Rigidity of Geodesic Planes in Hyperbolic Manifolds

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Main Theorem

1. $G = \mathrm{PSL}(2, \mathbb{C})$, $H = \mathrm{PSL}(2, \mathbb{R})$.
2. $\Gamma < G$ a discrete cocompact subgroup.

Theorem (Ratner-Shah)

Every orbit of H on G/Γ is either closed or dense.

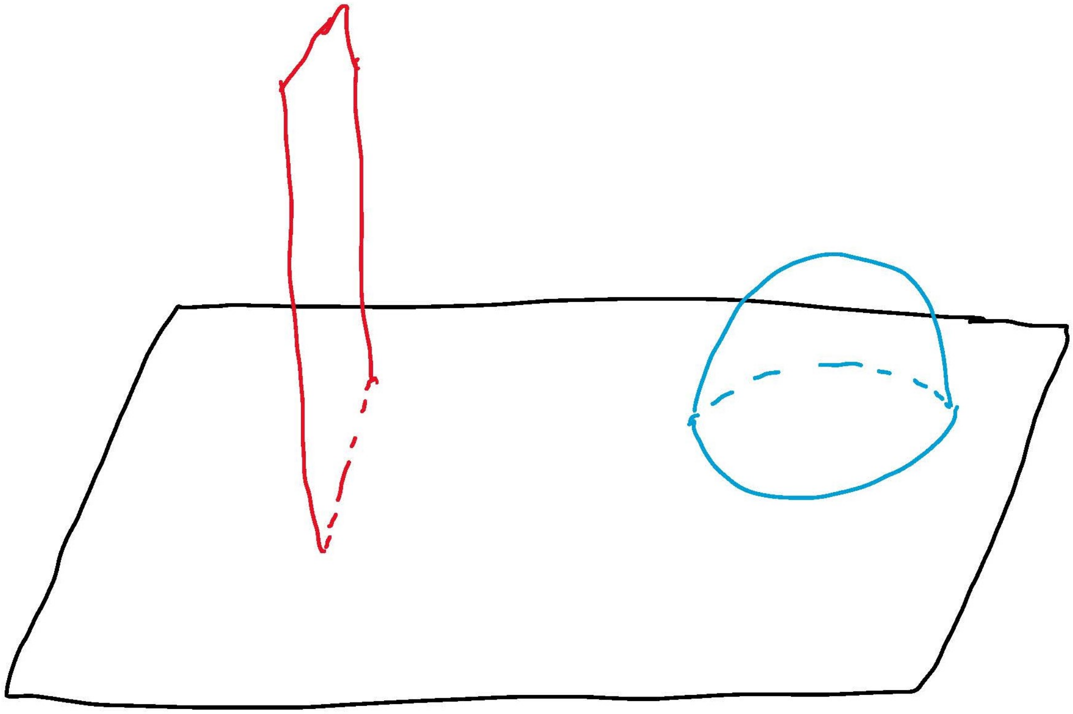
Geometry

- $K = \mathrm{PU}(2)$ maximal compact in G, $K \backslash G \cong \mathbb{H}^3$.
- $G = \mathrm{Isom}^+(\mathbb{H}^3)$.
- $\partial \mathbb{H}^3 \cong \mathbb{S}^2 \cong \mathbb{C} \cup \{\infty\} =: \hat{\mathbb{C}}$.
- $G \curvearrowright \partial \mathbb{H}^3$ by Mobiüs transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

Geometry

- $K\backslash Hg \longleftrightarrow$ the copy of $\mathbb{H}^2$ inside $\mathbb{H}^3$ whose boundary is $g \cdot \hat{\mathbb{R}} \subset \hat{\mathbb{C}}$.



Geomtric Reformulation

Theorem

Suppose P is a copy of $\mathbb{H}^2$ inside $\mathbb{H}^3$. Let $\pi : \mathbb{H}^3 \to \mathbb{H}^3/\Gamma$ be the quotient map. Then, $\pi(P)$ is either closed or dense.

Important 1-parameter subgroups

- $$A = \left\{ g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \right\}$$
- $$U = \left\{ u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : s \in \mathbb{R} \right\}, \qquad V = \left\{ v_s = u_{s\sqrt{-1}} : s \in \mathbb{R} \right\}.$$
- $N = UV$.
- $g_t u_s v_r g_{-t} = u_{e^t s} v_{e^t r}$.

P acts minimally

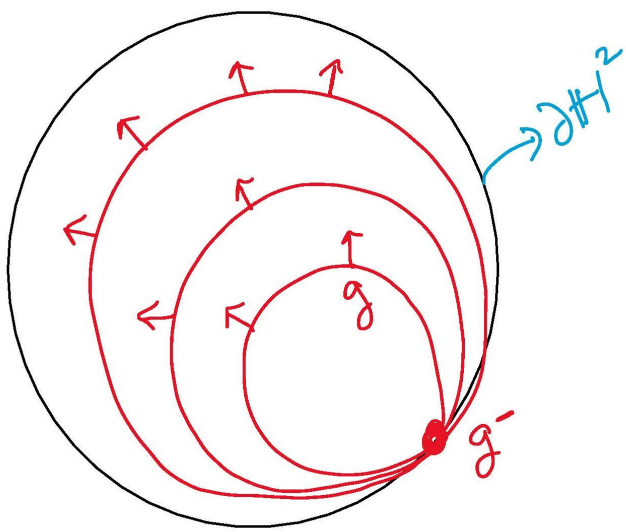
Lemma

Every orbit of the upper triangular group AN is dense.

- Since H contains AU, it remains to show that the closure of a non-closed orbit contains a V orbit.

P acts minimally - a picture

- Every Γ orbit on the boundary is dense + picture below.



P_g = all Circles tangent at g_∞ along with outward pointing $\perp$ vectors.

Producing additional invariance

- Let $x \in G/\Gamma$ and let $X = \overline{Hx}$.
- We can find $x_n \in X$ converging to x transversely; i.e.
  1. $x_n = g_n x$, $g_n \in G - H$.
  2. $g_n \to e$.
- Let's look at what happens to the unipotent orbits Ux_n and Ux (both contained in X, since $U \subset H$).

Principle

Unipotent orbits diverge primarily along the centralizer of the unipotent flow.

Polynomial Divergence

$$u_s x_n = (u_s g_n u_s^{-1}) \cdot u_s x.$$

- The deviation of $u_s x_n$ from $u_s x$ is measured by $u_s g_n u_s^{-1}$.
- From topology to linear algebra:

$$\underbrace{\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}}_{u_s} \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{g_n} \underbrace{\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}}_{u_s^{-1}} = \begin{pmatrix} a + sc & b + (d-a)s - cs^2 \\ c & d - sc \end{pmatrix}$$

Polynomial divergence $\Longrightarrow$ additional invariance

$$\underbrace{\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}}_{u_s} \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{g_n} \underbrace{\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}}_{u_s^{-1}} = \begin{pmatrix} a + sc & b + (d-a)s - cs^2 \\ c & d - sc \end{pmatrix}$$

$$p(s) = b + (d-a)s - cs^2.$$

- Remember: $g_n \notin H \Longrightarrow \mathrm{Im}(p(s)) \not\equiv 0$.
- Remember $g_n \to e$ so $a, d \to 1$ and $b, c \to 0$.
- p is a polynomial $\Longrightarrow$ one can pick s so that $a + sc, d - sc, \mathrm{Im}(p(s)) \asymp 1$.

Producing an orbit of V

- Taking suitable limits helps produce $y \in X$ and $v \in V - \{e\}$ so that $yv \in X$.
- To produce a whole orbit of V, we need the idea of minimal sets:

Definition

Suppose F is a group acting on a metric space X. Then, $Y \subseteq X$ is an F-**minimal set** if Y is closed, F-invariant and every F-orbit in Y is dense in Y.

- Minimal sets exist by Zorn's lemma and compactness.

Proof - reductions

- $X = \overline{Hx}$ is not a closed orbit. $Y \subset X$ is H-minimal and $Z \subset Y$ is U-minimal.
- Fix $z \in Z$ and a small neighborhood Ω of e in G.
- If $\Omega \cdot z \cap Y \subseteq Hz$, then Hz is open in Y (by homogeneity).
- So $Y \setminus Hz$ is closed and H-invariant. But, Y is H-minimal!
- There is $g \in \Omega$, $g \notin H$: $gz \in Y$.

- $\mathrm{Lie}(G) = \mathrm{Lie}(H) \oplus \sqrt{-1}\mathrm{Lie}(H)$.
- If g is small, then $g = h \exp(iq)$, with $h \in H$, $q \in Lie(H)$.
- Y is H-invariant $\Longrightarrow \exp(iq)z \in Y$. May assume $g = \exp(iq)$.

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Proof - reductions

By taking Ω arbitrarily small, we proved

Lemma

There exists $q_n \in \mathrm{Lie}(H)$, $q_n \to 0$, $\exp(iq_n)z \in Y$.

Let us write:

$$q_n = \begin{pmatrix} a_n & b_n \\ c_n & -a_n \end{pmatrix}.$$

Polynomials are important

$$u_s q_n u_s^{-1} = \begin{pmatrix} a_n + sc_n & b_n - 2a_n s - c_n s^2 \\ c_n & -a_n - sc_n \end{pmatrix}$$

$$p_n(s) = b_n - 2a_n s - c_n s^2, \qquad \ell_n(s) = a_n + sc_n.$$

Lemma

If either $a_n \neq 0$ or $c_n \neq 0$ for infinitely many n, then there is $s_n \to \infty$ so that $p_n(s_n) \to \xi \neq 0$ and $\ell_n(s_n) \to 0$ along a subsequence.

Proof.

Case: $c_n a_n \neq 0$, and $a_n/\sqrt{|c_n|} \to 0$. Take $s_n = 1/\sqrt{|c_n|}$.
Case: $c_n a_n \neq 0$, and $a_n/\sqrt{|c_n|} \not\to 0$. Take $s_n = \varepsilon_n a_n / c_n$.
Case: $c_n a_n = 0$, easy.

□

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Case: $|a_n| + |c_n| \neq 0$

Let's promote the previous lemma:

Lemma

If either $a_n \neq 0$ or $c_n \neq 0$ for infinitely many n, then $p_n(s_n t) \to \xi(t)$ and $\ell_n(s_n t) \to 0$ along a subsequence, where $\xi(t)$ is a non-constant polynomial and s_n is as in the previous lemma.

Polynomial divergence + minimal sets = done

1. Suppose $a_n \neq 0$ or $c_n \neq 0$ for infinitely many n.
2. Let $\xi(t)$ be the limiting polynomial from previous lemma. So, $u_{s_n t} \exp(iq_n) u_{s_n t}^{-1} \to v_{\xi(t)}$.
3. Let $z' \in Z$ be limit of $u_{s_n t} z$ along a subsequence:

$$u_{s_n t} \exp(iq_n) z = u_{s_n t} \exp(iq_n) u_{s_n t}^{-1} \cdot u_{s_n t} z \to v_{\xi(t)} z'.$$

4. Remember U and V commute and Z is U-minimal.

$$v_{\xi(t)} Z = v_{\xi(t)} \overline{U z'} = \overline{U v_{\xi(t)} z'} \subset Y.$$

5. Since this works for all t and $\xi(t)$ is non-constant ($+\epsilon$ more work):

$$VZ \subset Y,$$

i.e. the orbit closure contains an orbit of V.

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The case $a_n = c_n = 0$ is impossible

If $x = g\Gamma$, then the stabilizer $G_x = g\Gamma g^{-1}$.

Lemma

If $a_n = c_n = 0$ for all n, then there is a point z' whose stabilizer $G_{z'}$ inside G contains an upper triangular matrix with diagonal entries arbitrarily close to 1.

1. Since G/Γ is compact, there is $\Omega \subset G$ a neighborhood of identity: $G_x \cap \Omega = \{e\}$ for **all** x.
   - Γ has a compact fundamental domain $D \subset G$ so elements of D cannot contract Γ too much.
2. For all g and x, $G_{gx} = gG_xg^{-1}$.

The case $a_n = c_n = 0$ is impossible

1. If G_x contains an element of the form an, where a is diagonal and close to identity, and n is strictly upper triangular, then

$$ag_{-t}ng_{-t} \in G_{g_{-t}x}$$

2. But, $g_{-t}ng_t \to e$ as $t \to \infty$. Contradiction.

Producing upper triangular stabilizers

Principle

Recurrence of unipotent orbits along the normalizer is very special.

- If $a_n = c_n = 0$ for all n, this means that for all g close to identity, $gz \in Y$, then $g \in HV$.
- Also, it means there is a sequence $v_n \to e$ in V: $v_n z \in Y$.
- Z is U-minimal implies can find $u_n \in U$ with $u_n \to \infty$ and $u_n z \to z$.
- Fix one such u, $uz = hvz$, $h \in H$, $v \in V$ are small.
- But,
$$Y \ni uv_n z = v_n uz = v_n hvz.$$
- If v_n, h, v are small enough, then $v_n hvz \in \Omega z$, so $v_n hv = h'v'$.

Producing upper triangular stabilizers

- For $\varepsilon_n \neq 0$:

$$v_n = \begin{pmatrix} 1 & i\varepsilon_n \\ 0 & 1 \end{pmatrix}, \qquad vv'^{-1} = \begin{pmatrix} 1 & i\delta \\ 0 & 1 \end{pmatrix}.$$

- h is small so $h = \exp(q)$, $q = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$:

$$v_n q v v'^{-1} = \begin{pmatrix} a + i\varepsilon_n c & i(\delta - \varepsilon_n)a - \delta\varepsilon c + b \\ c & i\delta c - a \end{pmatrix}$$

- The only way this can be a real matrix is if $c = 0$.
- That means: $uz = hvz \Longrightarrow u^{-1}hvz = z$ and everything in sight is upper triangular!

Historical Remarks

- The above proof works for classification of closures of codimension 1 totally geodesic hyperplanes inside hyperbolic manifolds of higher dimensions. Compactness is not necessary, finite volume is enough.
- Ratner's proof is not topological and uses her classification of invariant measures of unipotent flows.
- Shah gave a topological proof in all codimensions.
- Very recently, McMullen, Mohammadi and Oh proved similar results for certain classes of infinite volume hyperbolic manifolds.

Thanks!