Elementary totally disconnected locally compact groups, after Wesolek

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Abstract

We present the main properties of the class of elementary totally disconnected locally compact (tdlc) groups, recently introduced by Wesolek, along with a decomposition result by the same author which shows that these groups along with topologically characteristically simple groups may be seen as building blocks of totally disconnected locally compact groups.

The class $\mathcal{E}$ of elementary tdlc (totally disconnected locally compact) Polish[1] groups was recently introduced by Wesolek in [Wes15] and is defined below.

Definition A. The class $\mathcal{E}$ of **elementary groups** is the smallest class of tdlc Polish groups such that

(E1) $\mathcal{E}$ contains all Polish groups which are either profinite or discrete;

(E2) whenever $N \leqslant G$ is a closed normal subgroup of a tdlc Polish group G, if $N \in \mathcal{E}$ and G/N is profinite or discrete then $G \in \mathcal{E}$;

(E3) if a Polish tdlc group G can be written as a countable increasing union of open subgroups belonging to $\mathcal{E}$, then $G \in \mathcal{E}$.

Much like in the case of amenable elementary groups, the class of elementary groups enjoys strong closure properties: it is closed under group extension, taking closed subgroups, Hausdorff quotients, and inverse limits.

Examples of elementary groups include solvable tdlc Polish groups. It is not known whether every tdlc Polish amenable group is an elementary group. A wealth of non-elementary groups is provided by compactly generated, topologically simple, non-discrete groups. As a consequence, for all $n \geqslant 3$, neither the group of automorphisms of the n-regular tree nor the special linear group of dimension n over $\mathbb{Q}_p$ are elementary.

The most remarkable feature of elementary groups is that they (along with topologically characteristically simple non-elementary groups) may be seen as building blocks for general tdlc Polish groups. To be more precise, using results of Caprace and Monod [CM11], Wesolek proved the following structure theorem.

[1] Or equivalently second-countable, see Sec. 1.

Theorem B ([Wes15, Thm. 1.6])**.** Let G be a compactly generated tdlc Polish group. Then there exists a finite increasing sequence

$$H_0 = \{e\} \leqslant \cdots \leqslant H_n$$

of closed characteristic subgroups of G such that

1. G/H_n is an elementary group and
2. for all $i = 0, \ldots, n-1$, the group $(H_{i+1}/H_i)/\mathrm{Rad}_{\mathcal{E}}(H_{i+1}/H_i)$ is a finite quasi-product[2] of topologically characteristically simple non-elementary subgroups, where $\mathrm{Rad}_{\mathcal{E}}(H)$ denotes the elementary radical[3] of H.

Our main goal here is to present the proof of the above result in details, which requires us to study the aforementioned closure properties of the class of elementary groups closely. To prove these closure properties, Wesolek makes heavy use of the ordinal-valued construction rank, which is basically a tool for using induction on elementary groups. Here, we try to avoid the construction rank as much as possible so as to make the proofs simpler and to highlight where this rank is really needed. In particular, we prove the following results directly.

- Every topogically simple compactly generated elementary group is discrete (Prop. 2.6).
- $\mathcal{E}$ is closed under extension (Prop. 3.1).
- $\mathcal{E}$ is closed under taking closed subgroups (Prop. 3.2).

However, in order to prove that the class of elementary groups is closed under taking Hausdorff quotients, we could not avoid the use of ordinals. We thus took the opportunity to give a gentle introduction to ordinals and define the construction rank in Section 4. Moreover, we use this construction rank only once, so as to get a general scheme for proving results by induction on elementary groups (Thm. 4.11). It is worth noting here that Wesolek defines a second rank on elementary groups, called the decomposition rank, which turns out to be more useful for studying elementary groups once their closure properties have been established (see [Wes15, Sec. 4.3]). We will not need to use it here and so we will not define it.

With this induction scheme in hand, we then prove the technical Lemma 5.4 which has the following two consequences.

- Every Hausdorff quotient of an elementary group is elementary (Thm. 5.5) .
- If $N_1, ..., N_k$ are elementary closed normal subgroups of a tdlc group G, then $\overline{N_1 \cdots N_k}$ is elementary (Cor. 5.7).

The latter result is fundamental, since it allows us to define the elementary radical of a tdlc Polish group G as the biggest closed normal elementary subgroup

[2]For the definition of quasi-products, see Def. 6.3.

[3]The elementary radical is the greatest elementary closed normal subgroup; see Thm. 6.1.

of G (Thm. 6.1). This radical is then used along with a fundamental result of Caprace and Monod to prove Theorem B in Section 6.

Section 1 contains some basic definitions and gives an introduction to Cayley-Abels graphs. In Section 2, we prove several propositions which lead to easy examples and non-examples of elementary tdlc Polish groups. Sections 3 and 5 cover closure properties of $\mathcal{E}$, with a discussion of ordinals and some background on the construction rank contained in Section 4. In Section 6, we discuss the elementary radical and prove the decomposition theorem for tdlc Polish groups.

1 Prerequisites

A topological group is **Polish** if it is separable and its topology admits a compatible complete metric. Polish groups should be regarded as nice groups: the completeness of a compatible metric allows for Baire category arguments, while the separability allows for constructive arguments. Polish locally compact groups are characterized as follows.

Theorem 1.1 (see [Kec95, Thm. 5.3])**.** *For a locally compact group G, the following are equivalent.*

(i) G is Polish.

(ii) G is second-countable.

(iii) The topology of G is metrizable.

For general topological groups, being Polish is the strongest of the three properties above, so the totally disconnected locally compact groups satisfying these conditions will be called **tdlc Polish groups**.

We will also need Cayley-Abels graphs, and we first recall a lemma which gives an easy way of understanding them (see [Wes15, Prop. 2.4]).

Lemma 1.2 (Folklore)**.** *Let G be a compactly generated tdlc Polish group, and let U be a compact open subgroup of G. Then there exists a finite symmetric*[4] *set $A \subseteq G$ such that $AU = UAU$ and*

$$G = \langle A \rangle U.$$

Moreover, if D is a dense subgroup of G, one can choose A as a subset of D.

Proof. Let S be a compact symmetric generating set of G, and let D be a dense subgroup of G. Then $\{xU : x \in D\}$ is an open cover of S, so we may find a finite symmetric subset $B \subseteq D$ such that $S \subseteq BU$. Now, UB is also compact, and $UBU \cap D$ is dense in UBU, which contains UB. So we may find a finite symmetric subset A of $UBU \cap D$ such that $UB \subseteq AU$, and we may assume that A contains B. Now, since $A \subseteq UBU$ and U is a group, we actually have

$$UAU = UBU \subseteq AUU = AU,$$

and we conclude by induction that for all $n \geqslant 1$, $(UAU)^n = A^n U$. Since S is symmetric and $S \subseteq BU \subseteq UAU$ we deduce that $G = \langle A \rangle U$. □

[4] A subset A of G is **symmetric** if $A^{-1} = A$.

Now, given a compact open subgroup U of a tdlc Polish group G and a finite symmetric set $A \subseteq G$ such that $G = \langle A \rangle U$ and $UAU = AU$, the Cayley-Abels graph $\mathcal{C}_{A,U}(G)$ is defined the following way:

- its set of vertices is G/U and
- its set of edges is $\{(gU, gaU) : a \in A, g \in G\}$.

Then the left action of G on G/U extends to a continuous transitive action of G on $\mathcal{C}_{A,U}(G)$ by graph automorphisms. In particular, all the vertices in $\mathcal{C}_{A,U}(G)$ have the same degree, and we will call that fixed number the **degree** of $\mathcal{C}_{A,U}(G)$. The fact that $UAU = AU$ ensures that the set of neighbors of U is exactly the set of aU for $a \in A$. But A is finite, so the degree of the Cayley-Abels graph $\mathcal{C}_{A,U}(G)$ is finite.

Suppose in addition that N is a closed normal subgroup of G, and let $\pi : G \to G/N$ be the natural projection. Then $G/N = \langle \pi(A) \rangle \pi(U)$ and $\pi(U)\pi(A)\pi(U) = \pi(A)\pi(U)$, so we may form the Cayley-Abels graph $\mathcal{C}_{\pi(A),\pi(U)}(G/N)$.

Observe that $\mathcal{C}_{\pi(A),\pi(U)}(G/N)$ is just the quotient of $\mathcal{C}_{A,U}(G)$ by the action of N by graph automorphisms, hence its degree is smaller or equal to the degree of $\mathcal{C}_{A,U}(G)$. Moreover, by definition of the quotient graph, if two distinct neighbors of a vertex $v \in \mathcal{C}_{A,U}(G)$ are in the same N-orbit, then the degree of $\mathcal{C}_{\pi(A),\pi(U)}(G/N)$ is strictly smaller than the degree of $\mathcal{C}_{A,U}(G)$. This observation will be crucial to the proof of the decomposition theorem (see Lem. 6.6).

2 Examples and non-examples

Since it is central, we recall here the definition of the class of elementary groups.

Definition 2.1. The class $\mathcal{E}$ of elementary groups is the smallest class of tdlc Polish groups satisfying the following properties.

(E1) The class $\mathcal{E}$ contains all Polish groups which are either profinite or discrete[5].

(E2) Whenever $N \leqslant G$ is a closed normal subgroup of a tdlc Polish group G, if $N \in \mathcal{E}$ and G/N is profinite or discrete[6] then $G \in \mathcal{E}$.

(E3) If a Polish tdlc group G can be written as a countable increasing union of open subgroups belonging to $\mathcal{E}$, then $G \in \mathcal{E}$.

By definition, a topological group is **SIN**[7] if it admits a basis of open neighborhoods of the identity, each of which is invariant under conjugacy. Note that this condition is automatically satisfied in an abelian topological group. Now, if G is a tdlc Polish SIN group, then by van Dantzig's theorem we may find a compact open subgroup $K \leqslant G$. The SIN condition ensures that the intersection U of the conjugates of K is still open. Then G/U must be a discrete group, so G is elementary by (E2). We have proved the following proposition.

[5] A profinite group is Polish if and only if it is metrizable, while a discrete group is Polish if and only if it is countable.

[6] Since G/N is automatically Polish (see [Gao09, Thm. 2.2.10]), we see that equivalently, one could ask that G/N is profinite metrizable, or countable discrete.

[7] SIN stands for "small invariant neighborhoods".

Proposition 2.2. *Every SIN tdlc Polish group is elementary. In particular, any abelian tdlc Polish group is elementary.*

As we will see in the next section (Prop. 3.1), the class of elementary groups is closed under extension, which yields that every solvable tdlc Polish group is elementary by the previous proposition.

Question 2.3 (Wesolek)**.** *Is every amenable*[8] *tdlc Polish group elementary?*

Remark. Since amenability passes to closed subgroups and to quotients, the question may be reformulated by first defining the class of "amenable elementary groups" to be the smallest class containing profinite amenable groups and discrete amenable groups, stable under extension and exhaustive open unions. The question then becomes: is every amenable tdlc Polish group amenable elementary?

A tdlc Polish group is **residually discrete** if for every finite subset $F \subseteq G \setminus \{1\}$, there is an open normal subgroup disjoint from F. It is **residually elementary** if for every finite subset $F \subseteq G \setminus \{1\}$, there is a closed normal subgroup N, disjoint from F, such that G/N is elementary.

Proposition 2.4. *Every residually discrete tdlc Polish group is elementary.*

Proof. Every tdlc Polish group can be written as an increasing union of compactly generated open subgroups (see Lem. 5.1). Moreover, being residually discrete passes to open subgroups, so by (E3) we only have to prove the proposition in the case when G is compactly generated. It is a theorem of Caprace and Monod that every locally compact, compactly generated, residually discrete group is SIN [CM11, Cor. 4.1], so we can conclude that G is elementary by Proposition 2.2. □

In fact, adapting the proof of Caprace and Monod's aforementioned result, Wesolek was able to show the following remarkable result, which we state without proof.

Theorem 2.5 (see [Wes15, Thm. 3.14])**.** *Every residually elementary group is elementary. In particular, any tdlc Polish group which can be written as the inverse limit of elementary groups is elementary.*

Note that the above result fails completely for the class of elementarily amenable discrete groups, since free groups are residually finite.

Let us now turn to non-examples of elementary groups. The following fact is closely related to Lemma 4.10, but as a warm-up to the next section we give here a detailed "ordinal-free" proof.

Proposition 2.6. *Every topologically simple compactly generated elementary group has to be discrete.*

Remark. As a consequence, the existence of a topologically simple compactly generated amenable nondiscrete tdlc Polish group[9] would provide a negative answer to Question 2.3.

[8] A good reference on amenability for locally compact groups is Appendix G in [BdlHV08].

[9] In other words, we seek a non-discrete analogue of the derived groups of topological full groups of minimal subshifts (these are finitely generated infinite amenable simple groups, see [Mat06, JM13]).

Proof of Proposition 2.6. Let G be a topologically simple compactly generated nondiscrete group, and let $\mathcal{F}$ be the class of elementary groups which are not isomorphic to G. We want to show that $\mathcal{F}$ contains the class $\mathcal{E}$ of elementary groups, and so it suffices to check that $\mathcal{F}$ satisfies the same defining properties as $\mathcal{E}$ (see Def. 2.1) in order to conclude that $\mathcal{E} \subseteq \mathcal{F}$ by minimality.

Because G is neither discrete nor profinite[10], $\mathcal{F}$ contains profinite and discrete groups, so (E1) is satisfied. The class $\mathcal{F}$ satisfies (E2) because G is topologically simple. Suppose that (E3) is not satisfied. Then we can write $G = \bigcup_{i\in\mathbb{N}} G_i$, where $(G_i)_{i\in\mathbb{N}}$ is an increasing chain of open subgroups of G and every G_i belongs to $\mathcal{F}$, hence is different from G. Let S be a compact generating set for G. Then $(G_i)_{i\in\mathbb{N}}$ is an open cover of S, so by compactness and the fact that $(G_i)_{i\in\mathbb{N}}$ is increasing, there exists $i \in \mathbb{N}$ such that G_i contains S. But then $G_i = G$, which is a contradiction. So (E3) is also satisfied by $\mathcal{F}$, which ends the proof. □

Let us now apply Proposition 2.6 and give some non-examples of elementary groups.

Proposition 2.7. *Let $n \geqslant 3$. Neither the group $\mathrm{Aut}^+(T_n)$ generated by the vertex stabilizers in the automorphism group of the n-regular tree, nor the projective linear group $PSl_n(\mathbb{Q}_p)$ are elementary.*

Proof. By Proposition 2.6, it suffices to show that these are topologically simple nondiscrete compactly generated groups. It is well-known that both groups are simple: for $\mathrm{Aut}^+(T_n)$ this is a result of Tits [Tit70], while for $PSl_n(\mathbb{Q}_p)$ a proof may be found in [Die71, Ch. II, §2]. That these groups are not discrete is clear. $\mathrm{Aut}^+(T_n)$ acts properly and cocompactly on the tree T_n, and $PSl_n(\mathbb{Q}_p)$ acts properly and cocompactly on its Bruhat-Tits building, and hence both are compactly generated (a more direct proof for $PSl_n(\mathbb{Q}_p)$ may be found in [dlHdC]). □

Remark. Since $\mathrm{Aut}^+(T_n)$ is an index 2 subgroup of $\mathrm{Aut}(T_n)$, we deduce from the above proposition and Corollary 3.3 that $\mathrm{Aut}(T_n)$ is non-elementary as well. And since every Hausdorff quotient of an elementary group is elementary, the special linear group $Sl_n(\mathbb{Q}_p)$ is also non-elementary.

3 Closure properties

Let us first present and prove some of the closure properties of the class of elementary groups which do not require the use of ordinals.

Proposition 3.1. *$\mathcal{E}$ is closed under extension: whenever G is a tdlc Polish group, and $H \triangleleft G$ is a closed normal subgroup such that both H and G/H are elementary, then G is elementary.*

Proof. Fix an elementary group H, and consider the class $\mathcal{F}_H$ of tdlc Polish groups Q such that whenever G is a tdlc Polish group with $Q = G/H$, then G is elementary. By property (E2), $\mathcal{F}_H$ contains all profinite Polish groups and discrete Polish groups, so we have to show that $\mathcal{F}_H$ has the other two defining properties of $\mathcal{E}$.

[10] A non-discrete profinite group cannot be topologically simple!

- Suppose $N \in \mathcal{F}_H$ is a closed normal subgroup of some tdlc Polish group Q, and that Q/N is a profinite or discrete group. Moreover, suppose that G is a tdlc Polish group such that $Q = G/H$. We want to show that $G \in \mathcal{E}$, and thus that $Q \in \mathcal{F}_H$. To this aim we let $\pi : G \to Q$ denote the quotient map, then $\pi^{-1}(N)$ is a closed normal subgroup of H, and $\pi^{-1}(N)/H = N$. Because $N \in \mathcal{F}_H$, the group $\pi^{-1}(N)$ is elementary. But $G/\pi^{-1}(N) = Q/N$ is profinite or discrete Polish, so G is elementary.

- Suppose $Q = G/H$ is written as a countable increasing union of open subgroups O_i belonging to $\mathcal{F}_H$. Then $\pi^{-1}(O_i)$ belongs to $\mathcal{E}$ because $O_i \in \mathcal{F}_H$. We deduce that $G = \bigcup_i \pi^{-1}(O_i)$ is elementary. □

Proposition 3.2. *Let G be an elementary group. If H is a tdlc Polish group such that there exists a continuous injective homomorphism $\pi : H \to G$, then H is elementary. In particular, any closed subgroup of an elementary group is elementary.*

Proof. Let $\mathcal{F}$ be the class of elementary groups G such that if H is a tdlc Polish group with a continuous injective homomorphism $\pi : H \to G$, then H is elementary.

First, $\mathcal{F}$ clearly contains discrete Polish groups. Let us show that it contains profinite Polish groups. Suppose G is profinite, and $\pi : H \to G$ is continuous and injective. Then H is residually discrete, hence it is elementary by Proposition 2.4.

Next, suppose that G is a tdlc Polish group, and $N \in \mathcal{F}$ is a closed normal subgroup of G such that G/N is profinite or discrete. Let $\pi : H \to G$ be a continuous injective morphism. Then π induces a continuous injective homomorphism $\tilde{\pi} : H/\pi^{-1}(N) \to G/N$. Because $\mathcal{F}$ contains discrete and profinite groups, we deduce that $H/\pi^{-1}(N)$ is elementary. But $\pi^{-1}(N)$ injects continuously into N, which belongs to $\mathcal{F}$, hence $\pi^{-1}(N)$ is elementary. We deduce from Proposition 3.1 that H is elementary.

Finally, suppose that G is a countable increasing union of open subgroups O_i which belong to $\mathcal{F}$. We must show that G also belongs to $\mathcal{F}$. Let $\pi : H \to G$ be a continuous injective homomorphism. Then the restriction of π to $\pi^{-1}(O_i)$ is also a continuous injective homomorphism, so the open subgroups $\pi^{-1}(O_i)$ are elementary and thus $H = \bigcup_{i\in\mathbb{N}} \pi^{-1}(O_i)$ is elementary. □

Corollary 3.3. *Let G be a tdlc Polish group, and let $H \leqslant G$ be a closed subgroup of finite index. Then G is elementary if and only if H is elementary.*

Proof. First, if G is elementary then H is elementary by the previous proposition. Conversely, if H is elementary, let N be the kernel of the action of G on G/H by left translation. The set G/H is finite, so N has finite index in G. Moreover, N is a closed subgroup of H, hence is elementary by the previous proposition. The group G/N is finite, so by property (E2) of the class of elementary groups, G also has to be elementary. □

The next closure property will require the use of an inductive argument based on the construction rank, which we now introduce in detail. The reader who is allergic to ordinals may just take Theorem 4.11 for granted and move directly to Section 5. However, he or she might also try to cure this allergy by reading what follows.

4 The construction rank on elementary groups

In order to motivate this section, let us digress a bit and discuss an easy example of a rank which may already be familiar. Suppose we have a finitely generated group Γ and a fixed finite symmetric generating set S containing the identity element. Then we can reconstruct every element of Γ as follows: start with $\Gamma_0 = S$, and then define by induction $\Gamma_{n+1} = \Gamma_n S$. Because Γ is generated by S, which is symmetric and contains the identity element, we have that $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$. The rank of an element $\gamma \in \Gamma$ is then defined to be the smallest $n \in \mathbb{N}$ such that $\gamma \in \Gamma_n$. Of course, the rank of $\gamma \in \Gamma$ is just $d_S(\gamma, e) + 1$ where d_S is the word metric defined by S. It measures how hard it is to construct γ using S and the group multiplication.

Now, let us try to define a rank on elementary groups the same way: we know that the class of elementary groups is generated by profinite and discrete Polish groups, so we let $\mathcal{E}_0$ be the class of Polish groups which are either profinite or discrete. For a Polish tdlc group G and $n \in \mathbb{N}$, we then say that $G \in \mathcal{E}_{n+1}$ if either

- there exists $N \in \mathcal{E}_n$ such that G/N is either profinite or discrete, or
- there exists a countable increasing family $(G_i)_{i\in\mathbb{N}}$ of open subgroups of G such that for all $i \in \mathbb{N}$, we have $G_i \in \mathcal{E}_n$, and moreover $G = \bigcup_{i\in\mathbb{N}} G_i$.

The problem is that, as opposed to the group case where every element is a product of finitely many elements of S, here the second operation needs countably many elementary groups in order to build a new one. Thus, there is no reason to have $\bigcup_{n\in\mathbb{N}} \mathcal{E}_n = \mathcal{E}$, and indeed one can show that the inclusion $\bigcup_{n\in\mathbb{N}} \mathcal{E}_n \subseteq \mathcal{E}$ is strict [Wes15, Sec. 6.2] (see also the recent preprint [RW] for compactly generated examples). So we need a rank taking values in something bigger than the integers: in particular, we want a rank which takes values into something "stable under countable supremums", and that is exactly what the set of countable ordinals will do for us.

4.1 The well-ordered set of countable ordinals

We will proceed with a crash course on (countable) ordinals. But first, let us define the fundamental property of the set into which a rank takes values, allowing for inductive arguments.

A strictly ordered set $(A, <)$ is **well-ordered** if every nonempty subset of A has a minimum. Note that every subset of A is a well-ordered set for the induced order.

Example 4.1. The set of rational numbers with the usual order $(\mathbb{Q}, <)$ is not well-ordered. The set $(\mathbb{N}^2, <_{lex})$, where $<_{lex}$ denotes the lexicographic order, is well-ordered.

Given a well-ordered set A and a property $P(x)$, to prove that $P(x)$ is true for all $x \in A$, one may use a proof by induction. Such a proof boils down to showing that the following holds:

$(*)$ For all $x \in A$, if $P(y)$ is true for all $y < x$, then $P(x)$ is true.

Let us see why $(*)$ implies that $P(x)$ is true for all $x \in A$. Consider the set $B = \{x \in A : P(x) \text{ is not true}\}$. If B were nonempty, then it would have a minimum x_1, and by definition, for all $y < x_1$, $P(y)$ would be true. But because $x_1 \in B$, this is a contradiction.

Remark. Let x_0 be the minimum of A. Then the assertion "$P(y)$ is true for all $y < x_0$" is necessarily verified, so if we want to show that $(*)$ holds, we will have to show that $P(x_0)$ is true.

Example 4.2. For the well-ordered set $\mathbb{N}$ of non-negative integers, we recover what is sometimes called the "principle of generalized recurrence".

The following proposition should be regarded by the non-set theory inclined as an axiom. Its refinements form the foundations of the theory of ordinals, a nice exposition of which can be found in [Kri71].

Proposition 4.3. *There exists a unique (up to order isomorphism) uncountable well-ordered set $(\omega_1, <)$ such that for all $\alpha \in \omega_1$, the well-ordered set*

$$\{\beta \in \omega_1 : \beta < \alpha\}$$

is countable. Moreover, ω_1 is stable under countable supremums, meaning that every countable family $(\alpha_i)_{i\in\mathbb{N}}$ of elements of ω_1 has a smallest upper bound in ω_1, denoted by $\sup\{\alpha_i : i \in \mathbb{N}\}$.

We will call ω_1 the **set of countable ordinals**. It will allow us to define a *rank* on elementary groups, and this rank will be useful to prove some of the permanence properties of this class of groups by induction.

Every countable ordinal $\alpha \in \omega$ is the strict supremum[11] of the set $\{\beta \in \omega_1 : \beta < \alpha\}$ of its predecessors . In this way, every element of ω_1 can actually be thought of as a well-ordered set. Moreover, for every countable well-ordered set C, there exists a unique $\alpha \in \omega_1$ such that C is order-isomorphic to the set of predecessors of α.

So the elements of ω_1 are precisely the isomorphism classes of countable well-orders. In particular, ω_1 contains the isomorphism class of the set of integers $\mathbb{N}$ with its usual order, which is denoted by ω. Identifying every $n \in \mathbb{N}$ with its set of predecessors $\mathbf{n} := \{0, ..., n-1\}$ equipped with the induced order, we see that the set of predecessors of ω in ω_1 is $\{\mathbf{n} : n \in \mathbb{N}\}$.

4.2 The construction rank

Now that we have our well-ordered set $(\omega_1, <)$ of countable ordinals at hand, let us see how to define functions by induction on ω_1. In general, one builds functions by induction on a well-ordered set $(A, <)$ the same way one proves assertions by induction: if we suppose that $f(y)$ is defined for all $y < x$, we have to prescribe a way of building $f(x)$. When A is the set of integers, this is the usual construction by induction of a function f, and we often only need to know $f(n)$ in order to define

[11] The strict supremum of a subset $A \subseteq B$ of an ordered set $(B, <)$ is, if it exists, the smallest $x \in B$ such that $x > y$ for all $y \in A$.

$f(n+1)$. For ω_1, the situation is complicated by the fact that not every element is of the form $\alpha+1$. Let us make sense of this last sentence and do a tiny bit of ordinal arithmetic.

Given a countable ordinal $\alpha \in \omega_1$, the nonempty set $\{\beta \in \omega_1 : \beta > \alpha\}$ has a minimum, which we denote by $\alpha+1$. The reader may check that, when seeing $\alpha+1$ as a well-ordered set, it is obtained by adding to the well-ordered set α an element $+\infty$ bigger than every element in α. Moreover, such a notation is consistent with the addition on $\mathbb{N}$.

Elements of the form $\alpha+1$ are called **successors**. But not every element is a successor: for instance ω is not a successor.

Proposition 4.4. *A countable ordinal $\alpha \in \omega_1$ is not a successor if and only if it is the supremum of its set of predecessors: $\alpha = \sup_{\beta<\alpha} \beta$.*

Proof. Suppose that $\alpha \in \omega_1$ is a successor, and write $\alpha = \beta+1$. Then the supremum of the set of predecessors of α is equal to β, hence it is not equal to α.

Conversely, if $\alpha \in \omega_1$ is not a successor, recall that α is the strict supremum of its set of predecessors. So the supremum of the set of predecessors of α has to be either α or a predecessor β of α. Let us see why the latter case cannot happen. If the supremum of the set of predecessors of α were some $\beta < \alpha$, because α is not a successor, we would have $\alpha \neq \beta+1$. Then $\alpha > \beta+1 > \beta$, contradicting the fact that β was the supremum of the set of predecessors of α. $\square$

Definition 4.5. A *non-zero* countable ordinal $\alpha \in \omega_1$ which is not a successor is called a **limit**.

Let us now go back to our initial problem and define by induction the class of elementary groups. Here our function f will map a countable ordinal α to a class of tdlc Polish groups $\mathcal{E}_\alpha$.

Definition 4.6. Let $\mathcal{E}_0$ be the class of Polish groups which are either profinite or discrete. Then, if $\alpha \in \omega_1$ is given and $\mathcal{E}_\beta$ is defined for all $\beta < \alpha$, define $\mathcal{E}_\alpha$ as follows.

- If $\alpha = \beta+1$ is a successor, we then say that $G \in \mathcal{E}_{\beta+1}$ if either
  - there exists $N \in \mathcal{E}_\beta$ such that G/N is either profinite or discrete, or
  - there exists a countable increasing family $(G_i)_{i\in\mathbb{N}}$ of open subgroups of G such that for all $i \in \mathbb{N}$, we have $G_i \in \mathcal{E}_\beta$, and moreover $G = \bigcup_{i\in\mathbb{N}} G_i$.
- If $\alpha = \sup_{\beta<\alpha} \beta$ is a limit, we let $\mathcal{E}_\alpha = \bigcup_{\beta<\alpha} \mathcal{E}_\beta$.

Note that by construction, for all $\beta < \alpha$ we have $\mathcal{E}_\beta \subseteq \mathcal{E}_\alpha$. Let us check that we have exhausted the class $\mathcal{E}$ of elementary groups. The proof is easy, but we give full details for the reader who is unacquainted with ordinals.

Proposition 4.7. *We have $\mathcal{E} = \bigcup_{\alpha\in\omega_1} \mathcal{E}_\alpha$.*

Proof. First, let us prove by induction that for all $\alpha < \omega_1$, we have $\mathcal{E}_\alpha \subseteq \mathcal{E}$. So let $\alpha \in \omega_1$, and suppose that for all $\beta < \alpha$, $\mathcal{E}_\beta \subseteq \mathcal{E}$.

- If $\alpha = 0$, because $\mathcal{E}_0$ is the class of profinite or discrete Polish groups, it is contained in $\mathcal{E}$.
- If $\alpha = \beta + 1$ is a successor, by construction of $\mathcal{E}_{\beta+1}$ and the stability properties of $\mathcal{E}$, the assumption that $\mathcal{E}_\beta \subseteq \mathcal{E}$ implies that $\mathcal{E}_{\beta+1} \subseteq \mathcal{E}$.
- If $\alpha = \sup_{\beta<\alpha} \beta$ is a limit, we have $\mathcal{E}_\alpha = \bigcup_{\beta<\alpha} \mathcal{E}_\beta \subseteq \mathcal{E}$.

This concludes our proof by induction, so we have $\bigcup_{\alpha\in\omega_1} \mathcal{E}_\alpha \subseteq \mathcal{E}$. By minimality of the class $\mathcal{E}$, we now only have to check that the class $\mathcal{F} := \bigcup_{\alpha\in\omega_1} \mathcal{E}_\alpha$ shares the defining properties of $\mathcal{E}$ in order to conclude that $\mathcal{E} = \mathcal{F}$ (see Def. A).

- The class $\mathcal{F} = \bigcup_{\alpha\in\omega_1} \mathcal{E}_\alpha$ contains $\mathcal{E}_0$, which is the class of profinite or discrete Polish groups.
- Let G be a tdlc Polish group, suppose that $N \in \mathcal{F}$ is a closed normal subgroup of G is such that G/N is profinite or discrete Polish. Then, let $\alpha \in \omega_1$ such that $N \in \mathcal{F}$. By definition of $\mathcal{E}_{\alpha+1}$, we have $G \in \mathcal{E}_{\alpha+1} \subseteq \mathcal{F}$.
- Suppose that G is written as a countable increasing union of open subgroups $(G_i)_{i\in\mathbb{N}}$ belonging to $\mathcal{F}$, and for all $i \in \mathbb{N}$, pick $\alpha_i \in \omega_1$ such that $G_i \in \mathcal{E}_{\alpha_i}$. Using Proposition 4.3, we may define $\alpha = \sup_{i\in\mathbb{N}} \alpha_i \in \omega_1$, and again by definition of $\mathcal{E}_{\alpha+1}$, we have $G \in \mathcal{E}_{\alpha+1} \subseteq \mathcal{F}$. □

Definition 4.8. Let G be an elementary group. Its **construction rank** is the smallest ordinal $\alpha \in \omega_1$ such that $G \in \mathcal{E}_\alpha$.

Since whenever α is a limit ordinal we have $\mathcal{E}_\alpha = \bigcup_{\beta<\alpha} \mathcal{E}_\beta$, the rank of an elementary group is always a successor ordinal, except when it is equal to zero.

4.3 Using the construction rank

Proposition 4.9. *Let G be an elementary group, and let $H \leqslant G$ be an open subgroup. Then H is elementary and the rank of H is smaller or equal to the rank of G.*

Proof. The proof is by induction on the construction rank, which is either null or a successor ordinal.

- The statement is clearly true for rank 0 elementary groups (recall that these are the Polish groups which are either profinite or discrete).
- Suppose G has rank $\alpha+1$ and that the proposition is true for every elementary group of rank at most α. Let H be an open subgroup of G. We have two subcases to consider.
  - Either G has a closed normal subgroup N of rank at most α such that G/N is profinite or discrete. Then $N \cap H$ is a closed normal subgroup of H. But $N \cap H$ is also an open subgroup of N, so by our induction hypothesis it has rank at most α. Since $H/N \cap H$ is topologically isomorphic to the open subgroup $HN/N \leqslant G/N$, we deduce that H has rank at most $\alpha + 1$, which is smaller or equal to the rank of G.

– Or G may be written as an increasing union of open subgroups $(G_i)_{i\in\mathbb{N}}$ of rank at most α. Then for all $i \in \mathbb{N}$, $G_i \cap H$ is an open subgroup of G_i, so by our induction hypothesis it has rank at most α. Again, we deduce that H has rank at most $\alpha + 1$, hence smaller or equal to the rank of G. □

The theorem that ends this section is the only place in this survey where we need the construction rank. Let us first present one useful lemma on elementary groups from which the result will follow easily.

Lemma 4.10. *Let G be a compactly generated elementary group of rank $\alpha+1$. Then G has a closed normal subgroup N of rank α such that G/N is profinite or discrete.*

Proof. Because G has rank $\alpha+1$, by definition either G has a closed normal subgroup N of rank α such that G/N is profinite or discrete, or G can be written as an increasing union of open subgroups $(G_i)_{i\in\mathbb{N}}$ of rank at most α. We want to show that the latter case is contradictory. Let S be a compact generating set of G. Then $(G_i)_{i\in\mathbb{N}}$ is an open cover of S. Because $(G_i)_{i\in\mathbb{N}}$ is increasing and S is compact, there exists $i \in \mathbb{N}$ such that G_i contains S. But then $G_i = G$, which contradicts the fact that G_i has rank at most α. □

Theorem 4.11. *Let $P(G)$ be a property. Then to show that $P(G)$ is true for every elementary group G, it suffices to show that:*

(i) $P(G)$ is true whenever G is a Polish group which is either profinite or discrete;

(ii) if G is an elementary group and N is a closed normal subgroup of G such that $P(N)$ is true and G/N is profinite or discrete, then $P(G)$ is true;

(iii) if G is an elementary group, then there exists an increasing chain $(G_i)_{i\in\mathbb{N}}$ of open compactly generated subgroups of G such that the implication

$$(\forall i \in \mathbb{N}, P(G_i)) \Rightarrow P(G)$$

holds.

Proof. The proof is by induction on the construction rank, which is either zero or a successor ordinal.

- By assumption (i), $P(G)$ is true for every rank 0 elementary group.
- Suppose that G has rank $\alpha + 1$ and that $P(H)$ is true for all H of rank at most α. By asumption (iii), we may fix an increasing chain $(G_i)_{i\in\mathbb{N}}$ of open compactly generated subgroups of G such that the implication

$$(\forall i \in \mathbb{N}, P(G_i)) \Rightarrow P(G)$$

holds. Let $i \in \mathbb{N}$. By Proposition 4.9, the group G_i has rank at most $\alpha + 1$. The previous lemma provides a closed normal subgroup N_i of G_i which has rank at most α, and such that G_i/N_i is profinite or discrete. By our induction hypothesis, we have that $P(N_i)$ is true, and by asumption (ii), this implies that $P(G_i)$ is true. So $P(G_i)$ is true for all $i \in \mathbb{N}$, and by asumption (iii) we conclude that $P(G)$ is true. □

5 More permanence properties

The main goal of this section is to show that every Hausdorff quotient of an elementary group is elementary (Thm. 5.5). Along the way, we will also prove some results needed for the decomposition theorem (Thm. B).

The following lemma echoes item (iii) in the previous theorem.

Lemma 5.1. *Let G be a tdlc Polish group. Then there exists an increasing sequence $(G_i)_{i\in\mathbb{N}}$ of open compactly generated subgroups of G such that $G = \bigcup_{i\in\mathbb{N}} G_i$. Moreover, if K is a compact group acting continuously on G by automorphisms, one can choose the G_i so that each of them is setwise fixed by K.*

Proof. Let U be an open compact subgroup of G, and let $(g_i)_{i\in\mathbb{N}}$ enumerate a countable dense family of elements of G. For every $i \in \mathbb{N}$, we let $S_i = U \cup \{g_0, ..., g_i\}$. Then $K \cdot S_i$ is compact and open, and we let G_i be the group generated by $K \cdot S_i$. Clearly the sequence $(G_i)_{i\in\mathbb{N}}$ is increasing, and each G_i is compactly generated. The sequence $(G_i)_{i\in\mathbb{N}}$ is also exhaustive by density of (g_i) and the fact that U is an open subgroup of G. Furthermore, the fact that K acts by automorphisms on G guarantees that each G_i is setwise fixed by K. □

Definition 5.2. A tdlc group G is called **quasi-discrete** if it has a dense subgroup whose elements have an open centralizer.

Caprace and Monod have shown that every compactly generated quasi-discrete group is SIN [CM11, Prop. 4.3]. The following lemma is very close to their result.

Lemma 5.3. *Let G be a quasi-discrete tdlc Polish group. Then G is elementary.*

Proof. Let U be a compact open subgroup of G, let $(g_i)_{i\in\mathbb{N}}$ be a countable dense set such that for all $i \in \mathbb{N}$, the centralizer of g_i is open. For all $n \in \mathbb{N}$, let $V_n \triangleleft U$ be an open subgroup such that every element of V_i commutes with $g_1, ..., g_n$. Then V_n is a compact open normal subgroup of $G_n = \langle U, g_1, ..., g_n\rangle$. Because it has such a compact open normal subgroup, G_n is an elementary open subgroup of G, hence $G = \bigcup_{n\in\mathbb{N}} G_n$ is elementary. □

We now need a technical lemma which will be superseded later by Corollary 5.6.

Lemma 5.4. *Let G be a tdlc Polish group, and let M, L be two closed normal subgroups of G intersecting trivially. If M is elementary, then $\overline{ML}/L$ also is.*

Proof. This is done using the inductive scheme provided by Theorem 4.11, where the property $P(M)$ we want to prove is "whenever M arises as a closed normal subgroup of some tdlc Polish group G, and L is another closed normal subgroup of G intersecting M trivially, then $\overline{ML}/L$ is elementary". To make the proof lighter, we won't make any reference to the ambient group G.

Note that the fact that $M \cap L$ is trivial implies $[M, L] = \{e\}$. So the conjugacy of L on M is trivial, in particular L normalizes any subgroup of M.

(i) First suppose that M is profinite. Then ML is closed, so $\overline{ML}/L$ is a continuous quotient of M, hence elementary. Next, if M is discrete, each element of M has an open centralizer in $\overline{ML}$, and so we may apply Lemma 5.3 to $\overline{ML}/L$ and deduce that $\overline{ML}/L$ is elementary.

(ii) Suppose that there exists a closed normal subgroup N of M such that $P(N)$ holds and M/N is either profinite or discrete. Then $\overline{NL}/L$ is elementary, moreover it is a closed normal subgroup of $\overline{ML}/L$, so because the class of elementary groups is stable under group extension (Proposition 3.1), we only need to show that

$$(\overline{ML}/L)/(\overline{NL}/L) \simeq \overline{ML}/\overline{NL}$$

is elementary. But M/N is profinite or discrete, so by case (i) applied to $M' = M/N$ and $L' = \overline{NL}/N$, we have that $\overline{M'L'}/L'$ is elementary. By the isomorphism theorem, the latter is isomorphic to $\overline{ML}/\overline{NL}$, which is thus elementary.

(iii) Let K be a compact open subgroup of $\overline{ML}$. Then K acts continuously on M by conjugation, so Lemma 5.1 provides an exhaustive increasing chain $(M_i)_{i\in\mathbb{N}}$ of open compactly generated subgroups of M such that each M_i is normalized by K. Suppose that $P(M_i)$ is true for every $i \in \mathbb{N}$, and fix $i \in \mathbb{N}$. Since the closed subgroup M_i is normalized by L, it is a normal subgroup of $\overline{M_iL}$. So we can apply $P(M_i)$ and deduce that the group $\overline{M_iL}/L$ is elementary. Recall that K normalizes M_i, so K normalizes M_iL. Because $(KM_iL/L)/(\overline{M_iL}/L)$ is a quotient of K, hence profinite, we deduce that KM_iL/L is elementary.

Now, $\overline{ML}/L = KML/L$ arises as the increasing union of the elementary open subgroups KM_iL/L, hence it is elementary. □

Theorem 5.5. *Let G be an elementary tdlc Polish group, and let N be a closed normal subgroup of G. Then G/N is elementary.*

Proof. Consider the smallest class $\mathcal{F}$ of elementary tdlc Polish groups G such that for all closed $N \triangleleft G$, the group G/N is elementary. We will show that $\mathcal{F}$ satisfies the same properties as the class of elementary groups, hence coincides with it. First, $\mathcal{F}$ clearly contains profinite and discrete groups.

Then, let G be an elementary group and assume that $M \in \mathcal{F}$ is a normal subgroup of G such that G/M is either profinite or discrete. Let N be a normal subgroup in G, consider the group $\tilde{G} = G/M\cap N$. Then if we let $\tilde{M} = M/M\cap N$ and $\tilde{N} = N/M \cap N$, the groups $\tilde{M}$ and $\tilde{N}$ are closed normal subgroups of $\tilde{G}$ intersecting trivially. Moreover, since $M \in \mathcal{F}$ the group $\tilde{M}$ is elementary. Thus, we may apply Lemma 5.4 to them and deduce that $\overline{\tilde{M}\tilde{N}}/\tilde{N}$ is elementary. But then, $\overline{MN}/N$ is isomorphic to $\overline{\tilde{M}\tilde{N}}/\tilde{N}$, hence elementary. Since $(G/M)/(\overline{MN}/N)$ is isomorphic to the group $G/\overline{MN}$ which is a quotient of the profinite or discrete group G/N, we deduce that $(G/N)/(\overline{MN}/N)$ is elementary. But elementariness is stable under extensions (Prop. 3.1), so G/N is elementary.

In order to conclude the proof, we need to deal with increasing unions: let G be an elementary group, written as an increasing union $G = \bigcup_{i\in\mathbb{N}} G_i$ of open subgroups belonging to $\mathcal{F}$. Let N be a closed normal subgroup in G. Then for all $i \in \mathbb{N}$, $N\cap G_i$ is a closed normal subgroup of G_i, hence $G_i/(N \cap G_i)$ is elementary, but now G/N may be written as an increasing union of the projections of the G_i's onto G/N . Moreover, each projection of G_i onto is isomorphic to $G_i/(N \cap G_i)$, so G/N is an increasing union of open elementary subgroups, hence elementary. □

Corollary 5.6. *Let G be a tdlc Polish group, let M, L be two closed normal subgroups of G. If M is elementary, then $\overline{ML}/L$ also is.*

Proof. Consider as in the proof of Theorem 5.5 the quotient group $\tilde{G} = G/(M \cap L)$, inside which $\tilde{M} := M/(M \cap L)$ and $\tilde{L} := L/(M \cap L)$ are closed, normal and have trivial intersection. By Theorem 5.5 the group $\tilde{M}$ is elementary, so we may apply Lemma 5.4 and deduce that $\overline{\tilde{M}\tilde{L}}/\tilde{L}$ is elementary. But the latter is isomorphic to $\overline{ML}/L$, which concludes the proof. □

Corollary 5.7. *Let G be a tdlc Polish group, and let L, M be two elementary closed normal subgroups of G. Then $\overline{ML}$ is elementary.*

Proof. By the previous corollary, $\overline{ML}/L$ is elementary. But because L is elementary and the class of elementary groups is stable under extension (Prop. 3.1), the group $\overline{LM}$ is also elementary. □

Here is the last permanence property that we will need in order to prove the decomposition theorem. Actually, we only need it for the much easier case where all the C_i's are normalized by some fixed open subgroup of G.

Theorem 5.8. *Let G be a Polish tdlc group. Suppose that there exists an increasing sequence of elementary subgroups $(C_i)_{i\in\mathbb{N}}$ of G such that each C_i has an open normalizer in G, and that $G = \overline{\bigcup_{i\in\mathbb{N}} C_i}$. Then G is elementary.*

Proof. By Lemma 5.1, we can write G as an increasing countable union of open compactly generated subgroups. Since these subgroups will satisfy the same assumption as G, and since an increasing union of open elementary subgroups is elementary, we only have to show that the theorem holds for G compactly generated.

So assume that G is compactly generated and fix a compact open subgroup U of G. By Lemma 1.2 applied to the dense subgroup $D = \bigcup_{i\in\mathbb{N}} C_i$, there exists $i \in \mathbb{N}$ and a finite subset $A \subseteq C_i$ such that $G = \langle A \rangle U$. Let V be an open subgroup of U that normalizes C_i, and let B be the compact reunion of the V-conjugates of A. Then $\overline{\langle B \rangle}$ is normalized by V, and it it is a closed subgroup of C_i, hence elementary by Proposition 3.2. Because $\overline{\langle B \rangle}V/\overline{\langle B \rangle}$ is a quotient of V, hence profinite, we deduce that $\overline{\langle B \rangle}V$ is elementary. But V has finite index in U and B contains A, so $\overline{\langle B \rangle}V$ has finite index in $G = \langle A \rangle U$. Having an elementary closed subgroup of finite index, G has to be elementary by Corollary 3.3. □

6 The decomposition theorem

We are now almost ready to understand how a compactly generated tdlc Polish groups can be decomposed into elementary and topologically characteristically simple non-elementary pieces. The main tool for doing this is the existence, inside any tdlc Polish group, of a maximum elementary closed normal subgroup.

Theorem 6.1 ([Wes15, Thm 1.5])**.** *Let G be a tdlc Polish group. Then the family of closed elementary normal subgroups of G has a unique maximum with respect to inclusion.*

This maximum is called the **elementary radical** of G, and denoted by $\mathrm{Rad}_{\mathcal{E}}(G)$. Note that it is a topologically characteristic subgroup of G, meaning that every continuous automorphism of G fixes $\mathrm{Rad}_{\mathcal{E}}(G)$ setwise.[12]

Moreover, by Proposition 3.1, the quotient $G/\mathrm{Rad}_{\mathcal{E}}(G)$ must have trivial elementary radical.

Proof of Theorem 6.1. Let $(U_n)_{n\in\mathbb{N}}$ be a countable basis of open subsets of G. Let $\mathcal{F}$ be the set of $n \in \mathbb{N}$ such that there exists a closed elementary normal subgroup intersecting U_n. For each $n \in \mathcal{F}$, choose such a closed elementary normal subgroup N_n intersecting U_n. Enumerate[13] $\mathcal{F} = \{n_k : k \in \mathbb{N}\}$, and let

$$N = \overline{\bigcup_{k\in\mathbb{N}} N_{n_1} \cdots N_{n_k}}.$$

By Corollary 5.7, each $\overline{N_{n_1} \cdots N_{n_k}}$ is elementary, so by Theorem 5.8, N is an elementary closed normal subgroup of G. By definition, N intersects every basic open set U_n which intersects some elementary closed normal subgroup. So every U_n that does not intersect N must intersect no elementary closed normal subgroup. But because N is closed, its complement may be written as a reunion of such U_n's. This implies that N is the unique maximum of the class of closed elementary normal subgroups of G. □

The second tool was developed by Caprace and Monod, and provides a way to decompose the Polish tdlc compactly generated groups which have a trivial elementary radical.

Definition 6.2. A tdlc Polish group G is **locally elliptic** if every finite subset of G generates a group with compact closure.

A result of Platonov asserts that every locally elliptic tdlc Polish group is an increasing union of open compact subgroups, so that in particular it is elementary (see [Wes15, Sec. 2.4]).

Definition 6.3. Let G be a tdlc Polish group. One says that G is the **quasi-product** of the closed normal subgroups $N_1, ..., N_k \leqslant G$ if the product map $N_1 \times \cdots \times N_k \to G$ which maps $(n_1, ..., n_k)$ to $n_1 \cdots n_k$ is injective and has dense image.

Theorem 6.4 (Caprace-Monod, [CM11, Thm. B][14])**.** *Let G be a compactly generated tdlc group. Then one of the following holds:*

(1) G has an infinite discrete normal subgroup;

(2) G has a non-trivial locally elliptic closed normal subgroup;

(3) G has exactly $0 < n < \infty$ minimal non-trivial closed normal subgroups.

[12] Continuous automorphisms of G are also homeomorphisms, since G is Polish.

[13] If $\mathcal{F}$ is finite, we allow for repetitions in the enumeration.

[14] As noted by Wesolek, the result in this paper contains a mistake, so we give here the corrected version.

Corollary 6.5 (Wesolek). *Let G be a non-trivial compactly generated tdlc Polish group with trivial elementary radical. Then G contains a topologically characteristic closed subgroup which decomposes as a quasi-product of $0 < n < \infty$ non-elementary closed normal subgroups.*

Proof. We apply the previous theorem to G. Because G has a trivial elementary radical, cases (1) and (2) cannot hold, so we deduce that G has exactly $0 < n < \infty$ minimal closed normal subgroups $N_1, ..., N_n$. Let $H = \overline{N_1 \cdots N_n}$. The subgroup H is topologically characteristic for its definition is clearly invariant under continuous group automorphisms.

Observe that the N_i's pairwise commute since for all $i \neq j$, $[N_i, N_j]$ is contained in $N_i \cap N_j$, which is trivial by minimality.

We now prove that for all $1 \leqslant m < n$, $\overline{N_1 \cdots N_m} \cap N_{m+1}$ is trivial. Indeed, if it is not trivial, it must contain N_{m+1} by minimality. But N_{m+1} commutes with every N_i for $i \leqslant m$, so in particular $\overline{N_1 \cdots N_m}$ has a non-trivial center C. Now such a center C is a closed characteristic subgroup of $\overline{N_1 \cdots N_m}$, hence C is a non-trivial abelian closed normal subgroup of G. Since abelian tdlc Polish groups are elementary, this contradicts the fact that G has a trivial elementary radical.

So for all $1 \leqslant m < n$, the groups $\overline{N_1 \cdots N_m}$ and N_{m+1} intersect trivially, which easily yields by induction that the product map $N_1 \times \cdots \times N_n \to H$ is injective. In other words, $H = \overline{N_1 \cdots N_n}$ is the quasi-product we seek. □

The next lemma is where Cayley-Abels graphs show up (see Lem. 1.2 and the paragraph thereafter for a reminder about these).

Lemma 6.6. *Let G be a nontrivial tdlc Polish group with trivial elementary radical, and let N be a non-trivial closed normal subgroup of G. Denote by $\pi : G \to G/N$ the natural projection. Let U be a compact open subgroup of G, and A be a finite subset of G such that $G = \langle A \rangle U$. Then the Cayley-Abels graph $\mathcal{C}_{A,U}(G)$ has a degree strictly greater than the degree of $\mathcal{C}_{\pi(A),\pi(U)}(G/N)$.*

Proof. First, the action on the Cayley-Abels graph $\mathcal{C}_{A,U}(G)$ has compact kernel, so since G has trivial elementary radical, this action must be faithful. Now consider the action of N on $\mathcal{C}_{A,U}(G)$. Because G has trivial elementary radical, N is non-discrete, hence there exists $g \in N \cap U \setminus \{1\}$. Such a g fixes the vertex $v := U$. Now, since the N-action is faithful, there exists another vertex $w \in \mathcal{C}_{A,U}(G)$ which is not fixed by g. In particular, because $\mathcal{C}_{A,U}(G)$ is connected, the boundary[15] of the set of g-fixed vertices has to be nonempty, implying that there exists a vertex in $\mathcal{C}_{A,U}(G)$ with two distinct neighbors belonging to the same N-orbit. As observed at the end of section 1, this yields that the degree of $\mathcal{C}_{\pi(A),\pi(U)}(G/N)$ is strictly less than the degree of $\mathcal{C}_{A,U}(G)$. □

Now the path to a decomposition theorem is clear. If G is a compactly generated tdlc Polish group, we apply the following algorithm to it until we get a Cayley-Abels graph of degree 0, that is, a compact (hence elementary) group:

[15]The boundary of a set F of vertices is the set of vertices not belonging to F, but connected to an element of F.

(A) quotient G by its elementary radical to obtain a group G' with trivial elementary radical. If $G = G'$ then stop. If $G \neq G'$, then

(B) quotient G' by the non-trivial quasi-product provided by Corollary 6.5, obtaining a new group H. If H is not compact, proceed to step (A), replacing G by H.

Indeed, the previous lemma guarantees that the degree of the associated Cayley-Abels graphs drops by at least one each time we apply both steps (in particular, the length of the characteristic series obtained by running this algorithm is bounded above by the minimal degree of the Cayley-Abels graphs of G). By lifting the obtained quasi-products back to G, we obtain a proof of the following theorem, also known as Theorem B.

Theorem 6.7. *Let G be a compactly generated tdlc Polish group. Then there exists a finite increasing sequence*

$$H_0 = \{e\} \leqslant \cdots \leqslant H_n$$

of closed characteristic subgroups of G such that

1. *G/H_n is an elementary group, and*
2. *For all $i = 0, ..., n-1$, the group $(H_{i+1}/H_i)/\mathrm{Rad}_{\mathcal{E}}(H_{i+1}/H_i)$ is a finite quasi-product of topologically characteristically simple non-elementary subgroups.*

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