LAB 13 - STABILITY OF EQUILIBRIA

MATH 1170

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In this lab, we'll explore the stability of an equilibrium of a differential equation. Specificially, we'll

1. use an infection model as a case study
2. explore how equilibria (and their stability) change due to parameter changes
3. etablish basic ideas in epidemiology like a basic reproduction number (R_0).

Setup

In class, you talked about an infection model and here we'll explore a slightly different version. Consider a population of N people[1] where some infection is spreading. Denote the *average*[2] number of people infected at time t by $I(t)$. Then, $I(t)$ satisfies the differential equation

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \alpha I(N - I) - \mu I,$$

where remember this comes from the idea that the rate at which the infected individuals change is

$$\text{per capita rate at which suceptive individual is infected} = \alpha I$$

and the number of suceptible individuals is those that aren't infected

$$\text{suceptible individuals} = \text{total} - \text{infected} = N - I$$

leading to the first term. That is, α basically corresponds to the infection "rate" normalized per-person. The parameter μ is similiar. It is the rate at which people recover.[3]

In this lab, we're going to fix $\alpha = .1$ and $\mu = 0.5$, and explore how the equilibria changes as N changes.

[1] this class, maybe?

[2] we use the "average" here because it's often not possible to predict *exactly* how many people are infected, but we can still get a rough idea

[3] so this wouldn't be appropriate for say, a zombie model where nobody recovers

Simulations

I've provided an unfinished code [4] `lab13_infection.r` which gives you a function called `infect_sim(N)`. So an example usage would be

```
> infect_sim(4)
```

which would simulate the infection for $N = 4$ people.

[4] you only have to do `source('lab13_infection.r')` once

Question 1

Complete the code using Euler's method as the true solution approximation.

Question 2

Set N to be $3,4,5,6,7$. Describe the plot by answering the following questions. What are the equilibria in this case? What is their stability? Include your plots with reasonable size for clean presentation.

Question 3

Afetring trying out different values of N. When do you see a dramatic change in behavior? Explain biologically why this special N might exist. This is called a "critical community size", a well studied quantity in epidemiology[5].

[5] **fun fact:** measles has one of the highest known critical community size at about 250-300 thousand people

Question 4

Change yoru code so that the function `infect_sim(N)` will plot out 10 initial condtions evenly ranged from 0 to N. Then include the plot with the case $N = 10$.

Question 5

Recall that the stability theorem says that an equilibrium $x^\star$ is stable for a differential equation

$$\frac{dx}{dt} = f(x) \quad \text{if and only if} \quad f'(x^\star) < 0$$

and unstable otherwise.

Here, we're interested in the "disease-free state" equilibrium $I^\star = 0$. We want to know when the disease free state is stable or unstable.

1. Use the stability therorem to derive the condition for the disease free state to be stable.

2. For $\alpha = .1, \mu = .5$, what is the condition on N so that the disease-free ($I^\star = 0$) is stable?

Question 6

The quantity in the previous question is often called R_0:

$$\frac{N\alpha}{\mu} = R_0$$

It's called the **basic reproduction number** and is a quantity of huge interest in epidemiology. We can think of it as: *on average* how many *other* people each person infected infects. [6]

[6] here's a fun clip from the film *Contagion* that talks about R_0 of different diseases `https://www.youtube.com/watch?v=VrATMF_FB9M`

With this word story, what is R_0 for Question 5? Why does the condition you derived from Question 4 make sense?