Bratteli-Vershik Models for Cantor and Borel dynamical systems

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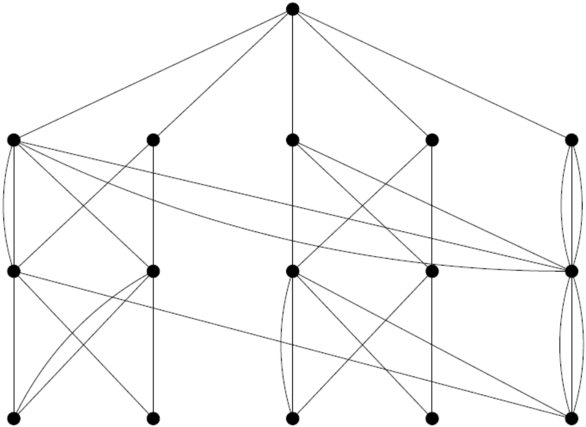
Goals

- Introduction to Bratteli diagram and Vershik map.
- Construct Bratteli-Vershik model of a minimal Cantor dynamical system.
- Some results in Cantor dynamics based on application of Bratteli-Vershik model.
- Bratteli-Vershik model for Borel dynamical system.

Cantor dynamics

- A **Cantor set** X is a 0-dimensional compact metric space without isolated points.
- (X, T) is called a **Cantor dynamical system (d.s.)** where $T : X \to X$ is a **homeomorphism**.
- $Orb_T(x) = \{T^n x : n \in \mathbb{Z}\}$ is called the T-**orbit** of x.
- T is **periodic** at x, if $|\{T^n x : n \in \mathbb{Z}\}| < \infty$, i.e., $\exists p$ s.t. $T^p x = x$.
- If every T-orbit is infinite, then T is called **aperiodic**.
- If every T-orbit is dense in X, then T is called **minimal**.

Example: a (non-simple, finite rank) Bratteli diagram



Definition of a Bratteli diagram

Definition

A Bratteli diagram is a graded infinite graph $B = (V, E)$ with the vertex set $V = \bigsqcup_{i \geq 0} V_i$ and edge set $E = \bigsqcup_{i \geq 1} E_i$:
1) $V_0 = \{v_0\}$ is a single point;

2) V_i and E_i are finite sets for every i;

3) edges E_i connect V_{i-1} to V_i: there exist maps r (range) and s (source) from E to V such that $r(E_i) \subseteq V_i, s(E_i) \subseteq V_{i-1}$, and $s^{-1}(v) \neq \emptyset$; $r^{-1}(v') \neq \emptyset$ for all $v \in V$ and $v' \in V \setminus V_0$.

- B is stationary if it repeats itself below the first level.
- B is of finite rank if for all $n \geq 1$, $|V_n| \leq k$ for some positive integer k.
- We say a finite rank diagram B has rank d if d is the smallest integer such that $|V_n| = d$ infinitely often.

Definition of a Bratteli diagram (cont.)

The incidence matrix F_n is a $|V_n| \times |V_{n-1}|$ matrix with entries

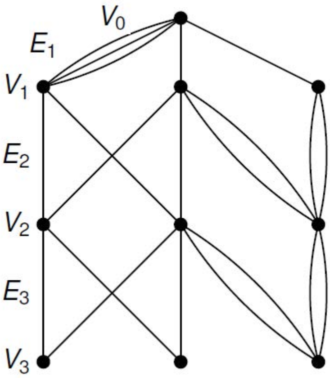
$$f_{v,w}^{(n)} = |\{e \in E_n : s(e) = w, r(e) = v\}|, \; v \in V_n, w \in V_{n-1}.$$

A Bratteli diagram is called simple if $\forall n \; \exists m > n$ such that $F_m \cdots F_{n+1} > 0$ (all entries are positive).

A finite or infinite sequence of edges $(e_i : e_i \in E_i)$ such that $r(e_i) = s(e_{i+1})$ is called a finite or infinite path. Let X_B be the set of infinite paths starting at the top vertex v_0. Then X_B a 0-dimensional compact metric space w.r.t. the topology generated by cylinder sets

$$[\overline{e}] := \{x \in X_B : x_i = e_i, \; i = 0, \dots, n\}.$$

Incidence matrix (Example)



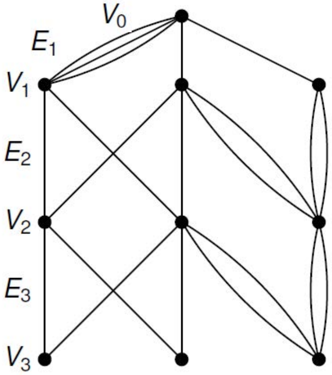
The diagram is *stationary* with incidence matrix

$$F = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}$$

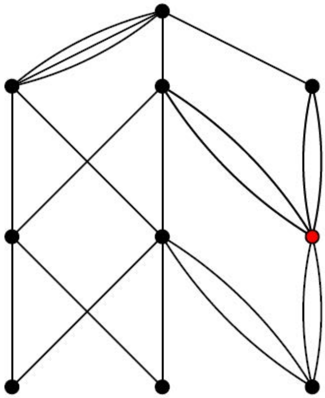
The sequence (F_n) of incidence matrices determine the structure of a Bratteli diagram.

Topology on the path space X_B: two paths are close if they agree on a large initial segment.

Ordered Bratteli diagrams

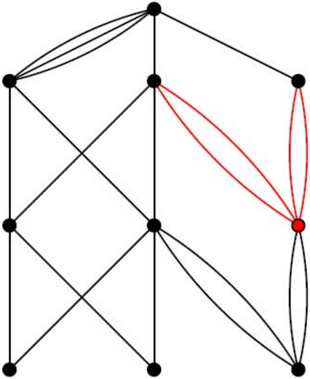


Ordered Bratteli diagrams



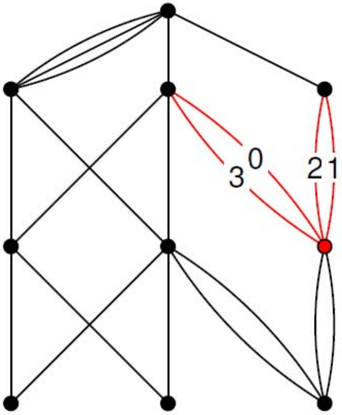
- Take a vertex $v \in V \setminus V_0$.

Ordered Bratteli diagrams



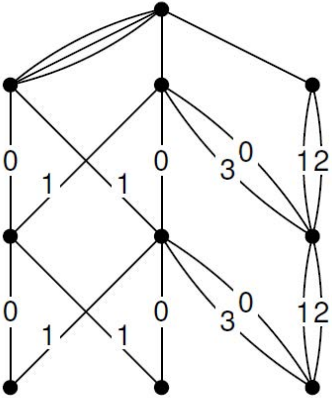
- Take a vertex $v \in V \setminus V_0$.
- Consider the set $r^{-1}(v)$.

Ordered Bratteli diagrams



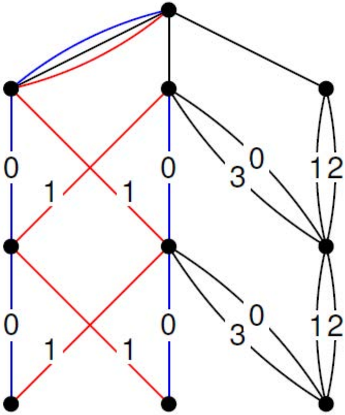
- Take a vertex $v \in V \setminus V_0$.
- Consider the set $r^{-1}(v)$.
- Enumerate edges from $r^{-1}(v)$

Ordered Bratteli diagrams



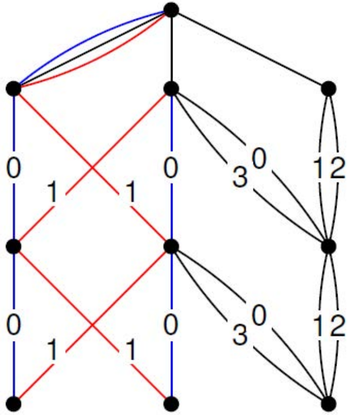
- Take a vertex $v \in V \setminus V_0$.
- Consider the set $r^{-1}(v)$.
- Enumerate edges from $r^{-1}(v)$
- Do the same for every vertex.

Ordered Bratteli diagrams

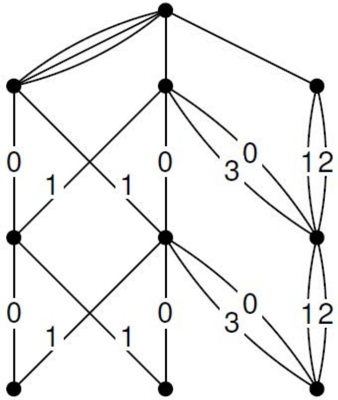


- An infinite path $x = (x_n)$ is called **maximal** if x_n is *maximal* in $r^{-1}(r(x_n))$. Similarly, **minimal** paths are defined.

Ordered Bratteli diagrams



- An infinite path $x = (x_n)$ is called **maximal** if x_n is *maximal* in $r^{-1}(r(x_n))$. Similarly, **minimal** paths are defined.
- The sets $X_{\max}$ and $X_{\min}$ of all maximal and minimal paths are non-empty and closed.

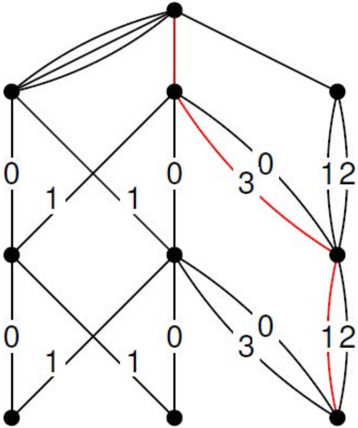


Define the Vershik map

$\varphi_B : X_B \setminus X_{\max} \to X_B \setminus X_{\min}$:

Ordered Bratteli diagrams

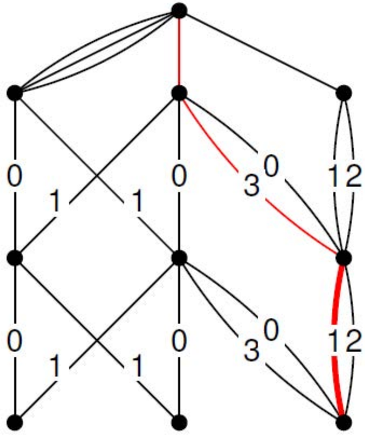
Vershik map



Define the Vershik map

$\varphi_B : X_B \setminus X_{\max} \to X_B \setminus X_{\min}$:

Fix $x \in X_B \setminus X_{\max}$.

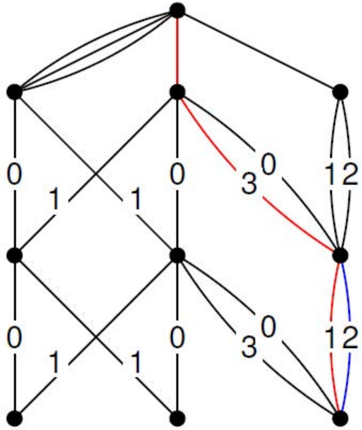


Define the Vershik map

$\varphi_B : X_B \setminus X_{\max} \to X_B \setminus X_{\min}$:

Fix $x \in X_B \setminus X_{\max}$.

Find the first k with non-maximal x_k.



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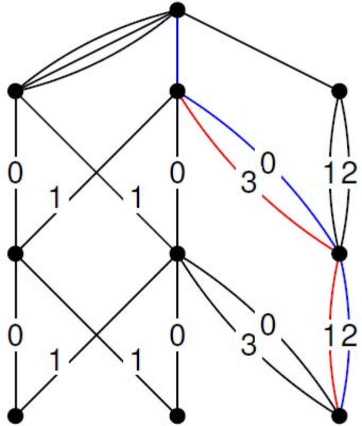
Take x_k to its successor $\overline{x}_k$.



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Find the first k with non-maximal x_k.

Take x_k to its successor $\overline{x}_k$.

Connect $s(\overline{x}_k)$ to the top vertex V_0 by the minimal path.

- φ_B is defined everywhere on $X_B \setminus X_{\max}$
- $\varphi_B(X_B \setminus X_{\max}) = X_B \setminus X_{\min}$

Definition

If the map φ_B can be extended to a homeomorphism of X_B such that $\varphi_B(X_{\max}) = X_{\min}$, then (X_B, φ_B) is called a Bratteli-Vershik system and φ_B is called the Vershik map.

Question:

Under what conditions on a Bratteli diagram does the Vershik map exist?

Answer:

If a Bratteli diagram B is simple, then the Vershik map **always** exists (e.g., use the left-to-right order).

Bratteli-Vershik model of minimal Cantor system

Theorem *A* (Herman, Putnam, and Skau '92):

For every minimal Cantor dynamical system (X, T), there exists a simple, ordered Bratteli diagram B such that the corresponding Vershik map φ_B is conjugate to T. (X_B, φ_B) is called **Bratteli-Vershik model** of (X, T).

Proof sketch: (Proof here is taken from F. Durand's survey paper.)

Four steps in the proof

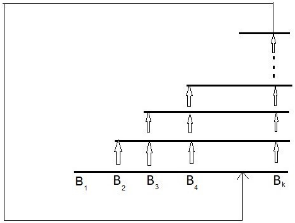
- *Step 1:* Construction of Nested sequence of Kakutani-Rokhlin Towers.
- *Step 2:* Construction of ordered Bratteli diagram B.
- *Step 3:* One to one correspondence in X and X_B
- *Step 4:* Showing that (X_B, φ_B) is conjugate to (X, T).

Definition

A *Kakutani-Rokhlin* partition of the minimal Cantor dynamical system (X, T) is a clopen (finite) partition $\mathcal{P}$ of the form

$$\mathcal{P} = \{T^j B_k |\, k \in V, 0 \le j < h_k\},$$

where V is (finite) index set, B_k is a clopen set and h_k (height) is a positive integer.



- k-th tower of $\mathcal{P}$: $\{T^j B_k |\, 0 \le j < h_k\}$
- Height of the k-th tower is h_k
- $T^j B_k$ is the j-th level
- Base of $\mathcal{P}$ is $B = \bigcup_{k \in V} B_k$

- Take a (nested) sequence of clopen sets $X = B(0) \supset B(1) \supset B(2) \supset \ldots$
- Let,
$$\mathcal{P}(1) = \{T^j B_i(1) | \, 0 \leq j < h_i(1), 1 \leq i < t(1)\}$$
be a KR-partition with base $B(1) = \bigcup_{i=1}^{t(1)} B_i(1)$.
- Since $B(2) \subset B(1)$, we can assume (refining $\mathcal{P}(1)$ if needed) that $B(2)$ is union of some sets in $B_i(1)$.
- Again construct KR-partition $\mathcal{P}(2)$,
$$\mathcal{P}(2) = \{T^j B_i(2) | \, 0 \leq j < h_i(2), 1 \leq i < t(2)\}$$
with base $B(2) = \bigcup_{i=1}^{t(1)} B_i(2)$.
- Apply this construction for every $B(n)$ to obtain nested sequence of KR-partitions $\mathcal{P}(n)$.

- Since $B(n+1)$ is union of sets $B_i(n)$ for $i \in \{0, ..., t(n)\}$, we can assume that $\mathcal{P}(n+1)$ refines $\mathcal{P}(n)$ for each n.

Thus we have the following lemma

Lemma B: There exists a sequence of KR-partitions $(\mathcal{P}(n))_n$ with

$$\mathcal{P}(n) = \{T^j B_i(n) | \, 0 \leq j < h_i(n), 1 \leq i < t(n)\}$$

satisfying:

- $\mathcal{P}(n+1)$ is finer than $\mathcal{P}(n)$ for each n.
- $\cup_n \mathcal{P}(n)$ generates the topology of X.

We describe the construction by an example : Let $(\mathcal{P}(n))_n$ be a sequence of KR-partitions satisfying Lemma B such that

(i) $\mathcal{P}_1 = \{B_1(1), TB_1(1), B_2(1), TB_2(1), T^2B_2(1)\}$

(ii) $\mathcal{P}(2) = \{\, T^jB_i(2) | \, 0 \leq j < h_i(2), 1 \leq i < t(2)\}$ with :

(a) $t(2) = 3,\ h_1(2) = 9,\ h_2(2) = 4,\ h_3(2) = 7,$

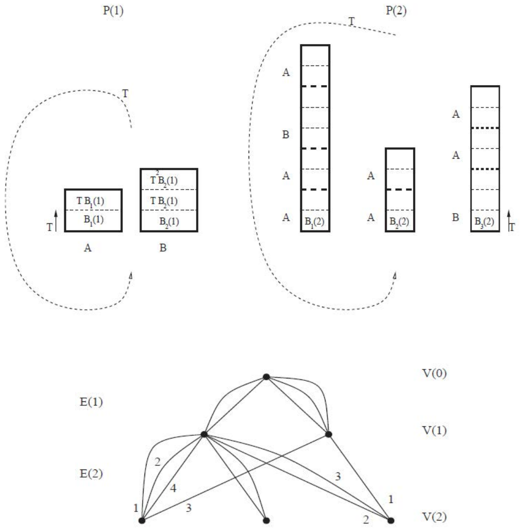
(b) $B_1(2) \subset B_1(1),\ T^2B_1(2) \subset B_1(1),\ T^4B_1(2) \subset B_2(1),$
$T^7B_1(2) \subset B_1(1),$

(c) $B_2(2) \subset B_1(1),\ T^2B_2(2) \subset B_1(1),$

(d) $B_3(2) \subset B_2(1),\ T^3B_3(2) \subset B_1(1),\ T^5B_3(2) \subset B_1(1).$

Proof of Theorem A (cont.)

Step 2: Constructing ordered Bratteli diagram



To summarize :

- Vertex set at each level given by number of towers :

$$V(n) = \{(n, 1), ..., (n, t(n))\}.$$

- Edge set $E(n)$ at each level is given by set of quadruples (n, t', t, j) satisfying :

$$T^j B_t(n) \subset B_{t'}(n-1) \tag{1}$$

where $1 \leq t' \leq t(n-1)$, $1 \leq t \leq t(n)$, $0 \leq j \leq h_t(n) - 1$ and $n \geq 1$.

- Source and Range maps :

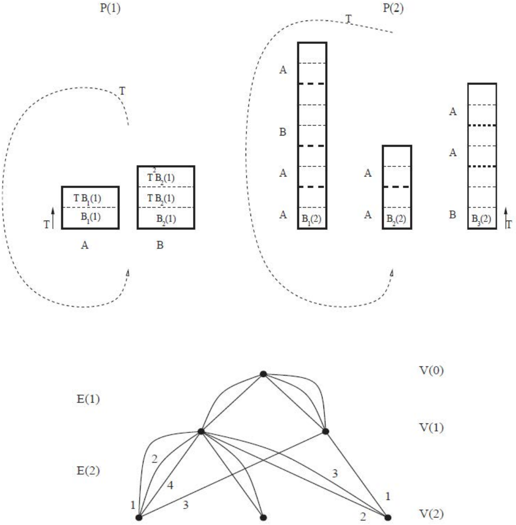
$$s(n, t', t, j) = (n-1, t') \; ; r(n, t', t, j) = (n, t)$$

.

- Order on the edges : $e_1 = (n_1, t'_1, t_1, j_1)$ and $e_2 = (n_2, t'_2, t_2, j_2)$ are comparable if $n_1 = n_2$ and $t_1 = t_2$. Define : $e_1 \geq e_2$ if $j_1 \geq j_2$.
- This makes $B = (V, E, \geq)$ an ordered Bratteli diagram.
- $((n, t'_n, t_n, j_n))_n$ is an infinite path iff $t_{n-1} = t'_n$ for all $n \geq 1$.
- $((n, t'_n, t_n, j_n))_n$ is a minimal edge iff $j_n = 0$ and maximal iff $j_n = h_{t_n}(n) - h_{t_{n-1}}(n-1)$.

Proof of Theorem A (cont.)

Step 2: Constructing ordered Bratteli diagram



Following Lemmma gives one to one correspondence in X and X_B.

Lemma C: TFAE

(i) $((n, t_{n-1}, t_n, j_n))_n$ is an infinite path in $B = (V, E, \geq)$.

(ii) $\bigcap_{n\geq 1} T^{(\sum_{i=1}^{n} j_i)} B_{t_n}(n) \neq \emptyset$ with $((n, t_{n-1}, t_n, j_n))_n$ satisfying (1).

(iii) $Card(\bigcap_{n\geq 1} T^{(\sum_{i=1}^{n} j_i)} B_{t_n}(n)) = 1$ with $((n, t_{n-1}, t_n, j_n))_n$ satisfying (1).

- X^B_{min} (and X^B_{max}) consist of single paths: minimal path is of the form $(n, t_{n-1}, t_n, 0)_n$. Hence

$$\bigcap_n T^0 B_{t_n}(n) = \bigcap_n B_{t_n}(n)$$

which is a single point (similarly for maximal path).

Proof of Theorem A (cont.)

Step 4: φ_B is conjugate to T

Consider the map $\psi : X_B \to X$ defined by

$$\psi((n, t_{n-1}, t_n, j_n)_n) = \{x\}, \text{ where } \{x\} = \bigcap_{n\geq 1} T^{(\sum_{i=1}^{n} j_i)} B_{t_n}(n)$$

- ψ is a well defined (Lemma C) homeomorphism.
- Need to show ψ commutes with the dynamics:

$$\begin{array}{ccc} e \in X_B & \xrightarrow{\varphi_B} & X_B \ni e' \\ \psi \downarrow & & \downarrow \psi \\ x \in X & \xrightarrow{T} & X \ni Tx \end{array}$$

Let $e = (e_n)_n = e_1 e_2 e_3..$ corresponds to x we will show that $\varphi_B(e) = (e'_n)_n = e'_1 e'_2 e'_3..$ corresponds to Tx.

Proof of Theorem A (cont.)

Step 4: φ_B is conjugate to T

Assume e is not maximal, then there exists n_0 such that e_{n_0} is not maximal and $\varphi_B(e) = (e'_n)_n = e'_1 e'_2 \ldots e'_{n_0-1} e'_{n_0} e'_{n_0+1} \ldots$

Since e_n is maximal for $n \in \{1, n_0 - 1\}$, $j_n = h_{t_n}(n) - h_{t_{n-1}}(n-1)$.

Since e'_{n_0} is successor of e_{n_0}, $j'_{n_0} = j_{n_0} + h_{t_{n_0-1}}(n_0 - 1)$.

Thus $\sum_{1 \leq n \leq n_0} j'_n = 0 + 0 + \ldots + j_{n_0} + h_{t_{n_0-1}}(n_0 - 1) = j_{n_0} + h_{t_{n_0-1}}(n_0 - 1)$.

On the other hand $\sum_{1 \leq n \leq n_0} j_n = \sum_{1 \leq n \leq n_0 - 1} h_{t_n}(n) - h_{t_{n-1}}(n-1) + j_{n_0}$.

$$= j_{n_0} + h_{t_{n_0-1}}(n_0 - 1) - 1$$

Thus $\varphi_B(e') = Tx$.

Main Results

- Giordano, Putnam, and Skau (1995) classified all minimal homeomorphisms of Cantor set with respect to **orbit equivalence.** By [HPS'92] it suffices to classify Vershik maps.
- Forrest (1997), Durand, Host, Skau (1999) described completely the class of dynamical systems that are represented by simple stationary Bratteli diagram. These are **minimal substitution dynamical systems**.
- Bezuglyi,Dooley and Medynets (2005), Medynets (2006): proved that for an **aperiodic** Cantor d.s. (X, T), there exists a **non-simple** ordered Bratteli diagram B such that T is conjugate to the Vershik map φ_B.
- Downarowicz and Karpel (2017) extended above result to homeomorphisms that may have **periodic points.**

Main Results (cont.)

- Aperiodic substitution d.s. are determined by non-simple stationary Bratteli diagrams (Bezuglyi-Kwiatkowski-Medynets (2009)).
- Ergodic invariant measures on non-simple stationary Bratteli diagrams are found in terms of Perron-Frobenius eigenvalue (and eigenvector) of its incidence matrix (Bezuglyi-Kwiatkowski-Medynets-Solomyak (2010)).
- Various criteria of unique ergodicity for finite rank Bratteli diagrams are proved (Bezuglyi-Kwiatkowski-Medynets-Solomyak (2013)).

Borel-Bratteli diagrams

Definition (Borel-Bratteli diagram)

A Borel-Bratteli diagram is an infinite graph $B = (V, E)$ such that $V = \bigsqcup_{i \geq 0} V_i$ and $E = \bigsqcup_{i \geq 1} E_i$ and
(i) $V_0 = \{v_0\}$ is a singleton, and every V_i and E_i are finite or countable sets;
(ii) there exist a range map r and a source map s from E to V such that $r(E_i) \subset V_i$, $s(E_i) \subset V_{i-1}$, $s^{-1}(v) \neq \emptyset$ for all $v \in V$, and $r^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_0$.
(iii) for every $v \in V \setminus V_0$, the set $r^{-1}(v)$ is finite.

Definition (Ordered Borel-Bratteli diagram)

Enumerate all edges from $r^{-1}(v)$ for all $v \neq v_0$. A Borel- Bratteli diagram B is called an ordered Borel-Bratteli diagram if the path space Y_B has no cofinal minimal and maximal paths.

Borel-Bratteli diagrams (cont.)

A few facts

- Path space Y_B is a **0-dimensional Polish space.**
- Incidence matrices $F_n = (f_{ik}^{(n)})$ have only finitely many non-zero entries at each row and $\sum_{k=1}^{\infty} f_{ik}^{(n)} = |r^{-1}(v_i(n))|$.
- Every **order on** B **defines a Vershik map** (homeomorphism) $\varphi_B : Y_B \to Y_B$.
- Given an ordered Borel-Bratteli diagram B, (Y_B, φ_B) is called **Borel-Bratteli dynamical system**.

Kakutani-Rokhlin partition

Let T be an aperiodic Borel automorphism of a standard Borel space $(X, \mathcal{B})$, and let A be a complete T-section whose points are T-recurrent. Then $\forall x \in A,\ \exists n_A(x) > 0$ such that $T^{n(x)}x \in A$ and $T^i x \notin A,\ 0 < i < n(x)$. Let $C_k = \{x \in A \mid n_A(x) = k\},\ k \in \mathbb{N}$, then $T^k C_k \subset A$ and $\{T^i C_k \mid i = 0, ..., k-1\}$ are pairwise disjoint. Thus,

$$X = \bigcup_{k=1}^{\infty} \bigcup_{i=0}^{k-1} T^i C_k$$

and X is partitioned into T-towers $\xi_k = \{T^i C_k \mid\ = 0, ..., k-1\},\ k \in \mathbb{N}$, where C_k is the base and $T^{k-1}C_k$ is the top of ξ_k. This partition of X is called Kakutani-Rokhlin partition.

Vanishing sequence of markers

Given an aperiodic automorphism T, there exists a sequence (A_n) of Borel sets such that
(i) $X = A_0 \supset A_1 \supset A_2 \supset \cdots$,
(ii) $\bigcap_n A_n = \emptyset$,
(iii) A_n and $X \setminus A_n$ are complete T-sections, $n \in \mathbb{N}$,
(iv) for $n \in \mathbb{N}$, every point in A_n is recurrent,
(v) for $n \in \mathbb{N}$, $A_n \cap T^i(A_n) = \emptyset, \ i = 1, ..., n-1$.

Definition

A sequence of Borel sets satisfying conditions (i) - (vi) is called a vanishing sequence of markers.

From Borel automorphisms to Borel-Bratteli diagrams

Let $(X, \mathcal{B}, T)$ be an aperiodic Borel d.s., and $X = A_0 \supset A_1 \supset A_2 \supset \cdots$ vanishing sequence of markers. Construct a Borel-Bratteli diagram for T using (A_n).
Let $(\xi_n = \{\xi_n(v) : v\})$ be the sequence of refining Kakutani- Rokhlin partitions constructed by (A_n), T. Towers of ξ_n correspond to vertices of V_n, and the i-th row of the incidence matrix F_n is determined by the intersection of $\xi_{n+1}(i)$ with towers of ξ_n. This automatically defines an order on $r^{-1}(v)$ for each v.

Theorem (Bezuglyi, Dooley, Kwiatkowski (2006))

Let T be an aperiodic Borel automorphism of $(X, \mathcal{B})$. Then there exists an ordered Borel-Bratteli diagram $B = (V, E, \geq)$ and a Vershik map $\varphi_B : Y_B \to Y_B$ such that (X, T) is isomorphic to (Y_B, φ_B).

Bratteli Diagrams and Translation Surfaces

Lindsey-Treviño (2016) and Treviño (2018) :

- Used **bi-infinite** Bratteli diagram $\mathcal{B}$ and two weight functions $w^{\pm}$ (which correspond to invariant measures on positive and negative part of $\mathcal{B}$), to construct a flat surface $S(\mathcal{B}, w^{\pm})$.
- Defined shift map σ on the space (denoted by $\mathcal{M}$) of bi-infinite Bratteli diagram $\mathcal{B}$.
  (By the functoriality property, we can think of $\mathcal{M}$ as a "moduli space" of flat surfaces of finite area.)
- A version of **Masur's criterion** in this setting : if the shift orbit of $\mathcal{B}$ has a "good" accumulation point then $S(\mathcal{B}, w^{\pm})$ will have a uniquely ergodic vertical flow.

Some questions

- Bratteli-Vershik Model for $\mathbb{Z}^d$ actions (both in Cantor and Borel setting).
- Following Lindsey-Treviño, can we construct translation surfaces from Borel-Bratteli diagrams? Can we obtain a version of Masur's criterion in this setting?

Thank You

MEMORIAL TRIBUTE

Ola Bratteli and His Diagrams

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Introduction

Magnus B. Landstad and George A. Elliott

Ola Bratteli is known first and foremost for what are now called Bratteli diagrams, a kind of infinite, bifurcating, graded graph. He showed how these diagrams (cousins of Coxeter–Dynkin diagrams) can be used to study algebras that are infinite increasing unions of direct sums of matrix algebras. They turned out to be very useful tools, giving a large class of examples, and later led to a *K*-theoretical classification both of the algebras just mentioned and, more recently, of an enormously larger class (all "well-behaved" simple amenable C*-algebras). According to MathSciNet, Trondheim (now NTNU) 1980–91, and since 1991 at the University of Oslo.

Ola's father, Trygve Bratteli, was a Norwegian politician from the Labour Party and prime minister of Norway in 1971–72 and 1973–76. During the Nazi invasion of Norway, he was arrested in 1942, and was a *Nacht und Nebel* prisoner in various German concentration camps from 1943 to 1945 but miraculously

Figure 1. Ola Bratteli (1946–2015).

https://www.ams.org/journals/notices/202005/rnoti-p665.pdf