LAB 11 - TAYLOR SERIES, NEWTON'S METHOD

MATH 1170

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In this lab, we'll explore a dynamical system using the new tools we've developed. Specifically, we will try to find the equilibria using

- Taylor series
- Newton's method

Discrete time dynamical system

After a very successful evening trick-or-treating, you decide to enjoy the fruits of your labor. You eat as much candy as possible, but due to a lack of self-control, you get sick and eject candy from your body and repeat this process. Suppose we can model the amount of candy in your stomach at hour t after Halloween as

$$x_{t+1} = e^{-x_t} - 0.5,$$

which would have the equilibrium

$$x^* = e^{-x^*} - 0.5.$$

Notice that, while the above equation to find the equilibrium is simple, we can't solve it algebraically. We will have to compute it using other methods, which is what we do in this lab.

Question 1:

Finish the provided code `lab10_discrete.R` to simulate this discrete time dynamical system for 100 iterations. Start with an initial guess of $x^* = 0$.[1]

From the result of the above simulation, answer the following questions. What is the value of the equilibrium? How many iterations **precisely** did it take to reach that value?

[1] if your code doesn't run, you probably didn't fill everything in. read the code carefully!

Taylor series

Next, we rewrite the equation from the previous for our system's equilibrium. Now it reads

$$e^{-x} = x + 0.5.$$

Using the idea of Taylor polynomial approximation, we will try to use a tangent line approximation and a quadratic approximation of e^{-x} to solve this problem.[2]

[2] note that the right hand side is a polynomial. If we approximate the left hand side with a polynomial, polynomial=polynomial is a good type of equation to have

Question 2

Use the **matplot** and **cbind** command to graph e^{-x} and $x + 1/2$ together on the same plot. From the plot, answer the following questions. Is there a solution for $e^{-x} = x + 1/2$? How can you tell?

Question 3

Calculate, by hand, the tangent line for e^{-x} at $x = 0$ and solve for the point where the tangent line intersects $x + 1/2$. Is the result a decent approximation? Why?

Question 4

Replace e^{-x} with its quadratic approximation at $x = 0$, and solve for the point where it is equal to $x + 1/2$. Is this a better or worse approximation? Why?

Newton's method

We will rewrite our equation one more time, so it reads

$$e^{-x} - x - 0.5 = 0.$$

This means that, if we define $f(x) = e^{-x} - x - 0.5$, then we have an equation of the form $f(x) = 0$. That is, the equilibrium is exactly the root of this equation!

Recall that Newton's Method tells us that in this scenario, we can approximate a solution iteratively using the following formula:[3]

[3] you will have to compute the derivative by hand

$$x_{\text{new}} = x_{\text{old}} - \frac{f(x_{\text{old}})}{f'(x_{\text{old}})}$$

In this section, we will use Newton's method to try to find the equilibrium, and compare our results to the previous methods.

Question 5.

Finish the provided code `lab10_newtons.R` to find the solution of $f(x) = 0$, with an initial guess $x = 0$. How many iterations **precisely** does it take to get to the equilibrium that we found in Question 1?

Question 6

Rerun your code for Newton's method from Question 5, this time with an initial guess of $x = 10^4$. Does Newton's method converge to the correct solution? If so, does it converge faster or slower than in Question 1? Why? What does this suggest about the usefulness of Newton's method?

[4] We know that this guess is very far off from the real solution.