An Invitation to "Entropy in dimension one" by W. Thurston

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"Entropy in dimension one" by Bill Thurston
In "Frontiers in Complex Dynamics" (2015)

- Thurston fell ill while writing it, passed away in 2012.
- Numerous people (in particular J. Milnor) helped prepare it for publication
- Beautiful, deep, inspiring, brilliant. Also unpolished, poorly written, hard to understand.

My presentation is an "invitation" - will cover some small pieces.

Topological entropy, $h_{top}(f)$,

where $f: X \to X$ is continuous, X a compact topological space.

- $h_{top}(f)$ is the sup of the measure-theoretic entropies
  (f-invariant, Borel, probability)

- open cover definition:

$$h_{top}(f) = \sup_{\substack{\text{finite open} \\ \text{covers } C \text{ of } X}} H(f, C)$$

$$H(f, C) = \lim_{n \to \infty} H(C \vee f^{-1}C \vee \dots \vee f^{-(n-1)}C)$$

H = log of cardinality of smallest finite subcover

- (n, ε)-separated set definition:
  restates above def for metric spaces in terms of growth rate (in a)
  of # of ε-distinguishable length-n orbits

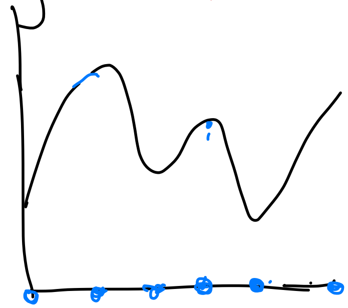
- topological entropy is a topological invariant
- If X is a d-dim metric space and f has expansion (Lipschitz) constant $K \geq 1$, then $h_{top}(f) \leq d \cdot \log(K)$.
- For f a subshift of finite type, $h_{top}(f)$ is the log of the $\underbrace{\text{spectral radius}}_{\text{leading eigenvalue}}$ of the incidence matrix.

  $M = [(m_{i,j})]$ $\quad m_{i,j} = \begin{cases} 1 & \text{if } i \to j \text{ allowed} \\ 0 & \text{else} \end{cases}$
- pseudo-Anosov surface diffeomorphisms admit Markov partitions: decompose the surface into "rectangles" so that each rectangle is mapped onto a finite union of the other rectangles. (like a subshift of finite type)

The log of the dilatation ("stretch factor") = topological entropy.

i.e. $e^{h_{top}(f)}$ = dilatation. (call $e^{h_{top}(f)}$ the "growth rate")

- By a multimodal self-map of an interval, I mean something like:



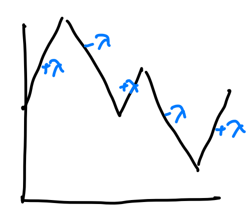
i.e., continuous, finitely many topological critical points.

Such a map is postcritically finite (PCF) if the forward orbit of the critical points is a finite set.

- If you partition the interval by cutting it at all points in the postcritical set, you get a Markov partition.
  growth rate = $\cancel{\log}$ (spectral radius)

- A uniform (λ)-expander is a continuous, piecewise linear self map of an interval whose derivative on each piece is $\pm\lambda$.

$e^{h_{top}(f)} = \lambda$



Theorem (Milnor–Thurston):
Every multimodal interval self-map with entropy $h > 0$ is semi-conjugate to a uniform λ-expander with the same topological entropy, $h = \log \lambda$.
Furthermore, if the map is PCF, so is the uniform expander.

pseudo-Anosovs: have a linear model that is "uniform expanders" for (2D) surfaces
admit Markov partitions,
growth rate = stretch factor/dilatation.

PCF multimodal interval maps: linear model is a PCF uniform λ-expander
admit markov partitions,
growth rate = stretch factor $= \lambda$.

Big question: Which numbers are realized as the growth rates of pseudo-Anosovs?

Thurston answered the 1D version: which numbers are realized as the growth rates of PCF multimodal interval maps?

Recall that the Perron-Frobenius Theorem says the spectral radius of a matrix like our incidence matrices (entries in $\mathbb{N} \cup \{0\}$) is 1) unique and 2) a special kind of number: a weak Perron number.

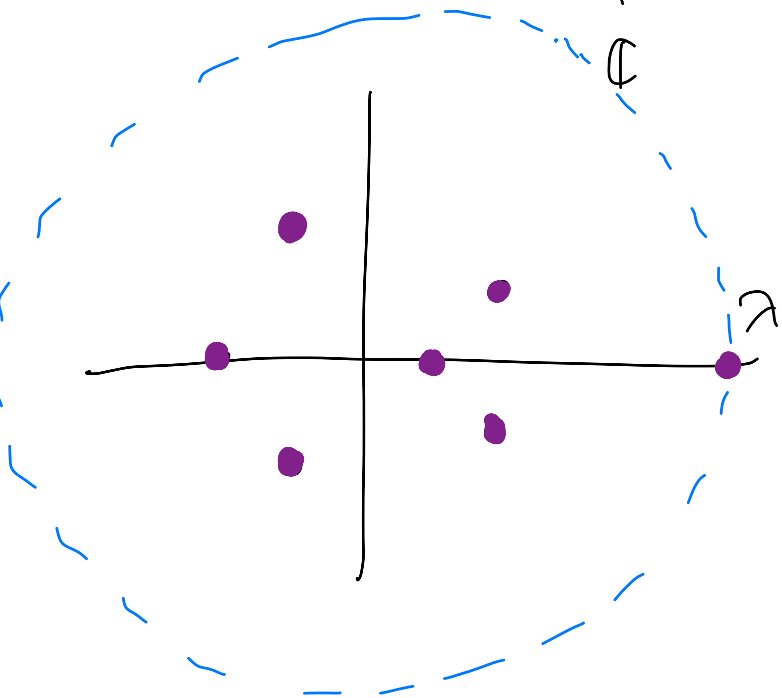
A <u>weak Perron number</u> is a real, algebraic integer that is $\geq$ the norm of all its Galois conjugates.

An algebraic integer λ is defined by a polynomial with integer coeffs.
monic (leading coeff $= 1$) and irreducible

$x^2 - x - 1 = 0$

ex $\frac{1+\sqrt{5}}{2}$

The Galois conjugates of λ are the roots of this polynomial.



weak Perron: $\lambda \geq |\lambda^\alpha|$ for all Galois conjugate λ^α of λ.

Theorem (Thurston): A positive real # λ is the growth rate of a PCF multimodal interval map $\iff$ λ is weak Perron.

(Also: same conclusion for ergodic train track representatives of outer automorphisms of free groups.)

I will tell you a bit about what goes into this proof.
But there is much more in the paper. Touches on:

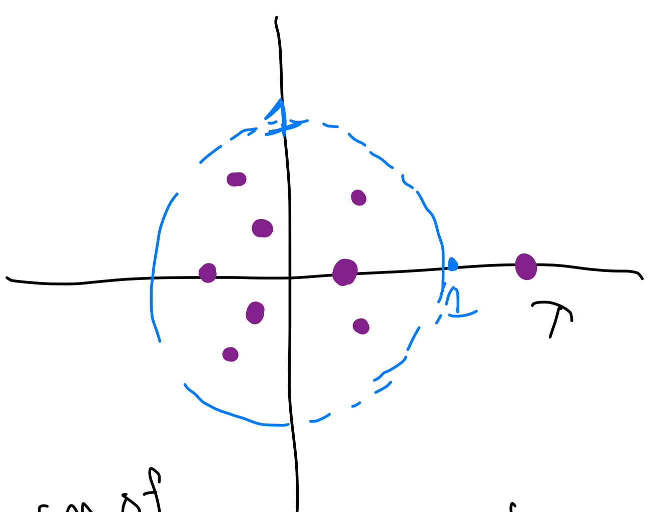
- entropy in bounded degree?
- for outer automorphisms, which pairs of weak Perron numbers can be growth rates of ϕ and ϕ^{-1}?
- entropies of self-maps of graphs (Hubbard trees $\Rightarrow$ core entropy)
- A mysterious example in which he constructs a pA from an interval map.

(ask my PhD student Ethan Farber about this!)

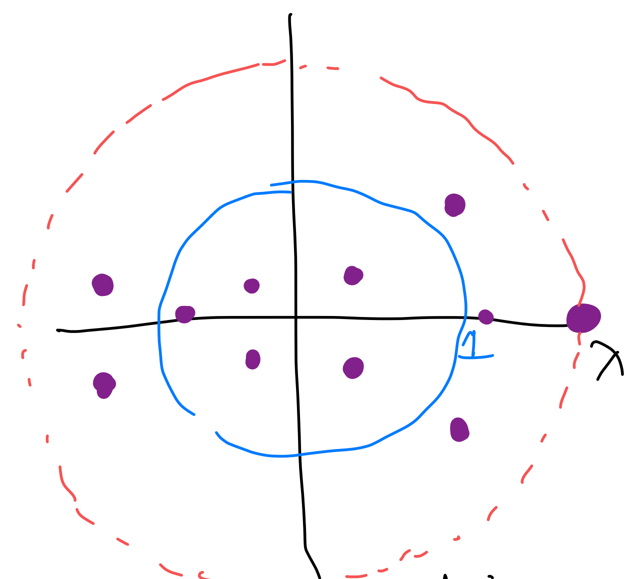
Thm: If λ is a weak Perron #, λ is the growth rate of a PCF interval map.

3 cases: easiest $\longrightarrow$ harder

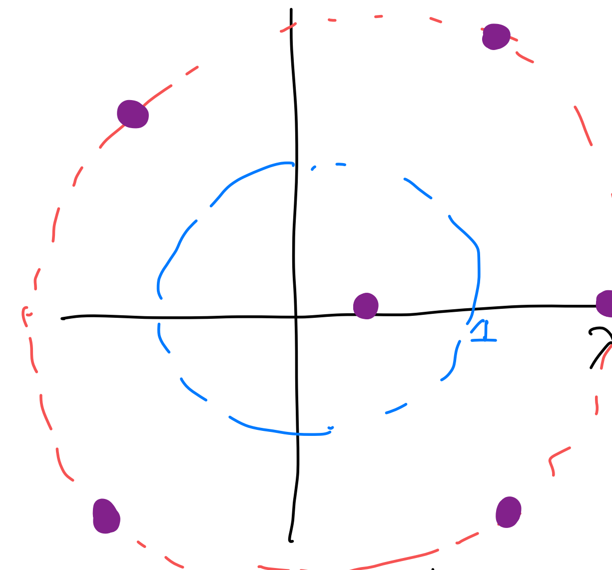
λ is a Pisot # | λ is Perron # | λ is a weak Perron #



norm of all Galois conjugates < 1



norm of all Galois conjugates $< |\lambda|$

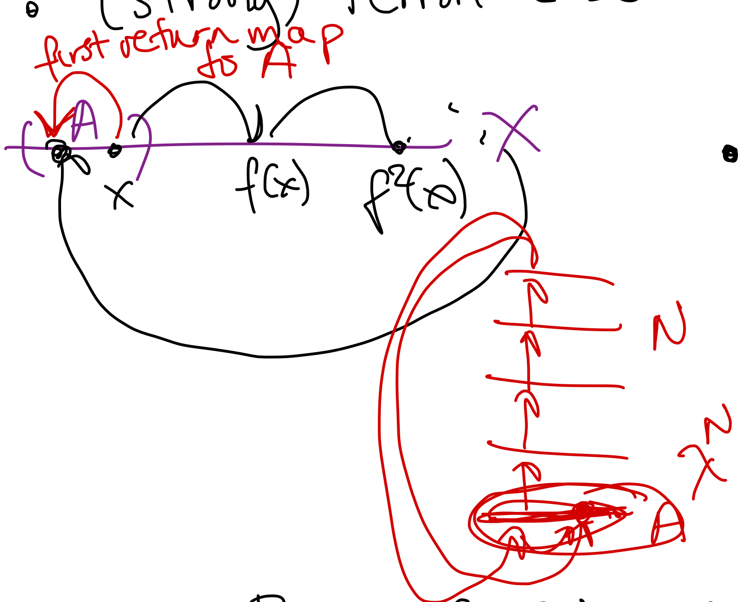


norm of all Galois conjugates $\leq |\lambda|$

- Pisot case: will discuss

  Thm: For any Pisot # λ, every uniform λ-expander whose critical points and critical values are all in $\mathbb{Q}(\lambda)$ is PCF.

- (strong) Perron case:
  - construct an incidence matrix that has eigenvalue λ^N for some large N.
  - Construct a map (basically a rotation of N pieces) so that the incidence matrix describes the dynamics of the first return map to one of the pieces.



- weak Perron case:
  - similar, uses fact that λ is weak Perron $\Longleftrightarrow$ some power λ^N is (strong) Perron.

Background: geometry of a number field $\mathbb{Q}(\lambda)$, λ an alg. integer.

- $\mathbb{Q}(\lambda)$ is a $\mathbb{Q}$ vector field with basis $\{1, \lambda, \dots, \lambda^{n-1}\}$ where n = degree of minimal polynomial for λ.

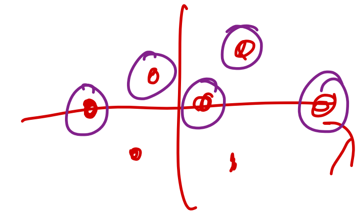
  i.e. $\mathbb{Q}(\lambda)$ consists of all things like

  $$a_0 + a_1\lambda + a_2\lambda^2 + \dots + a_{n-1}\lambda^{n-1}, \quad a_i \in \mathbb{Q}.$$

- The ring of integers $\mathcal{O}_{\mathbb{Q}(\lambda)}$ = elements of $\mathbb{Q}(\lambda)$ that are solutions to monic polys with $\mathbb{Z}$ coeffs.

  In particular, sums of form $b_0 + b_1\lambda + \dots + b_{n-1}\lambda^{n-1}$, $b_i \in \mathbb{Z}$, are in $\mathcal{O}_{\mathbb{Q}(\lambda)}$.

Geometry of a number field $\mathbb{Q}(\lambda)$



A neat result: Fix any alg. integer λ.

Let $\lambda_{(1)}, \dots, \lambda_{(r)}$ be the real Galois conjugates of λ

Let $\lambda_{(r+1)}, \dots, \lambda_{(r+s)}$ be one of each pair of complex conjugates that are Galois conjugates of λ.

Define $\Phi : \mathcal{O}_{\mathbb{Q}(\lambda)} \to \mathbb{R}^r \times \mathbb{C}^s \xrightarrow{\text{by}} \mathbb{R}^{r+2s}$ $\mathbb{C} = \mathbb{R} \times \mathbb{R}$

$$x \mapsto (\tau_1(x), \dots, \tau_{r+s}(x))$$

where $\tau_i : \mathcal{O}_{\mathbb{Q}(\lambda)} \to \mathbb{C}$ is the (Galois conjugation) map that replaces λ with $\lambda^{(i)}$

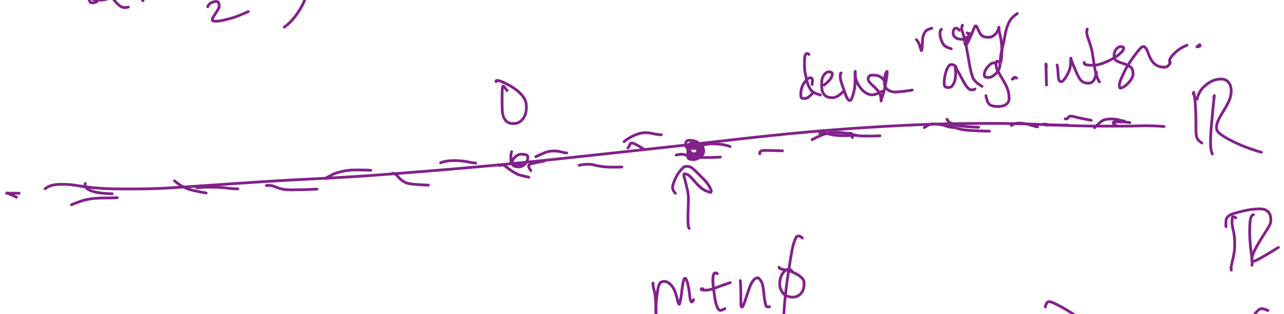
i.e. $\tau_i(a_0 + a_1\lambda + \dots + a_{n-1}\lambda^{n-1}) = a_0 + a_1\lambda_{(i)} + \dots + a_{n-1}\lambda_{(i)}^{n-1}$

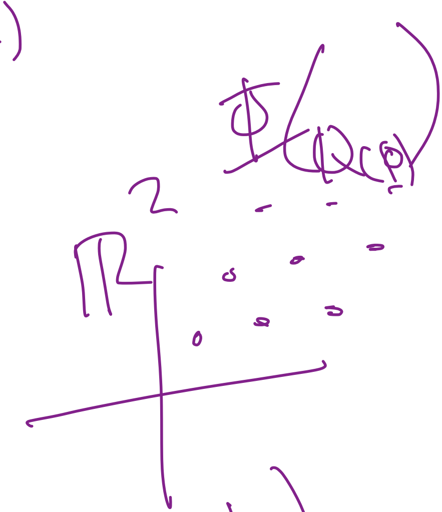
Then Φ is injective and $\Phi(\mathcal{O}_{\mathbb{Q}(\lambda)})$ is a lattice. $\to$ discrete additive subgroup. #gens = dim.

$\Longrightarrow$ A closed ball around the origin in $\mathbb{R}^r \times \mathbb{C}^s$ contains finitely many points of $\Phi(\mathcal{O}_{\mathbb{Q}(\lambda)})$!

$\phi_1 = \frac{1+\sqrt{5}}{2}$ Galois conj. $\frac{1-\sqrt{5}}{2} = \phi_{(2)}$

$\mathcal{O}_{\mathbb{Q}\left(\frac{1+\sqrt{2}}{2}\right)} = \{ m + n \cdot \phi : m, n \in \mathbb{Z} \}.$



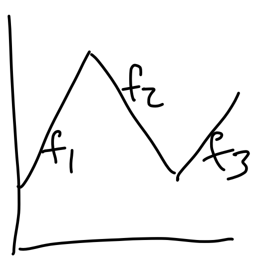


$\mathbb{R}^2$

$\Phi(m+n\phi) = (m+n\phi_1, m+n\phi_2)$

Claim: Let λ be Pisot. Let $f:[0,1]\to[0,1]$ be a uniform λ-expander whose critical points and crit values are in $\mathbb{Q}(\lambda)$.
Then f is PCF.

Thurston's proof: WLOG, we may assume all critical pts/values in $\mathbb{Z}[\lambda]$.
(Scale $[0,1]$ by an integer to clear denominators.).
Now, all pieces of f have the form $f_i(x) = a_i \pm \lambda x$ for some $a_i \in \mathbb{Z}[\lambda]$.



Let $\lambda_{(1)},\dots,\lambda_{(r)}$ be the real Galois conjs of λ
$\lambda_{(r+1)},\dots,\lambda_{(r+s)}$ be one of each pair of complex conj Galois conjs.
For each $\lambda_{(\alpha)}$ define $f_i^\alpha:\mathbb{C}\to\mathbb{C}$ by $f_i^\alpha(x) = \tau_{(\alpha)}(a_i) \pm \lambda_{(\alpha)} x$

Let z be a critical pt. The orbit of z under f is given by some sequence of compositions $f_{i_n}\circ\dots\circ f_{i_1}(z)$.
"Lift" this to an orbit in $\Phi(\mathcal{O}_{\mathbb{Q}(\lambda)}) \subset \mathbb{R}^r\times\mathbb{C}^s$, so you get the sequence of pts

$$\left(f_{i_n}^{(1)} \circ \dots \circ f_{i_1}^{(1)}(\tau_1(z)),\ f_{i_n}^{(2)} \circ \dots \circ f_{i_1}^{(2)}(\tau_2(z)),\ \dots,\ f_{i_n}^{(r+s)} \circ \dots \circ f_{i_1}^{(r+s)}(\tau_{r+s}(z)) \right)$$

i.e. in each coordinate you do the "same" sequence of maps, but using the appropriate Galois conjugate of λ

Key observation: for all $\lambda_{(\alpha)}$ except $\lambda_1 = \lambda$, all $f_i^{(\alpha)}$ maps are contractions.

For $\lambda_{(1)} = \lambda$ the orbit of z is bounded (since f is a self-map of an interval).

Therefore, the orbit of $\Phi(z)$ under the "lift" stays in some <u>bounded</u> subset of $\mathrm{Image}(\Phi) \subset \mathbb{R}^r \times \mathbb{C}^s$.

Since $\mathrm{Image}(\Phi)$ is a lattice, this means the orbit of $\Phi(z)$ under the "lift" hits only finitely many points.

$\therefore$ The orbit of z under f hits only finitely many points

$\therefore$ f is PCF.

Thank you!