Disjointness in Ergodic Theory, Minimal Sets, and a Problem in Diophantine Approximation

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0. Summary. The objects of ergodic theory—measure spaces with measure-preserving transformation groups—will be called processes, those of topological dynamics—compact metric spaces with groups of homeomorphisms—will be called flows. We shall be concerned with what may be termed the "arithmetic" of these classes of objects. One may form products of processes and of flows, and one may also speak of factor processes and factor flows. By analogy with the integers, we may say that two processes are relatively prime if they have no non-trivial factors in common. An alternative condition is that whenever the two processes appear as factors of a third process, then their product too appears as a factor. In our theories it is unknown whether these two conditions are equivalent. We choose the second of these conditions as the more useful and refer to it as disjointness.

ELEMENTARY PROOF OF FURSTENBERG'S DIOPHANTINE RESULT

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(Communicated by Andreas R. Blass)

ABSTRACT. We present an elementary proof of a diophantine result (due to H. Furstenberg) which asserts (in a special case) that for every irrational \( \alpha \) the set \( \{2^m3^n\alpha/m, n \geq 0\} \) is dense modulo 1. Furstenberg's original proof employs the theory of disjointness of topological dynamical systems.

1. Introduction

Throughout the paper by a semigroup we mean an infinite subset of positive integers which is closed under multiplication. Two integers \( p, q \) are called multiplicatively independent if both are \( \geq 2 \) and the ratio of their logarithms \( (\log p)/(\log q) \) is irrational. The equivalent requirement is that \( p \) and \( q \) should not be integral powers of a single integer.
Furstenberg's topological $\times 2 \times 3$ Theorem
(following Boshernitzan)

Aug 16 2022

Thin (Furstenberg '67)

\[
\mathbb{T} = \mathbb{R}/\mathbb{Z}
\]

\[m_2: \mathbb{T} \to \mathbb{T}, \quad m_2(x) = 2x \mod \mathbb{Z}\]

\[m_3: \mathbb{T} \to \mathbb{T}, \quad m_3(x) = 3x \mod \mathbb{Z}\]

If \(A \in \mathbb{T} \setminus \{0\}\) then \(\{m_2^k \circ m_3^j(x) : k, j \in \mathbb{N}\}\)

is dense.

Example ① \(x = \frac{1}{3}\)

\(\left\{ \frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5} \right\} \subset M_{20} \circ M_3 \sim \mathbb{N}\)

② \(\left\{ \frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{7}{15}, \frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5} \right\}\)

is \(M_{20} \circ M_3 \sim \mathbb{N}\).
(3) \( S \subseteq \mathbb{N} \) is a semigroup if 
\[ \alpha \beta \in S \implies \alpha \beta \in S. \]

(4) Suppose \( S = \mathbb{N} \)

So dense if \( \mathbb{Q} \subset \mathbb{N} \)

\[
\text{Def: } p, q \in \mathbb{N} \text{ are called } \text{multiplicatively independent } \text{ if } \frac{\log p}{\log q} \notin \mathbb{Q}.
\]

\[
\left( \frac{\log p}{\log q} = \frac{r}{s} \iff \log p^r = \log q^s \iff p^r = q^s \right)
\]

2, 3 are multi. ind.
6, 3 “ “ “ “

\[
\text{Def: A semigroup } S \subseteq \mathbb{N} \text{ is non-lacunary if } S \text{ contains too multiplicatively independent numbers.}
\]
Thus if $S \subseteq N$ non-lacunary and $K \subseteq \mathbb{N}$ is closed, $S$-invariant, infinite then $K = \mathbb{N}$.

Prop. The following are equivalent

(1) $S$ is non-lacunary
(2) there is no $r \in \mathbb{N}$ s.t.

$$S \subseteq \left\{ r^k : k \in \mathbb{N} \right\}$$

(3) If we write $S = \{ s_1, s_2, \ldots \}$

with $s_1 < s_2 < s_3 < \ldots$

$$\lim_{i \to \infty} \frac{s_{i+1}}{s_i} = 1$$

Ref. (Ex with hints)

Idea for $1 \implies 3$.

$WNLG$ $S = \left\{ \frac{k \cdot q^k}{\log q} : k \in \mathbb{N} \right\}$

where $\frac{\log q}{200} < 0$
Given $\exists \xi > 0$ want to find $c_{\xi} \in \mathbb{S}$. 

Fix $0 < \frac{S_{\xi + 1}}{S_{\xi}} < 1 + \xi$.

Find $r, s \in \mathbb{N}$ so that

$$\left| \frac{r}{s} - \frac{\log f}{\log q} \right| < \frac{\varepsilon}{2s \log q}$$

$$\frac{r}{s} < \frac{\log f}{\log q} < \frac{r}{s}$$

$$s_i = p \cdot q^e \quad p^e < q^r \quad \frac{q^r}{p^e} < 1 + \xi$$

$$S_{\xi + 1} \leq \frac{p \cdot q^e \cdot q^r}{p^e} = q^{e + r} \cdot p^{-s} < (1 + \xi) s_i$$

From now on $S \in \mathbb{N}$ non-lacunary semigp $K^{\infty}$ is $S$-inv if $\forall s \in \mathbb{S}$, $s(K) < k$.

Lemma 5. If $K^{\infty}$ closed, infinite, $S$ invariant, and contains a non-isolated rational number then $K = \mathbb{N}$. 
Lemma 2. If $K$ is closed, $S$-invariant, non-empty, then $K$ contains a rational point.

Proof of Lemma 1 assuming Lemma 2.

Let $K$ be infinite closed $S$-inv.

Want to show that $K = \overline{P}$.

Use Lemma 2 with $K'$ instead of $K$, where $K'$ is the set of accumulation points of $K$. By Lemma 2, $K'$ contains a rational point. Applying Lemma 1, we get the theorem.

Proof of Lemma 1. Suppose $0$ is an accumulation point of $K$. 

If $x \in S$
\[ x < \frac{1}{s} \]
$Sx$ is obtained by multiplying on $\mathbb{R}$.

By contradiction assume $K$ contains no rational points, is nonempty and $S$-in. We will show $K$ is dense to get a contradiction. Let $\varepsilon > 0$.

Let $p \in \mathbb{Q} \cap S$ mult. incl. in $S$.

Let $t$ be an integer satisfying
\[ t \varepsilon > 1 \quad \gcd(t, p) = 1 = \gcd(t, q) \]
There is \( u \in \mathbb{N} \) s.t. \( p^u \equiv q^u \equiv 1 \pmod{t} \).

Define inductively
\[
K = K_0 \supset K_1 \supset K_2 \supset \cdots \supset K_{t-1}
\]

where \( K_{i+1} = \left\{ x \in K_i : x + \frac{1}{t} \pmod{\mathbb{Z}} \in K_i \right\} \)

Then:

- Each \( K_i \) is inv. under both \( p^u \) and \( q^u \).

By induction:
\[
p^u + q^u = \frac{1}{t}
\]

for some \( r \)

- Each \( K_i \) is closed.

- If \( K_i \neq \emptyset \) then \( K_i \) is infinite.

- If \( K_i \neq \emptyset \) then \( K_i \) contains an irrational by contradiction assumption,
\[(p^u) \cap (q^u) \cdot x \text{ is infinite and in } K_i.\]

- \(K_i \neq \emptyset\).

To see this, consider \(D_i = K_0 - K_i\)
\(D_i\) closed (since \(K_i\) compact).
\(D_i\) has an accumulation point because \(K_i\) is infinite. \(D_i\) is inv. under \(S = \text{ semigroup generated by } p^u, q^u\)

\[\Rightarrow (\text{Lemma 1}) \quad D_i = \mathbb{T}^d\]

\[\Rightarrow \frac{1}{e} \epsilon D_i \Rightarrow K_i \neq \emptyset.\]

So \(K_i - 1 \neq \emptyset\).

Let \(y \in K_i - 1\). Then
\(y, y - \frac{1}{e}, y - \frac{2}{e}, \ldots, y - (1 - \frac{1}{e}) \in K_0\)

\(K_i, K_{i-1}\)

\[\Rightarrow K_0 \text{ is dense } \Rightarrow K_0 \text{ is}\]
Furstenberg x2 x3 conjecture

Let $S$ be a non-lacunary semigroup. Any $S$-invariant measure is either atomic or Lebesgue.

- **Atomic**: $\exists x_0 \text{ s.t. } \mu(\{x_0\}) > 0$.
- **Invariant**: $\forall s \in S$, $\forall A \subseteq \mathbb{B}$,

$$\mu(A) = \mu(S^{-1}(A)).$$