Consider $SL(2,\mathbb{R})$. It is a locally compact, topological group. It has 3 important subgroups: $v_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$, $u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ and $g_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}$.

We have the following relations:

$$g_t v_s = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ e^{-\frac{t}{2}} s & e^{-\frac{t}{2}} \end{pmatrix} = v_{e^{-t}s} g_t$$

and

$$g_t u_s = \begin{pmatrix} e^{\frac{t}{2}} & e^{\frac{t}{2}} s \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} = u_{e^t s} g_t.$$

Also, there exists a locally finite measure μ on $SL(2,\mathbb{R})$, unique up to scaling so that μ is invariant for the left (in fact both left and right) action of $SL(2,\mathbb{R})$.

Let Γ be a cocompact lattice in $SL(2, \mathbb{R})$.
That is, Γ is a discrete subgroup of $SL(2, \mathbb{R})$ so that $X := SL(2, \mathbb{R})/\Gamma$ is compact. Note that $SL(2, \mathbb{R})$ acts continuously on X from the left.

We can choose $D \subset SL(2, \mathbb{R})$, a fundamental domain for the Γ action and because $SL(2, \mathbb{R})/\Gamma$ is compact D can be chosen to have compact closure. Let be the measure on X defined by $\mu_X(A) = \mu(\{g \in D : g\Gamma \in A \neq \emptyset\})$. Note that $SL(2, \mathbb{R})$ preserves μ_X.

Theorem

(Furstenberg) v_s *is* μ_X *uniquely ergodic on* X.

Because invariant measures are the convex hull of ergodic measures, it suffices to show that v_s is uniquely ergodic.

Let $f \in C(X)$ and $M_t : X \to \mathbb{C}$ by $M_t(x) = \int_0^1 f(g_{-\log(t)}(v_s x))ds$.

Lemma

$\{M_t\}_{t \in \mathbb{R}^+}$ is an equicontinuous family of functions. That is, for all $\epsilon > 0$ exists $\delta > 0$ so that for all t if $d(x,y) < \delta$ then $|M_t(x) - M_t(y)| < \epsilon$.

$\bullet |M_t(x) - M_t(v_\ell x)| \leq 2\ell \|f\|_{\sup}$ for all ℓ, x.

–Indeed is $0 \leq \ell \leq 1$,

$|M_t(x) - M_t(v_\ell x)| = |\int_0^1 f(g_{-\log(t)} v_s x) - \int_0^1 f(g_{-\log(t)} v_{\ell+s} x) ds|$

$= |\int_0^\ell f(g_{-\log(t)} v_s x) - \int_1^{1+\ell} f(g_{-\log(t)} v_s x) ds|$.

•For all $\epsilon > 0$ there exists $\delta > 0$ so that $|M_t(x) - M_t(g_\ell x)| < \epsilon$ for all $0 \le \ell \le \delta$.
As before we expand M_t
$|M_t(x) - M_t(g_\ell x)| = |\int_0^1 (f(g_{-\log(t)} v_s x) ds - \int_0^1 f(g_{-\log(t)} v_r g_\ell x)) dr|$
$= |\int_0^1 (f(g_{-\log(t)} v_s x) ds - \int_0^1 f(g_\ell g_{-\log(t)} v_{re^\ell} x)) dr| =$
$|\int_0^1 \Big(f(g_{-\log(t)} v_s x) - e^{-\ell} f(g_\ell g_{-\log(t)} v_s x) \Big) ds -$
$\int_1^{e^\ell} e^{-\ell} f(g_{-\log(t)} v_s x) ds|$.
Now, because f is uniformly continuous we have that there exists $\delta > 0$ so that $|f(z) - f(g_\ell z)| < \epsilon$ for all $0 \le \ell \le \delta$ and $z \in X$.
If $0 \le \ell \le \delta$ then

$$|M_t(x) - M_t(g_\ell x)| < |\int_0^1 (e^{-\ell}\epsilon + (1 - e^{-\ell} \|f\|_{\sup})) ds + (e^\ell - 1) e^{-\ell} \|f\|_{\sup}.$$

$-\delta \le \ell \le 0$ is similar and we have the claim.

•For all $\epsilon > 0$ there exists $\delta > 0$ so that $|M_t(x) - M_t(u_\ell x)| < \epsilon$ for all $0 \leq \ell \leq \delta$.
We have:

$$v_s u_\ell = u_{\frac{\ell}{1+s\ell}} g_{-2\log(1+s\ell)} v_{\frac{s}{1+s\ell}}$$

–Applying $g_{-\log(t)}$ we get

$$M_t(u_\ell x) = \int_0^1 u_{\frac{\ell}{t(1+s\ell)}} g_{-\log(t)} g_{-2\log(1+s\ell)} v_{\frac{s}{1+s\ell}} ds.$$

By similar methods to above, we have the claim
•As $\{u_a g_b v_c x : -\delta < a, b, c < \delta\}$ is a neighborhood of x, the triangle inequality implies the lemma.

More conceptually

If $\alpha \in SL(2,\mathbb{R})$ is a close to the identity then there exists

- ϕ, C^1 close to the identity function $h(x) = x$.
- $\psi : [0,1) \to U_s G_\ell$, C^0 close the constant function sending everything to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$v_s\alpha = \psi(s)v_{\phi(s)}$. Because $g_{-\log(t)}u_a g_b = u_{\frac{a}{t}} g_b g_{-\log(t)}$, the equicontinuity of $\{M_t\}_{t\in\mathbb{R}^+}$ follows.

Lemma

(*)Any limit of M_{t_i}, h, is a v_s invariant function.

- $M_t = \frac{1}{t}\int_0^t f(v_s g_{-\log(t)} x) ds$ and so
  $M_{t_i} \circ g_{\log(t_i)} = \frac{1}{t_i}\int_0^{t_i} f(v_s x) ds$.
- So by the Von-Neumann ergodic theorem $M_{t_i} \circ g_{\log(t)}$ converges in L^2 norm to projection of f onto the v_s invariant functions, Pf.
- $\|M_{t_i} \circ g_{\log(t)} - Pf\|_2 = \|M_{t_i} - (Pf) \circ g_{-\log(t_i)}\|_2$ and so tends to zero.
- So $(Pf) \circ g_{-\log(t_i)}) \xrightarrow{L^2(\mu_X)} h$.
  –As $g_{-\log(t_i)} V_s g_{\log(t)} = V_s$ (as a set) $(Pf) \circ g_{-\log(t_i)}$ is v_s invariant like Pf is.
- So h is a limit of v_s invariant functions and thus it is v_s invariant as well.

Alternately

We want to show that any limit of M_{t_i} is a v_s invariant function. The previous proof relied on the fact that any limit of M_{t_i} is a $(L^2(\mu_X))$ limit of u_s invariant functions. Can we see this directly?

- $M_t(x) = \frac{1}{t}\int_0^t f(v_s g_{-\log(t)} x) ds$.
- Let $V(\epsilon, t) = \{z : |\frac{1}{t}\int_0^t f(v_s z) ds - \lim_{\ell \to \infty} \frac{1}{\ell}\int_0^\ell f(v_r x) dr| < \epsilon\}$.
- By the Birkhoff ergodic theorem and the Ergodic decomposition, $\mu_X(V(\epsilon, t)) \longrightarrow 1$.
- Let $H(x) = \lim_{t \to \infty} \frac{1}{\ell}\int_0^\ell f(v_r x) dr$.
  H is a v_s invariant function. It is Pf where P is projection onto the constants. $\|H \circ g_{-\log(t)} - M_t\|_2 \to 0$.

Proof of Furstenberg's Theorem

- Because $\{M_t\}$ is an equicontinuous family there exists $t_i \to \infty$ so that $M_{t_i} \to h$ in $\| \cdot \|_{\text{sup}}$.
- h is clearly continuous and by the previous lemma it is v_s invariant.
- Hedlund proved that every v_s orbit is dense and so h is constant.
- On the other have if there exist two v_s ergodic probability measures, ν_1, ν_2 then there exists a continuous function f so that $\int f d\nu_1 \neq \int f d\nu_2$.
- It is easy to see that for this f, M_t would never converge to a constant function.

Going further

Theorem

v_t is weakly mixing.

- If $f(v_s x) = e^{2\pi i \alpha s} f(x)$ then $f \circ g_\ell(v_s x) = f(v_{e^{-\ell}s} g_\ell x) = e^{2\pi \alpha e^{-\ell} s} f \circ g_\ell(x)$.
- So if we have a non-trivial eigenvalue we have uncountably many distinct eigenvalues.
- But eigenfunctions with different eigenvalues are orthogonal
- Since $L^2(\mu_X)$ is separable, this is a contradiction

Corollary

μ_X is v_ℓ ergodic as a $\mathbb{Z}$ dynamical system for all $\ell \neq 0$.

What about the time 1 map

Theorem

v_1 is μ_X uniquely ergodic.

- Let ν be a v_1 ergodic measure (as a $\mathbb{Z}$ system).
- $(v_s)_*\nu$ is too and as μ_X is v_s invariant, $(v_s)_*\nu = \mu_X$ iff $\nu = \mu_X$.
- $\int_0^1 (v_t)_*\nu dt$ is v_s invariant for all s.
- So it is μ_X.
- But since μ_X is an extreme point in the simplex of v_1 invariant measure, $(v_t)_* = \mu_X$ for almost every and thus every t.

Going further

Theorem

g_t is ergodic

Idea of proof:

- Let A be g_{-t} invariant set with $\mu_X(A) > 0$. There exists $\delta > 0$ so that
$$\mu_X(A \setminus v_s A) < \epsilon \mu_X(A)$$
for all $|s| < \delta$.
- Applying g_{-t} we have the same result for $|s| < e^t \delta$.
- Since ϵ is arbitrary we have $\mu_X(A \setminus v_s A) = 0$.
- Since v_s is ergodic, A has full measure.

Going further

What about $X' = SL(2,\mathbb{R})/SL(2,\mathbb{Z})$?

Theorem

$\mu_{X'}$ is the unique Borel probability measure ν, so that $\nu(\{x \in X' : g_t x \overset{t \to +\infty}{\longrightarrow} = \infty\}) = 0$.

Steps

Let $f \in C_c(X')$ and M_t be as above.

- $\{M_t\}_{t\in\mathbb{R}^+}$ with the topology of uniform convergence on compact sets is a precompact family of functions.
- Any limit is continuous and v_s invariant as above.
- As there is a dense orbit the limit is constant.
- If $g_t x \not\to \infty$ then there exists $z \in X$, $t_i \to \infty$ so that $g_{t_i} x = z$.
  –Passing to a further subsequence, we may assume that M_{t_i} converges.
  –$M_{t_i}(z)$ converges to $\frac{1}{t_k}\int_0^{t_k} f(v_s x)ds$.
  –So every point with a g_t limit is generic for $\mu_{X'}$.

Note that the complement of this set is exactly the periodic v_s orbits.