Theorem

Let $A : \mathbb{T}^2 \to \mathbb{T}^2$ *by* $A(x) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ *mod* 1.

A is mixing with respect to Lebesgue measure.

Recall that A is *mixing* if for any pair of measurable set S_1, S_2 we have that

$$\lim_{n\to\infty} Leb(A \cap A^{-n} S_1 \cap S_2) = Leb(S_1) Leb(S_2).$$

The proof is much more general.

Two facts

1. If A is not mixing then there exists $n_1, \ldots$ and $f \in L^2$, g non-constant so that

$$\frac{1}{m}\sum_{i=0}^{m} f \circ T^{n_i} \to g \text{ and } \frac{1}{m}\sum_{i=0}^{m} f \circ T^{-n_i} \to g$$

   almost everywhere.

2. Given g as in fact 1, there exists a set of full measure G so that if $x, y \in G$ and $\lim_{n\to+\infty} d(A^n x, A^n y) = 0$ then $g(x) = g(y)$ and if $(x, y) \in G$ and $\lim_{n\to-\infty} d(A^n x, A^n y) = 0$ then $g(x) = g(y)$.

Proof of Theorem assuming facts

- If $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + t \begin{pmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{pmatrix}$ mod 1 then $\lim_{n \to -\infty} d(A^n x, A^n y) = 0$.
- If $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + t \begin{pmatrix} 1 \\ \frac{\sqrt{5}+1}{2} \end{pmatrix}$ mod 1 then $\lim_{n \to \infty} d(A^n x, A^n y) = 0$.
- For almost every z we have $z + t \begin{pmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{pmatrix}$ is in G for a.e. t.
- For almost every t we have $z + t \begin{pmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{pmatrix} + s \begin{pmatrix} 1 \\ \frac{\sqrt{5}+1}{2} \end{pmatrix}$ is in G for a. e. s.
- For any such z we have $g(z') = g(z)$ for a.e. z'.
- So this contradicts fact 1.

Fact 1

Lemma

If $T : (X, \mu) \to (X, \mu)$ is not mixing then there exist, $f, g \in L^2(\mu)$ with $\int f d\mu = 0$ and $m_k \to \infty$ so that

$$f \circ T^{m_k} \to g \text{ and } f \circ T^{-m_k} \to g$$

weakly in $L^2(\mu)$.

This means that for any $h \in L^2(\mu)$ we have $\langle f \circ T^{m_k}, h\rangle \to \langle g, h\rangle$ and $\langle f \circ T^{-m_k}, h\rangle \to \langle g, h\rangle$.

Why weak limits? The weak topology on $L^2(\mu)$ of norm at most N is a compact metric space. (If $\{\phi_j\}$ is an orthonormal basis $d(f,g) = \sum_{j=1}^{\infty} |\langle f - g, \phi_j \rangle|$ is such a a metric.)

So we have limits (which may be projection onto constants)!

In fact if $f \in L^2(\mu)$ and T is μ-measure preserving then $f \circ V^i$ along a subsequence. More is true if T is μ measure preserving then the Koopman operator U_T^n has a subsequential limit in the weak operator topology, the topology of pointwise convergence in the weak topology.

This follows by observing that if $\{\phi_j\}$ is an orthonormal basis and $V_1, ..., V_\infty$ are unitary operators so that $\lim_{j \to \infty} \langle V_j \phi_\ell, g \rangle = \langle V_\infty \phi_j, g \rangle$ for all g in L^2 then $\underset{j \to \infty}{\langle} V_j h, g \rangle = \langle V_\infty h, g \rangle$ for all g, h in L^2.

Weak convergence for mixing transformations

(X, μ, T) is mixing iff U_T^n converges weakly to projection onto the constant functions, but it does not converge strongly to anything.

Indeed, because the span of characteristic functions of measurable sets is dense in L^2, T mixing implies that for all $f, g \in L^2$ we have that
$\langle f \circ T^n, g\rangle \to \langle f, 1\rangle\langle 1, g\rangle$.

In general, U_T^n does not need to converge to anything in the weak operator topology, but it does converge along a subsequence.

This proof uses the spectral theorem:

Theorem

Let $T : L^2(\mu) \to L^2(\mu)$ is unitary and $f \in L^2(\mu)$. Let $H_f = \overline{span\{f \circ T^n\}}$. For each $f \in L^2(\mu)$ there exists σ_f, a measure on $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $V : H_f \to L^2(\sigma_f)$, unitary, so that

$$\langle f, f \rangle = \int d\sigma_f \text{ and } \langle g \circ T, f \rangle = \int_{S^1} z \cdot (Vg) d\sigma_f.$$

- If f is an eigenfunction, with eigenvalue λ then $H_f = \mathbb{C}f$ and σ_f is point mass at λ.
- This theorem is convenient for the weak topology, because the statement is about inner products.
- Note that if $g = \sum a_j f \circ T^j$ we have $V(g) = \sum a_j z^j$ (as a function in $L^2(\sigma_f)$).

Proof of Lemma

- There exists such f so that $f \circ T^n \not\to 0$ weakly.
- So there exists $n_j \in \mathbb{Z}$ and $\phi \in L^2(\mu)$ so that $f \circ T^{n_j} \to \psi$ weakly.
- So by the spectral theorem there exists n_j so that $z^{n_j} \to \phi = V\psi$ weakly in $L^2(\sigma_f)$ the spectral measure (on S^1) associated to f.
- Note that $z^{-n_j} \to \bar{\phi}$ weakly.
- One can show that there exist m_j, m_j' so that $z^{n_{m_j} - n_{m_j'}} \to \phi \cdot \bar{\phi}$ in $L^2(\sigma_f)$. details
- Since $\phi \cdot \bar{\phi}$ is real valued, the same is true for $z^{-(n_{m_j} - n_{m_j'})}$.
- By the spectral theorem if $h = \lim \sum a_j f \circ T^j$ then

$$\langle f \circ T^n, h \rangle = \lim \sum a_j \int z^{n-j} d\sigma_f.$$

- So $f \circ T^{n_{m_j} - n_{m_j'}}$ converges weakly on $\overline{\text{span}\{f \circ T^n\}}$.
- It is trivial that it also converges weakly on $\overline{\text{span}\{f \circ T^n\}}^{\perp}$.

Theorem

(Banach-Saks) If $f_j \to g$ weakly then there exists n_j so that

$$\frac{1}{m}\sum_{j=1}^{m} f_{n_j} \to g \text{ a.e.}$$

- It suffices to show this for $g \equiv 0$.
- Given $n_1, ..., n_{j-1}$, choose n_j so that $\langle f_{n_j}, f_{n_\ell} \rangle < \frac{1}{2^j}$ for all $\ell < j$.
- Observe $\langle \sum_{j=1}^m f_{n_j}, \sum_{j=1}^m f_{n_j} \rangle = O(m)$ and so $\frac{1}{m}\sum_{j=1}^m f_{n_j} \to 0$ in L^2. details
- L^2 convergence implies convergence a.e. along a subsequence.

Fact 2

- By Lusin's theorem, for every $\epsilon > 0$ there exists k so that for all $\ell > k$ there is a set B_ℓ of measure at most ϵ so that if $x, y \notin B_\ell$ and $d(A^i x, A^i y) \to 0$ then

$$|\frac{1}{\ell}\sum_{i=1}^{\ell} f(A^{n_i}x) - \frac{1}{\ell}\sum_{i=1}^{\ell} f(A^{n_i}y)| < \epsilon.$$

- This gives that there is a full measure set G' so that if $x, y \in G'$ and $d(A^i x, A^i y) \to 0$ then $|\frac{1}{\ell}\sum_{i=1}^{\ell} f(A^{n_i}x) - \frac{1}{\ell}\sum_{i=1}^{\ell} f(A^{n_i}y)| \to 0$ along a subsequence.
- Because $\frac{1}{m}\sum_{i=1}^{m} f(A^{n_i})$ converges almost everywhere we have fact 2.

  -Let G be the full measure set where we have convergence intersected with G'.

Details

Lemma

If $z^{j_\ell} \to g$ and $z^{k_\ell} \to h$ weakly in $L^2(\sigma)$ then there exist subsequences j'_ℓ and k'_ℓ so that $z^{j'_\ell + k'_\ell} \to g \cdot h$ weakly.

- For any $r \in \mathbb{Z}$ and $\phi \in L^2(\sigma)$ we have that $\int g z^r \overline{\phi} d\sigma = \lim_{j_\ell} \langle z^{j_\ell + r}, \phi \rangle$.
- $\langle g \cdot h, \phi \rangle = \langle g, \bar{\phi} \cdot h \rangle = \lim_{\ell \to \infty} \langle z^{j_\ell} \cdot h, \phi \rangle$.
- Now for each $\epsilon > 0$, ϕ and ℓ there exists k_0 so that for all $r \geq r_0$ we have $|\langle z^{j_\ell} \cdot z^{k_r}, \phi \rangle - \langle z^{j_\ell} \cdot h, \phi \rangle| < \epsilon$.
- So there exists n_ℓ, m_ℓ subsequences of j_ℓ and k_ℓ respectively so that $\langle z^{n_\ell} z^{m_\ell}, \phi \rangle \to \langle g \cdot h, \phi \rangle$.
- Choosing $\phi_1, ..$ an ortho-normal basis for $L^2(\sigma)$ and applying a diagonal argument gives the lemma.

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Details

$$\langle \sum_{j=1}^{m} f_j, \sum_{j=1}^{m} \rangle \leq \sum_{j=1}^{m} \langle f_j, f_j \rangle + \sum_{k=1}^{j-1} 2Re\big(\langle f_j, f_k \rangle\big) \leq m \sup\{\|f\|_2^2\} + 2\sum_{j=1}^{m} \frac{j-1}{2^j}$$

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