Exam 2 Review

POTS, Solutions to Polynomial Equations, and Planar Transformations

Remember what POTS tells us:

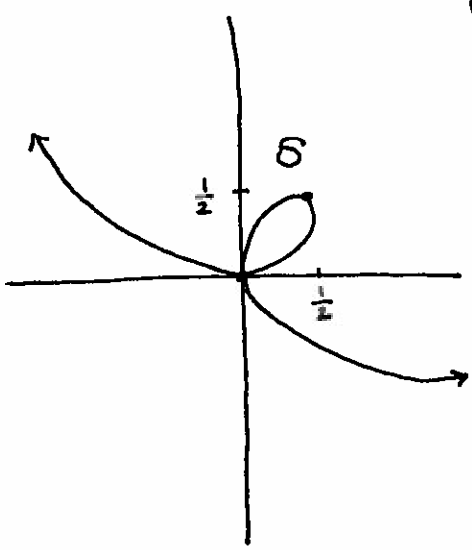
$$\boxed{\begin{array}{l}\text{If } S \text{ is the solutions to } p(x,y)=q(x,y)\\ \text{then } T(S) \text{ is the solutions to } p\circ T^{-1}(x,y)=q\circ T^{-1}(x,y)\end{array}}$$

$$S \xrightarrow{T} T(S)$$

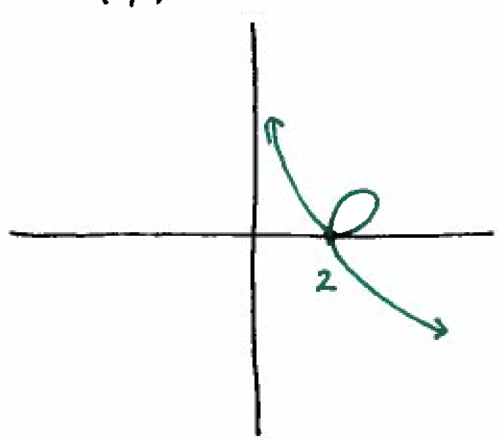
$$p(x,y)=q(x,y) \xrightarrow{T^{-1}} p\circ T^{-1}(x,y)=q\circ T^{-1}(x,y)$$

POTS is easiest to visualize when T is an addition function, flip, or a diagonal matrix, but it works for any transformation.

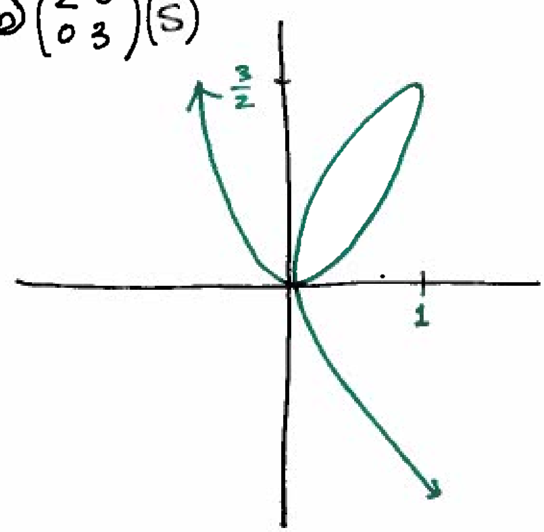
① The folium of Descartes is the set S of solutions to $x^3+y^3=xy$.
Draw $T(S)$ for the transformations below.
If you're not sure how to start, first describe in words what T does to the plane.



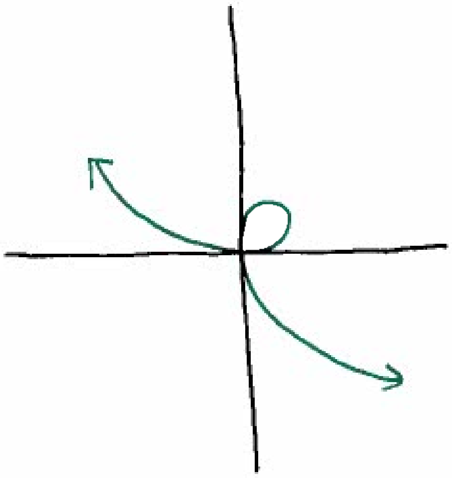
ⓐ $A_{(2,0)}(S)$



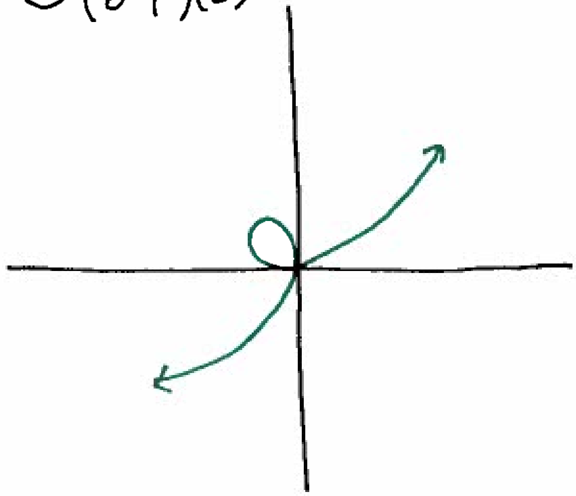
ⓑ $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}(S)$



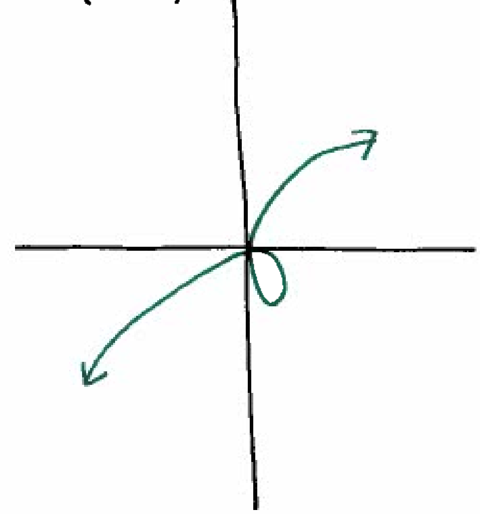
ⓒ $\begin{pmatrix}0&1\\1&0\end{pmatrix}(S)$



ⓓ $\begin{pmatrix}-1&0\\0&1\end{pmatrix}(S)$



ⓔ $\begin{pmatrix}1&0\\0&-1\end{pmatrix}(S)$



② Let's see how POTS applies in these 5 situations:

To find an equation for...	I should compose the equation...	with the transformation...
$A_{(2,0)}(S)$	$x^3+y^3=xy$	$A_{(2,0)}^{-1}$
$\begin{pmatrix}2&0\\0&3\end{pmatrix}(S)$	$x^3+y^3=xy$	$\begin{pmatrix}2&0\\0&3\end{pmatrix}^{-1}$
$\begin{pmatrix}0&1\\1&0\end{pmatrix}(S)$	$x^3+y^3=xy$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}^{-1}$
$\begin{pmatrix}-1&0\\0&1\end{pmatrix}(S)$	$x^3+y^3=xy$	$\begin{pmatrix}-1&0\\0&1\end{pmatrix}^{-1}$
$\begin{pmatrix}1&0\\0&-1\end{pmatrix}(S)$	$x^3+y^3=xy$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}^{-1}$

③ Find the inverses below:

ⓐ $A_{(2,0)}^{-1} = A_{(-2,0)}$

ⓑ $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$

ⓒ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

ⓓ $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

ⓔ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

④ For each transformation, find $T(x,y)$ & determine what T does to x & y.

ⓐ $A_{(-2,0)}(x,y) = (x-2, y)$

$x \longmapsto x-2$

$y \longmapsto y$

ⓑ $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x \\ \frac{1}{3}y \end{pmatrix}$

$x \longmapsto \frac{1}{2}x$

$y \longmapsto \frac{1}{3}y$

ⓒ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$

$x \longmapsto y$

$y \longmapsto x$

ⓓ $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$

$x \longmapsto -x$

$y \longmapsto y$

ⓔ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$

$x \longmapsto x$

$y \longmapsto -y$

⑤ Use your answers to ②-④ to find an equation for $T(S)$.

ⓐ Find an equation for $A_{(2,0)}(S)$

$$A_{(-2,0)}: \begin{array}{l} x \longmapsto x-2 \\ y \longmapsto y \end{array}$$

$$x^3+y^3=xy \Longrightarrow (x-2)^3+y^3=(x-2)(y)$$

ⓑ Find an equation for $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}(S)$

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}: \begin{array}{l} x \longmapsto \frac{1}{2}x \\ y \longmapsto \frac{1}{3}y \end{array}$$

$$x^3+y^3=xy \Longrightarrow \left(\tfrac{1}{2}x\right)^3+\left(\tfrac{1}{3}y\right)^3=\left(\tfrac{1}{2}x\right)\left(\tfrac{1}{3}y\right)$$

$$\frac{1}{8}x^3+\frac{1}{27}y^3=\frac{1}{6}xy$$

ⓒ Find an equation for $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(S)$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1}: \begin{array}{l} x \longmapsto y \\ y \longmapsto x \end{array}$$

$$x^3+y^3=xy \Longrightarrow y^3+x^3=yx$$

(same as original)

ⓓ Find an equation for $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}(S)$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{-1}: \begin{array}{l} x \longmapsto -x \\ y \longmapsto y \end{array}$$

$$x^3+y^3=xy \Longrightarrow (-x)^3+y^3=(-x)y$$

$$y^3-x^3=-xy$$

ⓔ Find an equation for $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(S)$.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1}: \begin{array}{l} x \longmapsto x \\ y \longmapsto -y \end{array}$$

$$x^3+y^3=xy \Longrightarrow x^3+(-y)^3=x(-y)$$

$$x^3-y^3=-xy$$

* Remember: On page 1, the equation for S is $x^3+y^3=xy$.