Let $\mathcal{A}$ denote a finite set. $\mathcal{A}^{\mathbb{N}}$ with the product topology is a compact metric space.
The left shift $L : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ by $L(\mathbf{x})_i = x_{i+1}$ is a continuous map.

We consider $X \subset \mathcal{A}^{\mathbb{N}}$ closed and so that $X = L(X)$.

Let

$$B_n(X) = \{(a_1, ..., a_n) \in \mathcal{A}^n : \exists \mathbf{x} \in X \text{ with } x_i = a_i \text{ for all } 1 \leq i \leq n\}.$$

Given $(a_1, ..., a_n) \in B_n(X)$ let
$C_{a_1,...,a_n} = \{\mathbf{x} \in X : x_i = a_i \text{ for all } i \leq n\}$.

Theorem

(Boshernitzan) Let $L : X \to X$ be a minimal, aperiodic shift dynamical system of linear block growth. That is, $|B_n(X)| < Cn$ for arbitrarily large n.

1. *(X, L) has at most $C - 1$ invariant ergodic Borel probability measures.*
2. *If there exists an L-invariant measure μ so that*

$$\limsup n \min\{\mu(C_{a_1,\ldots,a_n}) : a_1, \ldots, a_n \in B_n(X)\} > 0$$

   then (X, L) is uniquely ergodic.

The second result is called Boshernitzan's criterion for unique ergodicity and was proved by Veech for IETs (and Boshernitzan in general).

By Theorem 1, we may assume $\nu_1, ..., \nu_k$ are the ergodic invariant Borel probability measures of (X, L).
We assume that there is $a \in \mathcal{A}$ so that $\nu_1(C_a) < \nu_2(C_a) \leq \nu_i(C_a)$ for all $i > 2$.

The proof is based on combining two facts:

1: For every $\epsilon > 0$ there exists n_0 so that the μ-measure of the union of the cylinders of length n that have between

$$n\Big(\frac{2}{3}\nu_1(C_a) + \frac{1}{3}\nu_2(C_a)\Big) \text{ and } n\Big(\frac{1}{3}\nu_1(C_a) + \frac{2}{3}\nu_2(C_a)\Big)$$

occurrences of a is at most ϵ, for all $n \geq n_0$.

2: If $\gamma_n = \min\{\mu(C_{a_1,\dots,a_n}) : a_1, \dots, a_n \in B_n\}$ then the μ-measure of the union of the cylinders of length n that have between $\lceil n(\frac{2}{3}\nu_1(C_a) + \frac{1}{3}\nu_2(C_a))\rceil$ and $\lfloor n\big(\frac{1}{3}\nu_1(C_a) + \frac{2}{3}\nu_2(C_a)\big)\rfloor$ occurrences of a is at least

$$\lfloor n\Big(\frac{1}{3}\nu_2(C_a) - \frac{1}{3}\nu_1(C_a)\Big)\rfloor\gamma_n. \quad (*)$$

Proof of Theorem 2 assuming these two facts (and Theorem 1):
We are assuming

$$\limsup n \min\{\mu(C_{a_1,\dots,a_n}) : a_1, \dots, a_n \in B_n(X)\} > 0.$$

Given $\nu_2(C_a) > \nu_1(C_a)$, the lim sup of $(*)$ is positive.

This contradicts the first fact.

Our first fact comes from connecting the symbols and the dynamics.
By the ergodic theorem:

$$\nu_i(\{\mathbf{x} \in X : \frac{1}{n}|\{j \leq n : L^j(\mathbf{x}) \in C_a\}| \notin [\nu_i(C_a) - \epsilon, \nu_i(C_a) + \epsilon]\}) \overset{n\to\infty}{\longrightarrow} 0$$

The set of $\mathbf{x}$ is the union $(b_1, ..., b_n) \in B_n(X)$ so that $\{i \leq n : b_i = a\} \notin [n(\nu_i(C_a) - \epsilon), b(C_a + \epsilon)]$.

Completing the proof of the first fact:

- Because no ergodic measure for (X, L) ν_i satisfies

$$\nu_1(C_a) < \nu_i(C_a) < \nu_2(C_a),$$

  for each ν_i there exists n_i so that the ν_i-measure of the union of the cylinders of length n that have between

$$n(\frac{2}{3}\nu_1(C_a) + \frac{1}{3}\nu_2(C_a)) \text{ and } n(\frac{1}{3}\nu_1(C_a) + \frac{2}{3}\nu_2(C_a))$$

  occurrences of a
  is at most ϵ, for all $n \geq n_i$.
- Let n_0 be the maximum over all the n_i. Because any invariant measure (including μ) is a convex combination of ergodic measures, we have the first claim.

Our second fact:

- For each n, there exists $(a_1, ..., a_n) \in B_n(X)$ with at most $n\nu_1(C_a)$ occurrences of the letter a and there exists $(b_1, ..., b_n) \in B_n(X)$ with at least $n\nu_2(C_a)$ occurrences of the letter a.
- By minimality of (X, L) there is some m and word $(a_1, ..., a_n, w_1, ..., w_k, b_1, ..., b_n) \in B_m(X)$.
- Let $u_0, ..., u_{m-n}$ be the sequence of words $u_0 = (a_1, ..., a_n), u_1 = (a_2, ..., a_n, w_1), ..., u_{m-n-1} = (w_k, b_1, ..., b_{n-1}), u_{m-n} = (b_1, ..., b_n)$.
  Notice:
  - $u_i \in B_n(X)$ for all i.
  - The number of occurrences of a in u_{i+1} is at most one more than the number of occurrences of a in u_i.

Theorem

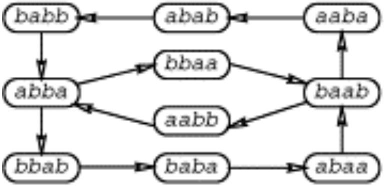
(Boshernitzan) Let $L : X \to X$ *be a minimal, aperiodic shift dynamical system of linear block growth. That is,* $|B_n(X)| < Cn$ *for arbitrarily large* n*.* (X, L) *has at most* $C - 1$ *invariant ergodic Borel probability measures.*

We will show that there are at most $2C$ ergodic measures.

Given X and $n \in \mathbb{N}$ we get a directed graph G_n.
Vertices$= B_n(X)$.
Two vertices $(a_1, ..., a_n), (b_1, ..., b_n)$ are connected if $a_i = b_{i-1}$ for $2 \leq i \leq n$ and $(a_1, a_2, ..., a_n, b_n) = (a_1, b_1, ..., b_n) \in B_{n+1}(X)$.

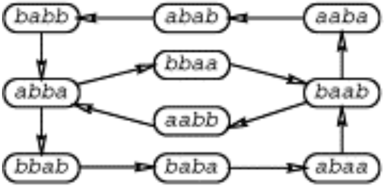
A Rauzy graph for the Thue-Morse system:

$X = \overline{\{L^i(a, b, b, a, b, a, a, b, b, a, a, b, a, b, b, a, ...)\}_{i \in \mathbb{N}}}$



Picture by Anna Frid

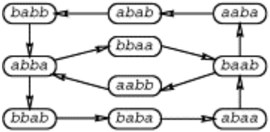
So $(a, a, b, a, b) \in B_5(X)$ but $(a, a, b, a, a) \notin B_5(X)$.



1. For any invariant measure μ, $\mu(C_{aaba}) = \mu(C_{abab}) = \mu(C_{babb})$.
2. Infinitely many n have G_n has at most $C - 1$ vertices with more than 1 outgoing edge and at most $C - 1$ vertices with more than 1 incoming edge.
   Because $|B_{n+1}| - |B_n| < C$ for infinitely many n.

Because ergodic measures are mutually singular, given any two ergodic measures, μ, ν for all large n we can find A, B

- disjoint clopen sets
- unions of n cylinders
- $\mu(A) > 1 - \epsilon$ and $\nu(A) < \epsilon$ and
- $\nu(B) > 1 - \epsilon$ and $\mu(B) < \epsilon$.



Lets call a “chain” a set of consecutive vertices so that each vertex has only one incoming and outgoing edge.
–The cylinders in each chain have the same measure for every invariant measure.

For any $\epsilon > 0$ and ergodic measure μ, ν there exists n_0 so that for all $n \geq n_0$, no in chain in G_n can have measure more than ϵ for both.

Moral: For any two ergodic measures, at most one of them can give definite weight to a chain that does not contain any vertices with two incoming arrows.
So the number of ergodic measures is at most that number of chains.

With a little more work, one can show the bound on ergodic measures is at most C.