea becomes to find a sequence \( f^{-n} \) such that

\[
\mu(A \cap f^{-n}A) \to \mu(A)^2
\]

In order to prove this, then we need 2 lemmas:

**Lemma 1:** If \( \mathcal{G} \) is an ITET of \( n \) intervals and \( \mu \)

is a non-atomic Borel measure inv. under \( f \),

then there exist an ITET of \( \mathbb{R} \) on an interval \( g : [0, 1] \subset \mathbb{R} \).
\[ S([0, 1], \lambda) \sim (\Sigma, \mu) \]

There exists a bijection between them up to subsets of measure 0.

R is the "isomorphism."

\[ R: I \rightarrow [0, 1] \text{ and this can be taken to be } \Psi \]

Shelah:

\[ R: [0, b] = I \rightarrow [0, 1] \]

\[ y \mapsto \mu([0, y]) \]

Since \( \mu \) is not atomic, \( R \) is continuous and surjective. \( \Psi \)

Generally this is not 1-1.
 \[ R_y = \lambda \]

\[ f \quad \xrightarrow{\quad} \quad I \]

\[ R \downarrow \quad C^0 \quad \downarrow \quad R \]

\[ \mathbb{I}^0 \rightarrow \mathbb{I}^0 \]

\[ x \quad \mapsto \quad y \]

\[ g(x) = R(f(y)) \quad \text{as } R_y \]

Some checks imply that \( g \) is

\[ \text{an IET} \]

\[ \Delta = I \quad \text{and let } f_g \text{ be the} \]

\[ \text{Lemma 3} \]

\[ f: I^D \text{ is an IET of an IET.} \]
induced SET.

\[ y \in \Delta \Rightarrow \exists t_{\text{first return}} \] 

\[ f^{-1} \circ t_{\text{first return}} \circ f \in \Delta \text{ for } f^{-1} \circ t_{\text{first return}} \circ f \in \Delta \]

The time of first return to \( \Delta \).

\( f^{-1} \) is an SET of at most \( m+2 \) intervals. Moreover

\[ \Delta = \Delta_1 \cup ... \cup \Delta_s \]

\[ r \leq s \leq m+2 \]

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Proof of Theorem 1

- We consider ergodic measures

- Lemma 1 \( \Rightarrow \) it is sufficient to prove the Theorem
for $A$ on $[0,\infty)$

Fix $\Delta \subseteq \mathcal{I}$

Lemma 2 \implies \mathcal{I} = \bigcup_{i=1}^{k_0} \bigcup_{i} \Delta_i$

where $\tau_i$ is the time of first return to $\Delta_i$

for each $\Delta_i$, we have $f_{\Delta_i}$

the induced IET on $\Delta_i$

and we apply Lemma 2 once more to $\Delta_i$

$\Delta_i = \bigcup_{j=1}^{s_i} \Delta_{ij} = \bigcup_{j=1}^{s_i} \bigcup_{i} \Delta_{ij}$

first time of return to $\Delta_{ij}$
\[ I = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} \Delta_{ij} \]

These are all disjoint intervals.

Note: \( \sum_{i} \Delta_i = \Delta_n \)

\[ \text{Proof:} \quad f_{ij}^n(\Delta_{ij}) \subset \Delta_n \quad (4) \]

\[ f_{ij}^n(\bigcup_{j=1}^{m} \Delta_{ij}) = \bigcup_{j=1}^{m} \left( f_{ij}^n(\Delta_{ij}) \right) \subset \Delta_n \]

\[ \Rightarrow \Delta_n = \bigcup_{j=1}^{m} \Delta_{ij} \]

\[ (4) \Rightarrow \Delta_n \subset \bigcup_{j=1}^{m} f_{ij}^n \Delta_{ij} \]
Recall

\[ I = \bigcup_{i=1}^{s} \bigcup_{j=1}^{t} f^{n} \Delta_{ij} \]

This is called "partition of \( \mathbb{E}^{n} \) \( \mathbb{E}_{\Delta} \). We say the AOTI is measurable with \( \mathbb{E}_{\Delta} \) if \( \Delta \)

is union of elements in \( \mathbb{E}_{\Delta} \)

Let \( A \) measurable with \( \mathbb{E}_{\Delta} \)

\[ A \subset \bigcup_{i=1}^{s} \bigcup_{j=1}^{t} f^{n} \Delta_{ij} \]

\( f \) is measure for each

\[ s \leq \text{mtz}, t \leq \text{mtz} \]
\[ \mu(A \cap f^i A) = \sum_{j} \mu(A \cap f^j A) \geq \frac{1}{(m+2)^2} \mu(A) \]

\[ \text{Fix } A \text{ and } \mu(A) < \frac{1}{10(m+2)^2} \]

\[ \text{Fix } N > N \]

Choose \( \Delta \subset I \) so that
\[ \exists A_\Delta \text{ measurable with } \xi_A \]

\[ \mu(A_\Delta A_\Delta) < \frac{1}{10} \mu(A)^2 \]

\[ t_i > N \quad \forall i \quad \text{this holds for any sub-interval of } \Delta \]

Pick \( A_\Delta \) for some \( t_j > t_i > N \)
\[ \mu(A \cap f^{-1}(A)) \geq \mu(A_\delta \cap f^{-1}(A_\delta)) - 2\mu(A_\delta A_\delta) \]

\[ \geq \frac{1}{(m+2)^2} \mu(A_\delta) - \frac{1}{5} \mu(A)^2 \]

\[ \text{Rmk} \]

\[ \mu(A) > \frac{8}{10} \mu(A) \]

\[ \mu(A) < \frac{1}{10(m+2)^2} \]

\[ \geq \left(\frac{8}{10}\right)^2 \frac{1}{(m+2)^2} \mu(A) - \frac{1}{5} \mu(A)^2 \]

\[ \geq \mu(A)^2 \left(\frac{10 - \frac{8^2}{10}}{10^2 - \frac{5}{5}}\right) \]
\[ > 2 \]
\[ > 2 \mu(A)^2 \]
\[ \implies f \text{ is not mixing.} \] 

\[ f: S \to \text{IET} \]

\[ h: \mathbb{R} \to \mathbb{R}^+ \text{ roof function} \]

3. determine a "vertical" flow
\[ \{ f_t \} \text{ on } S^h = \{ (x, t) \in I \times \mathbb{R} \mid 0 \leq t \leq h(x) \} \]
If the time of return to \( S \) is \( t = t_{1-} \) and \( h_{1} = \sum_{k=0}^{\infty} h(f^{k}(x)) \), then \( h_{1} \) is bounded below by some finite Borel measure on \( T \). Determine a measure \( \nu = \mu \times \lambda \) on \( T_{h} \).

\[ h \in C([0,1]) \]

\[ \nu(h) = \sup P \sum_{0 \leq i \leq m_{P}} |h(x_{i+1}) - h(x_{i})| \]

where \( P = \{ P = (x_{0}, \ldots, x_{m_{P}}) \} \).
\[ 0 \leq x_0 < x_1 < \ldots < x_n = b \]

\[ h \in BV([a,b]) \quad \text{if} \quad V(h) < \infty \]

Lauded
Variation

Thus (A. Koch)

\[ f \text{ is } \text{FET} \]

\[ h \in BV(h) \quad \text{v.e. Borel measure} \]

inv. wrt \( h \) \( \mathfrak{H} \) \{ "rectified flow" \}

Then \( \mathfrak{H} \) is not mixing.