Let $X = \mathsf{SL}(2,\mathbb{R})/\mathsf{SL}(2,\mathbb{Z})$ and $\mathbf{m}_X$ denote Haar measure on X.

Theorem 1. *Let μ be a N-ergodic and invariant Borel probability measure on X. Either N is $\mathbf{m}_X$ or it is supported on a periodic N orbit. Moreover if x is not a u periodic point then $\overline{\{\mathsf{u}^t x\}} = X$.*

1. Idea of proof

Lets try to show that

$$\frac{1}{T}\int_0^T f(\mathsf{u}^t x)dt \sim \int f d\mathbf{m}_X \tag{1}$$

under some assumptions to come later on x, T and f.

We will try to relate it to

$$\frac{1}{T\mathbf{m}_X(A)}\int_A \int_0^T f(\mathsf{u}^t y)dtdy \tag{2}$$

for $A \subset X$ with $\mathbf{m}_X(A) > 0$. This is good, because since $\mathbf{m}_X$ is N ergodic, for all $\epsilon > 0$ and $f \in L^1(\mathbf{m}_X)$ there exists T_0 so that for every set B with $\mathbf{m}_X(B) > \epsilon$ we have

$$\Big|\frac{1}{T\mathbf{m}_X(B)}\int_B \int_0^T f(\mathsf{u}^t y)dtdy - \int f d\mathbf{m}_X\Big|. \tag{3}$$

Exercise 1. Prove this. Hint: Use the Birkhoff ergodic theorem.

It is natural to replace (2) by

$$\frac{1}{T}\frac{1}{abc}\int_0^T \int_0^c \int_0^b \int_0^a f(\mathsf{u}^t\mathsf{v}^\ell\mathsf{g}^r\mathsf{u}^s x)dsdrd\ell dt. \tag{4}$$

This leads to two issues:

Issue 1: $(\ell, r, s) \to \mathsf{v}^\ell\mathsf{g}^r\mathsf{u}^s x$ should be injective for $(\ell, r, s) \in [0, c] \times [0, b] \times [0, a]$.

-This is easy to arrange. Because there is some non-zero injectivity radius at x there is some $c, b, a > 0$ so that $(\ell, r, s) \to \mathsf{v}^\ell\mathsf{g}^r\mathsf{u}^s x$ is injective for $(\ell, r, s) \in [0, c] \times [0, b] \times [0, a]$. This will eat up positive measure say δ and we can choose $T > T_0$ for $\epsilon = \delta$ as in (3).

Issue 2: Why on earth should (4) be close to $\frac{1}{T}\int_0^T f(u_t x)dt$?

-We can choose c small enough depending on T so that

$$\mathsf{D}(\mathsf{u}^t\mathsf{v}^\ell\mathsf{g}^r\mathsf{u}^s, \mathsf{g}^r\mathsf{u}^{e^{-2r}t}\mathsf{u}^s) < \epsilon \tag{5}$$

for all $0 \le \ell \le c$ and $0 \le t \le T$. Now the g^r is less of an issue, because by our assumptions on D, $\mathsf{D}(\mathsf{g}^r\mathsf{u}^{e^{-2r}t}\mathsf{u}^s, \mathsf{u}^{e^{-2r}t}\mathsf{u}^s) \le |r|$. This makes it natural to request f is 1-Lipschitz.

Now this presents another issue. T depends on the measure of

$$\bigcup_{(\ell,r,s)\in[0,c]\times[0,b]\times[0,a]} \mathsf{v}^\ell \mathsf{g}^r \mathsf{u}^s x$$

so changing c requires changing T which will then require changing c again...

So how will we make c small while keeping the measure the same?

Observe that

$$\bigcup_{(\ell,r,s)\in[0,c]\times[0,b]\times[0,a]} \mathsf{g}^d \mathsf{v}^\ell \mathsf{g}^r \mathsf{u}^s \mathsf{g}^{-d} x = \bigcup_{(\ell,r,s)\in[0,e^{-2d}c]\times[0,b]\times[0,e^{2d}a]} \mathsf{v}^\ell \mathsf{g}^r \mathsf{u}^s x.$$

So we can make c smaller by conjugating by g^d. Now g^d is a homeomorphism of X that preserves $\mathbf{m}_X$ so we just need that $(\ell, r, s) \to \mathsf{v}^\ell \mathsf{g}^r \mathsf{u}^s \mathsf{g}^{-d} x$ is injective for $(\ell, r, s) \in [0, c] \times [0, b] \times [0, a]$. This requires:

Lemma 1. *For all $\epsilon > 0$ there exists $c, b, d > 0$ so that for all $x \in \mathcal{K}_\epsilon$ so that $\mathsf{v}^\alpha \mathsf{g}^\beta \mathsf{u}^s x \neq \mathsf{v}^{\alpha'} \mathsf{g}^{\beta'} \mathsf{u}^{s'} x$ for all $(\alpha, \beta, s) \neq (\alpha', \beta', s')$ with $0 \leq \alpha < a, 0 \leq \beta < b, 0 \leq s < d$.*

Where $\mathcal{K}_\epsilon = \{x \in X :$ the shortest vector in the lattice x has length at least $\epsilon\}$ and it is compact for all $\epsilon > 0$. Moreover,

Lemma 2. *There exists ϵ_0 so that for all x that do not contain a horizontal vector, $\mathsf{g}^{-t} x \in \mathcal{K}_{\epsilon_0}$ for arbitrarily large t.*

Recall: x is N-periodic iff x thought of as a lattice x has a horizontal vector. So the assumption on x in the Lemma is natural to make.

1.1. **Round up:** Given ϵ_0 as in Lemma 1, choose $c, a > 0$ so that $\phi_z : [0, c] \times [0, \delta] \times [0, a]$ given by $(\ell, r, s) \to \mathsf{v}^\ell \mathsf{g}^r \mathsf{u}^s z$ is injective for each $z \in \mathcal{K}_{\epsilon_0}$. Let ϵ be the measure of the image of ϕ_z and choose T_0 for this ϵ. Now choose c' so that (5) holds for all $0 \leq \ell < c'$ and $0 \leq T \leq T_0$. We now choose d so that $g_{-d} x \in \mathcal{K}_{\epsilon_0}$ and $e^{-2d} c < c'$ which we can do by Lemma 1. By our choices

$$(6)\quad \left|\frac{1}{T}\frac{1}{a\delta c}\int_0^T \int_0^c \int_0^\delta \int_0^a f(\mathsf{u}^t \mathsf{g}^d \mathsf{v}^\ell \mathsf{g}^r \mathsf{u}^s \mathsf{g}^{-d} x) ds dr d\ell dt - \int f d\mathbf{m}_X\right| < \epsilon.$$

Now this is equivalent to

$$\left|\frac{1}{T}\frac{1}{a\delta c'}\int_0^T \int_0^{e^{-2d}c} \int_0^\delta \int_0^{e^{2d}a} f(\mathsf{u}^t \mathsf{v}^\ell \mathsf{g}^r \mathsf{u}^s x) ds dr d\ell dt - \int f d\mathbf{m}_X\right| < \epsilon.$$

Now

$$\left|\frac{1}{T}\frac{1}{a\delta c}\int_0^T \int_0^c \int_0^\delta \int_0^a f(\mathsf{u}^t \mathsf{v}^\ell \mathsf{g}^r \mathsf{u}^s x) ds dr d\ell dt - \frac{1}{Te^{2d}a}\int_0^T \int_0^{e^{2d}a} f(u_t u_s x) ds dt < 3\epsilon.$$

Now

$$\Big|\frac{1}{Te^{2d}a}\int_0^T\int_0^{e^{2d}af(u_tu_sx)} - \frac{1}{T+e^{2d}a}\int_0^{T+e^{2d}a} f(u_tx)dt < \frac{2T}{T+e^{2d}a}.$$

I have left out one issue: the distortion of the volume in our coordinates given by changing points in our compact set. This is not serious because g^d preserves the measure and u^t acts outside the set of positive measure in (3). One minor thing is we want to choose

$$\epsilon = \inf_{z\in\mathcal{K}_{\epsilon_0}} \mathbf{m}_X\big(\phi_z([0,c]\times[0,\delta]\times[0,a])\big).$$

One more significant thing is we need the following lemma:

Lemma 3. *Let $\phi:\mathbb{R}^3\to SL(2,\mathbb{R})$ by $\phi(\alpha,\beta,\gamma)\to \mathsf{v}^\alpha\mathsf{g}^\beta\mathsf{u}^\gamma$. There exists $\sigma>0,\delta>0$ so that $\frac{d\phi_*\lambda^3}{d\mathbf{m}_G}(\alpha,\beta,\gamma)\in[\sigma,2\sigma]$ for all $|\alpha|,|\beta|,|\gamma|<\delta$.*

Using this lemma, we can adapt the proof of Exercise 1 to obtain our result. (This is needed for (6).)

2. What we showed

Proposition 1. *If x has no horizontal vector and $f\in C_c(X)$ is 1-Lipshits then*

$$\liminf_{T\to\infty}\Big|\frac{1}{T}\int_0^T f(\mathsf{u}^tx)dt - \int f d\mathbf{m}_X\Big| = 0.$$

Proof of Theorem 1 assuming Proposition 1. Assume μ is a N ergodic invariant probability measure that is not Haar. Then there exists x a generic point for μ along N and $f\in C_c(X)$ so that $\int f d\mu \neq \int f d\mathbf{m}_X$. Since Lipshitz functions are dense in $C_c(x)$ with supremum norm and therefore L^1 norm, we may assume f is Lipshitz. We may scale f to be 1-Lipshitz and observe that since $\int cf d\mu\neq\int cf d\mathbf{m}_X$ the proposition implies x must have a horizontal vector. This establishes the measure classification part of the Theorem.

Now if x is not supported on a periodic N orbit, it does not have a horizontal vector. If $U\subset X$ is open, there exists $f\in C_c(X)$, 1-Lipschitz so that the support of f is a non empty set in U. By the assumption of the proposition f there exists a sequence $T_1,\ldots$ so that

$$\lim_{i\to\infty}\frac{1}{T_i}\int_0^{T_i} f(\mathsf{u}^tx)dt = \int f d\mathbf{m}_X$$

and so $\mathsf{u}^tx\in U$ for some t. Since U was an arbitrary open set, we have established the measure classification part of the theorem. □