

$H^2(\mathbf{SL}_3(\mathbb{Z}[t]); \mathbb{Q})$ is infinite dimensional

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Abstract

We prove that $H^2(\mathbf{SL}_3(\mathbb{Z}[t]); \mathbb{Q})$ is infinite dimensional. The proof follows an outline similar to recent results by Cobb, Kelly, and Wortman, using the Euclidean building for $\mathbf{SL}_3(\mathbb{Q}((t^{-1})))$ and a Morse function from Bux-Köhl-Witzel.

1 Introduction

Krstić-McCool proved that $\mathbf{SL}_3(\mathbb{Z}[t])$ is not finitely presented [KM99]. In [BMW10], Bux-Mohammadi-Wortman show that $\mathbf{SL}_n(\mathbb{Z}[t])$ is not FP_{n-1} . In general, a group G being of type FP_k implies that $H^k(G; M)$ must be finitely generated, where M is a $\mathbb{Z}G$ -module.

In [Wor13], Wortman exhibits a finite index subgroup $\Gamma \leq \mathbf{SL}_n(\mathbb{F}_q[t])$ such that $H^{n-1}(\Gamma; \mathbb{F}_p)$ is infinite dimensional. In [Cob15], Cobb shows that $H^2(\mathbf{SL}_2(\mathbb{Z}[t, t^{-1}]); \mathbb{Q})$ is infinite dimensional. In [Kel13], Kelly exhibits a finite index subgroup $\Gamma \leq \mathbf{B}_n(\mathbb{F}_q[t, t^{-1}])$ such that $H^2(\Gamma; \mathbb{F}_p)$ is infinite dimensional, where $\mathbf{B}_n(\mathbb{F}_p[t, t^{-1}])$ is the upper triangular subgroup of $\mathbf{SL}_n(\mathbb{F}_p[t, t^{-1}])$ and $p \neq 2$.

In this paper, we prove the following result:

Theorem 1. $H^2(\mathbf{SL}_3(\mathbb{Z}[t]); \mathbb{Q})$ is infinite dimensional.

We will let $\Gamma = \mathbf{SL}_3(\mathbb{Z}[t])$ and $G = \mathbf{SL}_3(\mathbb{Q}((t^{-1})))$.

First, we will use ideas from Bux-Köhl-Witzel [BKW13] to define an $\mathbf{SL}_3(\mathbb{Q}[t])$ -invariant piecewise linear Morse function on the Euclidean building for $\mathbf{SL}_3(\mathbb{Q}((t^{-1})))$. Then we will construct a 2-connected Γ -complex Y , which will be built from a connected subset of the Euclidean building by gluing cells as freely as possible until we arrive at a 2-connected complex. We will show that $H^2(\Gamma \backslash Y; \mathbb{Q})$ is infinite dimensional by constructing infinite linearly independent families of 2-cocycles and

2-cycles that pair nontrivially. Finally, we will use the equivariant homology spectral sequence with

$$E_{p,q}^2 = H_p(\Gamma \backslash Y; \{H_q(\Gamma_\sigma; \mathbb{Q})\}) \Rightarrow H_{p+q}(\Gamma; \mathbb{Q})$$

to show that the infinite dimension of $H^2(\Gamma \backslash Y; \mathbb{Q})$ implies that $H^2(\Gamma; \mathbb{Q})$ is infinite dimensional.

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2 Preliminaries

Let X be the Euclidean building for $\mathbf{SL}_3(\mathbb{Q}((t^{-1})))$. X is a 2-dimensional simplicial complex, with vertices corresponding to the homothety classes of 3-dimensional $\mathbb{Q}[[t^{-1}]$ -lattices (two lattices are in the same homothety class if one is a nonzero scalar multiple of the other) in $\mathbb{Q}((t^{-1}))^3$. A basis $\{v_1, v_2, v_3\}$ for $\mathbb{Q}((t^{-1}))^3$ gives rise to the $\mathbb{Q}[[t^{-1}]$ -lattice

$$v_1\mathbb{Q}[[t^{-1}]] \oplus v_2\mathbb{Q}[[t^{-1}]] \oplus v_3\mathbb{Q}[[t^{-1}]]$$

We will let $v_1 \oplus v_2 \oplus v_3$ denote the lattice above.

Note that $\mathbf{SL}_3(\mathbb{Q}((t^{-1})))$ acts linearly on the vector space $\mathbb{Q}((t^{-1}))^3$, and therefore on $\mathbb{Q}[[t^{-1}]$ -lattices, and this gives an $\mathbf{SL}_3(\mathbb{Q}((t^{-1})))$ -action on the vertices of X . Let x_0 represent the vertex corresponding to the equivalence class of the $\mathbb{Q}[[t^{-1}]$ -lattice generated by the standard basis, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. The $\mathbf{SL}_3(\mathbb{Q}((t^{-1})))$ -stabilizer of x_0 is $\mathbf{SL}_3(\mathbb{Q}[[t^{-1}]])$. Let \mathcal{A}_0 represent the apartment of X which is stabilized by the diagonal subgroup of $\mathbf{SL}_3(\mathbb{Q}((t^{-1})))$, and let \mathcal{C}_0 represent the chamber in \mathcal{A}_0 which contains x_0 and is stabilized by the subgroup of upper-triangular matrices in $\mathbf{SL}_3(\mathbb{Q}[[t^{-1}]])$. We will refer to \mathcal{A}_0 as the *standard apartment*, \mathcal{C}_0 as the *standard chamber*, and x_0 as the *standard vertex*.

The subgroup of permutation matrices (matrices with exactly one entry of ± 1 in each row and column, and all other entries 0) acts transitively on the 6 chambers in \mathcal{A}_0 which contain x_0 . There are 6 sectors in \mathcal{A}_0 based at x_0 , separated by the three walls in \mathcal{A}_0 which pass through x_0 , and the permutation subgroup acts transitively on these sectors. Let \mathcal{S}_0 be the sector which contains the standard chamber \mathcal{C}_0 . \mathcal{S}_0 is a strict fundamental domain for the action of $\mathbf{SL}_3(\mathbb{Q}[t])$ on X [Sou77].

Let $X_\Gamma = \Gamma \mathcal{S}_0$, and observe that $\mathcal{A}_0 \subset X_\Gamma$ because Γ contains the permutation matrices in $\mathbf{SL}_3(\mathbb{Z})$ which act transitively on the sectors of \mathcal{A}_0 based at x_0 .

2.1 Cell Stabilizers

In this section, we will discuss the Γ -stabilizers of cells in \mathcal{S}_0 . For simplicity, we will let $\Gamma_\sigma = \text{Stab}_\Gamma(\sigma)$ and $G_\sigma = \text{Stab}_G(\sigma)$ for a cell $\sigma \subset X$. (Recall that $G = \mathbf{SL}_3(\mathbb{Q}((t^{-1})))$ and $\Gamma = \mathbf{SL}_3(\mathbb{Z}[t])$.)

Lemma 2. *If x is a vertex in \mathcal{S}_0 , then Γ_x has one of the following forms, where $u, v, w \in \mathbb{Z}[t]$, $a, b, c, d \in \mathbb{Z}$ such that $|ad - bc| = 1$, and k and m are nonnegative integers which depend on x .*

1. If x_0 is the standard vertex of X , then $\Gamma_{x_0} = \mathbf{SL}_3(\mathbb{Z})$.

2. If x is a vertex in the interior of \mathcal{S}_0 , then

$$\Gamma_x = \left\{ \left(\begin{array}{ccc} \pm 1 & u & w \\ 0 & \pm 1 & v \\ 0 & 0 & \pm 1 \end{array} \right) \middle| \deg(u) \leq k, \deg(v) \leq m, \deg(w) \leq m + k \right\}$$

3. If x is a vertex in $\partial\mathcal{S}_0$, and $x \neq x_0$, then Γ_x has one of the following forms:

$$\Gamma_x = \left\{ \left(\begin{array}{ccc} a & b & w \\ c & d & v \\ 0 & 0 & \pm 1 \end{array} \right) \middle| \deg(w), \deg(v) \leq k \right\}$$

$$\Gamma_x = \left\{ \left(\begin{array}{ccc} \pm 1 & u & w \\ 0 & a & b \\ 0 & c & d \end{array} \right) \middle| \deg(u), \deg(w) \leq k \right\}$$

Proof. First, observe that $G_{x_0} = \mathbf{SL}_3(\mathbb{Q}[[t^{-1}]])$, and therefore

$$\Gamma_{x_0} = G_{x_0} \cap \Gamma = \mathbf{SL}_3(\mathbb{Z})$$

Any vertex x in \mathcal{S}_0 corresponds to a $\mathbb{Q}[[t^{-1}]$]-lattice of the form

$$t^i e_1 \oplus t^j e_2 \oplus e_3$$

for nonnegative integers $j \leq i$, where $\{e_1, e_2, e_3\}$ is the standard basis for $\mathbb{Q}((t^{-1}))^3$. Any vertex in $\partial\mathcal{S}_0$ corresponds to a lattice with either $j = 0$ or $i = j$. Letting

$$g = \begin{pmatrix} t^i & 0 & 0 \\ 0 & t^j & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we have

$$g(e_1 \oplus e_2 \oplus e_3) = t^i e_1 \oplus t^j e_2 \oplus e_3$$

Therefore, $\Gamma_x = (g\mathbf{SL}_3(\mathbb{Q}[[t^{-1}]])g^{-1}) \cap \Gamma$. Computing gAg^{-1} for an arbitrary matrix $A \in \mathbf{SL}_3(\mathbb{Q}[[t^{-1}]])$ gives

$$gAg^{-1} = g \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} g^{-1} = \begin{pmatrix} a_{11} & t^{i-j}a_{12} & t^i a_{13} \\ t^{j-i}a_{21} & a_{22} & t^j a_{23} \\ t^{-i}a_{31} & t^{-j}a_{32} & a_{33} \end{pmatrix}$$

where $a_{ij} \in \mathbb{Q}[[t^{-1}]]$. If $gAg^{-1} \in \Gamma$, then we obtain the following form for gAg^{-1} :

$$\begin{pmatrix} \deg = 0 & \deg \leq (i-j) & \deg \leq i \\ \deg \leq j-i & \deg = 0 & \deg \leq j \\ \deg \leq -i & \deg \leq -j & \deg = 0 \end{pmatrix}$$

If x is in the interior of \mathcal{S}_0 , then $i > j > 0$ and we take $k = i - j$ and $m = j$.

If x is in the boundary of \mathcal{S}_0 , then either $j = 0$ or $i = j$. If $j = i = 0$, then $x = x_0$, so we may assume $i \neq 0$. In either case ($j = 0$ or $i = j$) we take $k = i$. Depending on whether or not $j = 0$, we obtain one of the two forms for Γ_x stated in the lemma. \square

Lemma 3. *For σ a subcell of \mathcal{C}_0 , Γ_σ is of type F_1 .*

A much stronger result is proved in [BMW10], where it is shown that if σ is any cell in X , then Γ_σ is of type F_∞ . However, we only make use of the specific case above, and provide a short proof here:

Proof. First, recall that a group is type F_1 if and only if it is finitely generated. First, suppose σ is a 0-cell. It is easy to see that Γ_σ is finitely generated by Lemma 2.

Let $e_{ij}(a)$ represent the elementary matrix with a in the ij^{th} entry, 1's on the diagonal and 0's elsewhere.

Suppose σ is a 1-cell in \mathcal{C}_0 . If σ contains x_0 , then Γ_σ is a maximal parabolic subgroup of $\mathbf{SL}_3(\mathbb{Z})$ and is therefore finitely generated. If σ does not contain x_0 , then Γ_σ is upper-triangular and generated by $e_{12}(1), e_{23}(1), e_{13}(1), e_{13}(t)$, and the finite diagonal subgroup of $\mathbf{SL}_3(\mathbb{Z})$.

Finally, suppose $\sigma = \mathcal{C}_0$. In this case Γ_σ is the upper-triangular subgroup of $\mathbf{SL}_3(\mathbb{Z})$ and it is easy to see that this group is finitely generated. \square

2.2 Morse Function

If Z is a CW-complex, let $Z^{(i)}$ denote the i -skeleton of Z .

A function $h : X \rightarrow \mathbb{R}$ is a *piecewise linear Morse function* (or *Morse function*) if h restricts to an affine (height) function on every simplex, $h(X^{(0)})$ is discrete, and h is not constant on any simplex of dimension at least 1. Our goal in this section will be to define a Γ -invariant Morse function on X_Γ , and an $\mathbf{SL}_3(\mathbb{Q}[t])$ -invariant Morse function on X . Since the standard sector, \mathcal{S}_0 , is a strict fundamental domain for Γ acting on X_Γ (respectively, for $\mathbf{SL}_3(\mathbb{Q}[t])$ acting on X), any Morse function on \mathcal{S}_0 can be extended to a Γ -invariant Morse function on X_Γ (respectively, to an $\mathbf{SL}_3(\mathbb{Q}[t])$ -invariant Morse function on X).

The Morse function we define on X is essentially the same one defined by Bux-Köhl-Witzel [BKW13]. We will make this statement more precise in Remark 5.

Define a function \hat{h} on $\mathcal{S}_0^{(0)}$ by $\hat{h}(x) = d(x_0, x)$, where d is the Euclidean metric on \mathcal{A}_0 . A first attempt at extending \hat{h} to \mathcal{S}_0 would be to extend using barycentric coordinates on each simplex. However, there is a sequence of edges in the middle of the sector which are flat with respect to this extension. We denote this sequence by

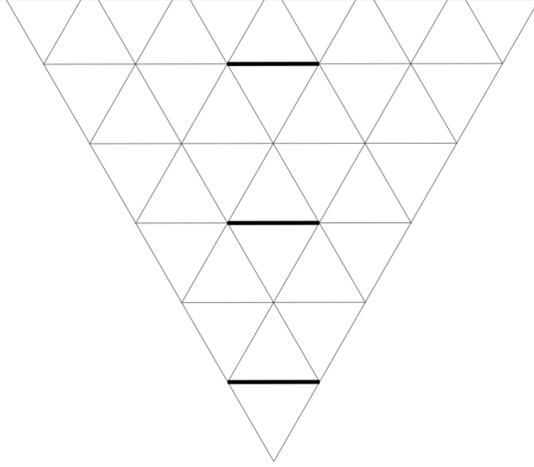


Figure 1: The sector \mathcal{S}_0 , with the edges which are flat under \hat{h} highlighted.

$\{\eta_n\}_{n \in \mathbb{N}}$. Specifically, η_n is the edge spanned by the vertices $t^{2n+1}e_1 \oplus t^n e_2 \oplus e_3$ and $t^{2n+1}e_2 \oplus t^{n+1}e_2 \oplus e_3$.

For each n , η_n is contained in two chambers of \mathcal{S}_0 . Let \mathcal{C}_n^\uparrow be the chamber in \mathcal{S}_0 which is above η_n (more precisely, the chamber with $\hat{h}(v) > \hat{h}(\eta_n^{(0)})$ for the vertex v which is not in η_n), and \mathcal{C}_n^\downarrow the chamber below η_n . Let \hat{X} denote the barycentric

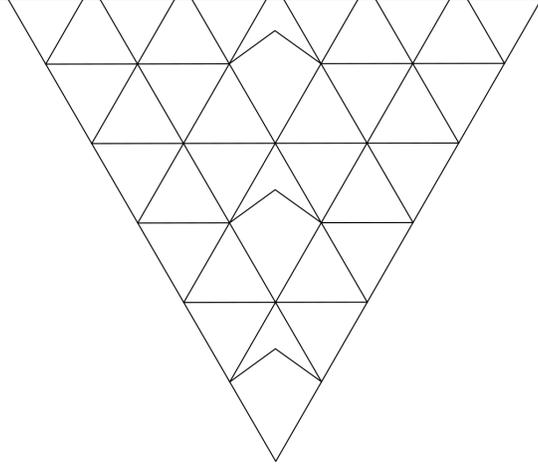


Figure 2: The sector \mathcal{S}_0 with $\hat{h}(\mathring{\eta}_n)$ redefined.

subdivision of X , and similarly let $\mathring{\sigma}$ denote the barycenter of a cell $\sigma \subset X$. We will extend \hat{h} to cells in $\mathring{\mathcal{S}}_0^{(0)}$ which do not intersect $\{\eta_n\}_{n \in \mathbb{N}}$ using barycentric coordinates, then choose $\hat{h}(\mathring{\eta}_n)$ such that

$$\hat{h}(\mathring{\partial}\eta_{n+1}) > \hat{h}(\mathring{\eta}_n) > \hat{h}(\mathring{\partial}\eta_n)$$

Finally, extend \hat{h} to cells which intersect $\{\eta_n\}_{n \in \mathbb{N}}$ using barycentric coordinates.

Note that \hat{h} is discrete on the vertices of $\mathring{\mathcal{S}}_0$ and bounded below by 0, so there is a function $h : \mathring{\mathcal{S}}_0^{(0)} \rightarrow \mathbb{Z}$ such that h and \hat{h} induce the same ordering on $\mathring{\mathcal{S}}_0^{(0)}$ and $h(x_0) = 0$. Extending h to the 1- and 2-cells of $\mathring{\mathcal{S}}_0$ by using barycentric coordinates, then Γ -invariantly to \mathring{X}_Γ , we obtain a Γ -invariant function on \mathring{X}_Γ . We may also extend h to an $\mathbf{SL}_3(\mathbb{Q}[t])$ -invariant function, \bar{h} , on \mathring{X} , because \mathcal{S}_0 is a strict fundamental domain for the action of $\mathbf{SL}_3(\mathbb{Q}[t])$ on X .

Let $y_n = \mathring{\eta}_n$.

Lemma 4. *h and \bar{h} are piecewise linear Morse functions.*

Proof. It suffices to show that $h|_{\mathring{\mathcal{S}}_0} = \bar{h}|_{\mathring{\mathcal{S}}_0}$ is Morse, since h and \bar{h} are respectively Γ - and $\mathbf{SL}_3(\mathbb{Q}[t])$ -invariant and $\mathring{\mathcal{S}}_0$ is a strict fundamental domain for the respective group actions on \mathring{X}_Γ and \mathring{X} . By construction, $h(\mathring{\mathcal{S}}_0^{(0)})$ is discrete in \mathbb{R} . Since h is defined on 1- and 2-simplices by using barycentric coordinates, h restricts to a height function on simplices.

Let $\sigma \in \mathring{\mathcal{S}}$ be a cell. We must show that if h is constant on σ then σ is a vertex. By construction, $h|_\sigma$ is constant if and only if $h|_{\sigma(0)}$ is constant. Therefore, it suffices to show that h is not constant on any 1-cells of $\mathring{\mathcal{S}}_0$.

Suppose σ is a 1-cell. If σ does not contain y_n , then $h|_\sigma$ is not constant because \hat{h} is not constant on any 2-cells, or on 1-cells which do not contain y_n . If σ contains y_n , then $h|_\sigma$ is not constant by our choice of $h(y_n)$. \square

Remark 5. We note that \bar{h} is essentially the same as the Morse function defined in [BKW13]. The proof of Bux-Köhl-Witzel requires only the input of a uniform, $\mathbf{SL}_3(\mathbb{Q}((t^{-1})))$ -invariant reduction datum for X . In the most general context of Bux-Köhl-Witzel, this reduction datum is supplied for arithmetic groups over function fields by Harder's reduction theory. However, in the specific case of $\mathbf{SL}_n(\mathbb{F}_p[t])$, there exists a reduction theory that is more precise than Harder's. Namely, the action of $\mathbf{SL}_n(\mathbb{F}_p[t])$ on its Euclidean building admits a strict fundamental domain for its action on its Euclidean building, and this fundamental domain is exactly a sector. A proof of this last statement is given by Soulé in [Sou77]. Notice that in the result of Soulé, that the fields of coefficients for the polynomial rings are arbitrary, and thus the same statement applies equally as well to $\mathbf{SL}_3(\mathbb{Q}((t^{-1})))$, thus supplying a uniform, $\mathbf{SL}_3(\mathbb{Q}((t^{-1})))$ -invariant reduction datum for X , and now the proof of Bux-Köhl-Witzel applies without modification.

Lemma 6. If y_n is defined as above, then

$$\Gamma_{y_n} = \left\{ \left(\begin{array}{ccc} \pm 1 & u & w \\ 0 & \pm 1 & v \\ 0 & 0 & \pm 1 \end{array} \right) \middle| \deg(u), \deg(v) \leq n, \deg(w) \leq 2n + 1 \right\}$$

Proof. Recall that y_n is the barycenter of the edge spanned by the vertices x and x' , corresponding to the lattices $t^{2n+1}e_1 \oplus t^n e_2 \oplus e_3$ and $t^{2n+1}e_1 \oplus t^{n+1}e_2 \oplus e_3$, respectively.

By Lemma 2,

$$\Gamma_x = \left\{ \left(\begin{array}{ccc} \pm 1 & u & w \\ 0 & \pm 1 & v \\ 0 & 0 & \pm 1 \end{array} \right) \middle| \deg(u) \leq n + 1, \deg(v) \leq n, \deg(w) \leq 2n + 1 \right\}$$

$$\Gamma_{x'} = \left\{ \left(\begin{array}{ccc} \pm 1 & u & w \\ 0 & \pm 1 & v \\ 0 & 0 & \pm 1 \end{array} \right) \middle| \deg(u) \leq n, \deg(v) \leq n + 1, \deg(w) \leq 2n + 1 \right\}$$

To complete the proof, we observe that $\Gamma_{y_n} = \Gamma_x \cap \Gamma_{x'}$. \square

Let $\mathcal{S}'_0 = \mathcal{S}_0 \cup \{y_n\}_{n \in \mathbb{N}}$ be the standard sector modified to include the vertices y_n . Note that both h and \bar{h} restrict to Morse functions on \mathcal{S}'_0 . Let \mathcal{A}'_0 be the standard

sector modified to include the vertices y_n and their images in each sector based at x_0 , and let $X' = \mathbf{SL}_3(\mathbb{Q}[t])\mathcal{S}'_0$ and $X'_\Gamma = \mathbf{SL}_3(\mathbb{Z}[t])\mathcal{S}'_0$ be the analogously modified versions of X and X_Γ , respectively. Rather than using the barycentric subdivisions \mathring{X} , \mathring{X}_Γ , and $\mathring{\mathcal{S}}_0$, we will use X' , X'_Γ , and \mathcal{S}'_0 . Note that h restricts to a Morse function on X'_Γ and \bar{h} restricts to a Morse function on X' . We will abuse notation and use h and \bar{h} to denote the restricted Morse functions on X'_Γ and X' .

2.3 The descending star and descending link

The *star* of a vertex in a CW-complex is the union of all cells which contain that vertex. We denote the of a vertex z in a CW-complex Z by $St(z, Z)$. If h is a piecewise linear Morse function on Z , then the *descending star* of z , denoted $St^\downarrow(z, Z)$, is the subset of $St(z, Z)$ which consists of cells on which h has a unique maximum at z :

$$St^\downarrow(z, Z) = \{ \text{cells } \sigma \in St(z, Z) \mid h(v) < h(z) \text{ for every vertex } v \in \sigma - \{z\} \}$$

The *link* of a vertex z in a CW-complex Z is the set of faces of cells in $St(z, Z)$ which have codimension 1 and do not contain z . We denote the link of z in Z by $Lk(z, Z)$. If σ is a cell in $St(z, Z)$ we will use $\bar{\sigma}$ to denote the faces of σ which are in $Lk(z, Z)$. The *descending link* of z is then

$$Lk^\downarrow(z, Z) = \{ \text{cells } \bar{\sigma} \in Lk(z, Z) \mid h(v) < h(z) \text{ for every vertex } v \in \bar{\sigma} \} = St^\downarrow(z, Z) \cap Lk(z, Z)$$

For simplicity of notation, when Z is X' , we will suppress the simplicial complex and write $Lk^\downarrow(x)$ for $Lk^\downarrow(x, X')$ and $St^\downarrow(x)$ for $St^\downarrow(x, X')$.

Lemma 7. $Lk^\downarrow(x)$ is connected for all $x \in X'$.

First we will prove a simpler lemma:

Lemma 8. $Lk^\downarrow(x) \cap \mathcal{A}'_0$ is connected for all $x \in \mathcal{A}'_0 - \{x_0\}$ and consists of either 1 or 2 edges of $Lk(x)$.

Proof. We may assume $x \in \mathcal{S}'_0$. When $x \neq y_n$, this lemma is a consequence of Euclidean geometry and the fact that the angle spanned by the cells of $St^\downarrow(x) \cap \mathcal{A}'_0$ is strictly less than π . If $St(x)$ does not contain y_n , then $St(x, X') = St(x, X)$. Since the chambers of X are equilateral triangles, with angles measuring $\frac{\pi}{3}$, there can be at most 2 chambers in $St^\downarrow(x) \cap \mathcal{A}'_0$. If there are exactly 2 chambers in $St^\downarrow(x) \cap \mathcal{A}'_0$, they must share an edge and therefore $Lk^\downarrow(x) \cap \mathcal{A}'_0$ is connected. If $St(x)$ contains y_n , then there is at most 1 chamber in $Lk^\downarrow(x) \cap \mathcal{A}'_0$. When $x = y_n$, note that there are two cells in \mathcal{S}'_0 which contain y_n . Exactly one of these is in $St^\downarrow(y_n)$, and we will denote it by \mathcal{C}_n^\downarrow . Note that \mathcal{C}_n^\downarrow is not a simplex, and so $Lk^\downarrow(y_n) \cap \mathcal{A}'_0$ consists of two edges. We will denote the union of these two edges by $\bar{\mathcal{C}}_n^\downarrow$. \square

Proof of Lemma 7. Let $\Gamma^{\mathbb{Q}} = \mathbf{SL}_3(\mathbb{Q}[t])$. It suffices to show that $\Gamma_x^{\mathbb{Q}}$ is generated by elements which fix at least one vertex in $Lk^{\downarrow}(x) \cap \mathcal{A}'_0$. Since the diagonal subgroup of $\Gamma^{\mathbb{Q}}$ acts trivially on X , we may ignore these generators of $\Gamma_x^{\mathbb{Q}}$. We may assume that $x \in \mathcal{S}'_0$.

First, assume $x = y_n$. Then $Lk^{\downarrow}(y_n)$ is $\bar{\mathcal{C}}_n^{\downarrow} = \mathcal{C}_n^{\downarrow} \cap Lk(y_n)$. By Lemma 6, $\Gamma_{y_n}^{\mathbb{Q}}$ has the following form:

$$\left\{ \left(\begin{array}{ccc} \pm 1 & u & w \\ 0 & \pm 1 & v \\ 0 & 0 & \pm 1 \end{array} \right) \mid u, v, w \in \mathbb{Q}[t], \deg(u), \deg(v) \leq n, \deg(w) \leq 2n + 1 \right\}$$

Note that $\Gamma_{y_n}^{\mathbb{Q}}$ is generated by diagonal matrices, which act trivially on $Lk^{\downarrow}(y_n)$, and elementary matrices $e_{12}(u)$, $e_{23}(v)$, and $e_{13}(w)$, where $u, v, w \in \mathbb{Q}[t]$ such that $\deg(u) \leq n$, $\deg(v) \leq n$, or $\deg(w) = 2n + 1$. Generators with u or v nonzero are in the stabilizer of at least one vertex adjacent to y_n . The two corresponding root subgroups fix families of walls in \mathcal{A}_0 which are parallel to the walls containing the boundary of \mathcal{S}_0 . Elements of these root subgroups which stabilize y_n must also fix $\mathcal{C}_n^{\downarrow}$. Generators with w nonzero stabilize the edge η_n of which y_n is the barycenter, and hence these elements stabilize the two vertices of $\bar{\mathcal{C}}_n^{\downarrow}$ which are adjacent to y_n .

Next, consider the case when x is in the interior of \mathcal{S}_0 . Note that x and x_0 are not in a common wall of \mathcal{A}_0 , so there is a unique sector \mathcal{S}_x of \mathcal{A}_0 based at x which contains x_0 . This sector intersects $Lk^{\downarrow}(x)$ in one edge, $\bar{\mathcal{C}}_x$. Note that $\Gamma_x^{\mathbb{Q}}$ is generated by diagonal matrices, and elementary matrices of the form $e_{12}(u)$ or $e_{23}(v)$, where $u, v \in \mathbb{Q}[t]$ (elementary matrices in $\Gamma_x^{\mathbb{Q}}$ of the form $e_{13}(w)$ are commutators of elementary matrices of the other two forms in $\Gamma_x^{\mathbb{Q}}$). Each of these groups fixes a wall of \mathcal{S}_x , and therefore fixes at least one vertex of $\bar{\mathcal{C}}_x$.

Now suppose x is in the boundary of \mathcal{S}_0 . There is a unique sector \mathcal{S}_x based at x which contains x_0 and intersects the interior of \mathcal{S}_0 . \mathcal{S}_x intersects $Lk^{\downarrow}(x)$ in an edge, $\bar{\mathcal{C}}_x$. There is a second sector based at x , \mathcal{S}'_x , which contains x_0 but is disjoint from the interior of \mathcal{S}_0 , and there is some $g \in \Gamma_x^{\mathbb{Q}} \cap \Gamma_{x_0}^{\mathbb{Q}}$ such that $\gamma\mathcal{S}_x = \mathcal{S}'_x$. Elements of $\Gamma_x^{\mathbb{Q}}$ which are also in $\Gamma_{x_0}^{\mathbb{Q}}$ fix the wall of \mathcal{A}_0 which contains x and x_0 (and therefore, fix a vertex of $\bar{\mathcal{C}}_x$). Elements of $\Gamma_x^{\mathbb{Q}}$ which have 0 in the upper right corner (i.e. those γ such that $\gamma_{13} = 0$) fix a vertex of $g\bar{\mathcal{C}}_x$. These two types of elements, along with diagonal matrices, generate $\Gamma_x^{\mathbb{Q}}$. □

3 Construction of a 2-connected Γ -complex

In this section we will prove the following proposition:

Proposition 9. *There is a 2-connected Γ -complex Y with a Γ -equivariant map $\psi : Y \rightarrow X$ such that $\psi(Y^{(1)})$ has bounded height with respect to h . Furthermore, there are only finitely many Γ -orbits in Y of cells with nontrivial stabilizers, and all nontrivial cell stabilizers are of type F_1 .*

3.1 An $\mathrm{SL}_3(\mathbb{Q}[t])$ -invariant subspace of the building

Proposition 10. *There is $\Gamma^{\mathbb{Q}}$ -invariant, connected subcomplex $Z \subseteq X$ whose distance from a single $\Gamma^{\mathbb{Q}}$ -orbit in X is bounded.*

Proof. This proposition is essentially proved by Bux-Köhl-Witzel in [BKW13]. Indeed, if \mathbb{Q} is replaced by \mathbb{F}_p in the above proposition, then the proposition is proved in Section 10 of [BKW13], and it is the means by which it is shown that $\mathrm{SL}_n(\mathbb{F}_p[t])$ is of type F_{n-2} . Furthermore, replacing \mathbb{F}_p by \mathbb{Q} makes no changes in their proof. \square

Proof of Proposition 9. Let \mathcal{C}_0 be the standard chamber, and let $Y_0 = \Gamma \cdot \mathcal{C}_0$. Y_0 is connected by Suslin's theorem, which states that Γ is (finitely) generated by matrices which fix at least one vertex of \mathcal{C}_0 . For any cell $\sigma \subset Y_0$, Γ_σ is a conjugate of Γ_{σ_0} for some subcell $\sigma_0 \subset \mathcal{C}_0$ and thus Γ_σ is of type F_1 .

If Y_0 is not simply connected, there is a map $f : S^1 \rightarrow Y_0$ with noncontractible image. For each $\gamma \in \Gamma$, attach a 2-cell Δ_γ^2 to Y_0 by identifying the boundary of Δ_γ^2 with $\gamma f(S^1)$. Note that Γ acts on these new 2-cells by permuting the indices.

If the resulting space is not simply connected, repeat the above process with any remaining nontrivial 1-spheres until the resulting space is simply connected. Call this space Y_1 . Define a Γ -equivariant map $\psi : Y_1 \rightarrow X$ by mapping Δ_γ^2 to the unique filling disk in X of $\gamma f(S^1)$. If σ is a cell in $Y_1 - Y_0$, then $\Gamma_\sigma = \{1\}$ by construction.

If Y_1 is not 2-connected, there is a map $f : S^2 \rightarrow Y_1$ with noncontractible image. Duplicate the process above, attaching a family of 3-disks to Y_1 along the Γ orbit of $f(S^2)$, and repeating the process if necessary until the resulting space is 2-connected. Call this space Y . Again, any cell in $Y - Y_1$ has trivial stabilizer. X is 2-dimensional and aspherical, and there are no nontrivial 2-spheres in $\psi(Y_1)$. Therefore, we may extend ψ by mapping each 3-disk continuously to the image of its boundary in X .

By construction, $Y^{(1)}$ has bounded height under h . Any cell in Y with nontrivial stabilizer is contained in Y_0 , and Y_0 is in the Γ -orbit of \mathcal{C}_0 and we have shown that the stabilizers of cells in this orbit are type F_1 .

\square

Remark 11. *Note that the application of Suslin's theorem above (to show that Y_0 is connected) is not necessary, although it is convenient. If Y_0 were not connected, one could construct a connected complex in the following way: let $p : [0, 1] \rightarrow X$ be a path in X between two components. By Proposition 10, p can be chosen so that its height under h is bounded, regardless of the choice of components. For each $\gamma \in \Gamma$, attach a 1-cell p_γ to Y_0 by identifying the endpoints of p_γ with the endpoints of $\gamma(p)$. Note that Γ acts on $\{p_\gamma\}$ by permuting the indices. If the resulting space is not connected, repeat the process with any remaining connected components. Call the connected space Y'_0 , and note that there is a Γ equivariant map $\psi : Y'_0 \rightarrow X$ such that $\bar{h} \circ \psi(Y'_0)$ is bounded. For any cell $\sigma \subset Y'_0$, $\Gamma_\sigma = \{1\}$ if $\sigma \notin Y_0$. The above proof of the existence of the complex Y can be adapted to a more general setting.*

4 Cocycles and Cycles in $\Gamma \backslash Y$

In this section, we prove the following:

Proposition 12. *$H_2(\Gamma \backslash Y; \mathbb{Q})$ is infinite dimensional.*

We will prove this proposition by defining an infinite family of independent cocycles $\{\Phi_n\}_{n \in \mathbb{N}} \subseteq H^2(\Gamma \backslash Y; \mathbb{Q})$. Then we will exhibit an infinite family of cycles in $H_2(\Gamma \backslash Y; \mathbb{Q})$, and use the cocycles Φ_n to show that these cycles are independent.

In order to define Φ_n , we will first discuss a quotient of X , and define a family φ_n of local cocycles on that quotient, then use φ_n to define the cocycles Φ_n on $\Gamma \backslash Y$.

4.1 Congruence Subgroups of $\mathbf{SL}_3(\mathbb{Q}[t])$

In this subsection, we will make a brief digression to discuss congruence subgroups of $\mathbf{SL}_3(\mathbb{Q}[t])$, in order to define a local cocycle in the next section.

There is a sequence of *congruence subgroups* of $\mathbf{SL}_3(\mathbb{Q}[t])$ given by

$$\mathbf{SL}_3(\mathbb{Q}[t], (t^n)) = \ker(\mathbf{SL}_3(\mathbb{Q}[t]) \rightarrow \mathbf{SL}_3(\mathbb{Q}[t]/(t^n)))$$

Let U denote the upper-triangular subgroup of $\mathbf{SL}_3(\mathbb{Q}[t])$ and let U_n denote the upper-triangular subgroup $U \cap \mathbf{SL}_3(\mathbb{Q}[t], (t^{n+1}))$. (Note that $U_n \trianglelefteq U$, and $U_n \backslash U$ can be identified with the upper-triangular subgroup of $\mathbf{SL}_3(\mathbb{Q}[t]/(t^{n+1}))$.)

Let $\pi_n : X \rightarrow U_n \backslash X$ be the quotient map. Since $U_n \trianglelefteq U$ and U acts on X , both U and $U_n \backslash U$ act on $U_n \backslash X$. The Morse function \bar{h} is $\mathbf{SL}_3(\mathbb{Q}[t])$ -invariant, and it induces a Morse function on $U_n \backslash X$, which we will also call \bar{h} .

Let z_n be the vertex in X which corresponds to the lattice

$$t^{2n}e_1 \oplus t^n e_2 \oplus e_3$$

Lemma 13. *The vertex $\pi_n(z_n)$ is stabilized by U .*

Proof. Let $u \in U$. Then

$$u = \begin{pmatrix} 1 & p_x & p_z \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{pmatrix}$$

where $p_x, p_y, p_z \in \mathbb{Q}[t]$. We will write $u = u_1 u_2$ where $u_1 \in U_n$ and u_2 stabilizes z_n . Any polynomial $p \in \mathbb{Q}[t]$ can be written as a sum $p = p' + p''$ where $p' \in (t^{n+1})\mathbb{Q}[t]$ and $\deg(p'') \leq n$. Write $p_x = p'_x + p''_x$ and $p_y = p'_y + p''_y$. Let $q_z = p_z - p'_x p''_y$ and write $q_z = q'_z + q''_z$. Note that $u = u_1 u_2$, where

$$u_1 = \begin{pmatrix} 1 & p'_x & p'_z \\ 0 & 1 & p'_y \\ 0 & 0 & 1 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 1 & p''_x & p''_z \\ 0 & 1 & q''_y \\ 0 & 0 & 1 \end{pmatrix}$$

Note that $u_1 \in U_n$. Since $\deg(p''_x), \deg(p''_y), \deg(q''_z) \leq n$, u_2 stabilizes z_n (by Lemma 2). Therefore

$$u\pi_n(z_n) = uU_n z_n = U_n u z_n = U_n u_1 u_2 z_n = U_n z_n = \pi_n(z_n)$$

□

We will abuse notation slightly and use z_n to denote both the vertex in X , and its image $\pi_n(z_n)$ in the quotient.

Lemma 14. *$Lk^\downarrow(z_n, U_n \setminus X)$ is a complete bipartite graph.*

Proof. First, we observe that $Lk^\downarrow(z_n, U_n \setminus X) = U_n \setminus Lk^\downarrow(z_n, X)$. We have previously shown that $Lk^\downarrow(z_n, X)$ is the orbit of a single 1-cell under elementary matrices $e_{12}(at^n)$, and $e_{23}(bt^n)$, where $a, b \in \mathbb{Q}$. Let \hat{e} denote the image of this edge in $U_n \setminus X$. In $U_n \setminus U$, $e_{12}(at^n)$ and $e_{23}(bt^n)$ commute, since their commutator is in U_n . Suppose $u \in U_n \setminus U$ stabilizes z_n . Then there are elements $u_1 = e_{12}(at^n)$ and $u_2 = e_{23}(bt^n)$ such that $u\hat{e} = u_1 u_2 \hat{e}$. Furthermore, u_1 and u_2 each fixes exactly one vertex of \hat{e} and moves the vertex which the other fixes. This gives a labelling of every vertex in $Lk^\downarrow(z_n, U_n \setminus X)$ by a rational number, and every edge by an ordered pair of rational numbers. Since there are no restrictions on a and b , all pairs of rational numbers are possible and because of the action on X , different ordered pairs of rational numbers give different edges. Hence $Lk^\downarrow(z_n, U_n \setminus X)$ is a complete bipartite graph. □

From this point on, we will let

$$\begin{aligned} S_n^\downarrow &= \text{Star}^\downarrow(z_n, X) \\ \hat{S}_n^\downarrow &= \text{Star}^\downarrow(z_n, U_n \setminus X) \\ L_n^\downarrow &= \text{Lk}^\downarrow(z_n, X) \\ \hat{L}_n^\downarrow &= \text{Star}^\downarrow(z_n, U_n \setminus X) \end{aligned}$$

4.2 Local cocycles

Lemma 15. *There is an infinite family of (nontrivial) U -invariant cocycles $\varphi_n \in H^2(\hat{S}_n^\downarrow, \hat{L}_n^\downarrow; \mathbb{Q})$.*

Proof. Relative cycles in $H_2(\hat{S}_n^\downarrow, \hat{L}_n^\downarrow; \mathbb{Q})$ correspond to cycles in $H_1(\hat{L}_n^\downarrow; \mathbb{Q})$. By Lemma 14, \hat{L}_n^\downarrow is a complete bipartite graph, the vertices of each type are parametrized by \mathbb{Q} , and the edges can be labeled by ordered pairs of rational numbers. (In fact, $\hat{L}_n^\downarrow = \mathbb{Q} \star \mathbb{Q}$.) We fix an orientation from one family of vertices to the other family, and define a function on the edges $\{\eta_{(q,r)}\}_{q,r \in \mathbb{Q}}$ of \hat{L}_n^\downarrow by taking $\varphi_n(\eta_{(q,r)}) = qr$.

To verify that φ_n is a cocycle, note that \hat{L}_n^\downarrow is a graph, so there are no nontrivial 2-coboundaries on \hat{L}_n^\downarrow .

Next, we will show that φ_n is U -invariant. The loops of length 4 in \hat{L}_n^\downarrow form a generating set for $H_1(\hat{L}_n^\downarrow, \mathbb{Q})$, so it suffices to check that φ_n is U -invariant on loops of length 4. If σ is a loop of length 4, then σ has the form

$$\eta_{(q_1, r_1)} - \eta_{(q_2, r_1)} + \eta_{(q_2, r_2)} - \eta_{(q_1, r_2)}$$

and $\varphi_n(\sigma) = (q_1 - q_2)(r_1 - r_2)$. If $u \in U$, then u stabilizes z_n and acts by addition of the degree n coefficient of the u_{12} and u_{23} entries on the coordinates of the subscript, so

$$\varphi_n(u\sigma) = ((q_1 + q) - (q_2 + q))((r_1 + r) - (r_2 + r)) = \varphi_n(\sigma)$$

U_n acts trivially on $U_n \setminus U\mathcal{S}_0$, so the value of φ_n is invariant under the action of U .

Finally, we will show that φ_n is nontrivial by exhibiting a cycle $\hat{\sigma}_n \in H_1(\hat{L}_n^\downarrow; \mathbb{Q})$ such that $\varphi_n(\hat{\sigma}_n) \neq 0$. Let $\hat{\sigma}_n = 2\eta_{(0,0)} + \eta_{(-1,0)} + \eta_{(-1,1)} - \eta_{(0,1)} - \eta_{(0,-1)} + \eta_{(1,-1)} - \eta_{(1,0)}$. Using the form of φ_n given above, we see that $\varphi_n(\hat{\sigma}_n) = -2$. \square

Lemma 16. *There is a relative 2-cycle $\sigma_n \in H_2(S_n^\downarrow, L_n^\downarrow; \mathbb{Q})$ such that $\pi_n(\sigma_n) = \hat{\sigma}_n$.*

Proof. Let \mathcal{C}_n be the chamber in $St^\downarrow(z_n, \mathcal{A}_0)$, $\bar{\mathcal{C}}_n$ the corresponding edge in $Lk^\downarrow(z_n, \mathcal{A}_0)$, and

$$u_1 = \begin{pmatrix} 1 & t^n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t^n \\ 0 & 0 & 1 \end{pmatrix}$$

Take

$$\sigma_n = \mathcal{C}_n - u_1^{-1}\mathcal{C}_n + u_1^{-1}u_2\mathcal{C}_n - u_1^{-1}u_2u_1\mathcal{C}_n + [u_1^{-1}, u_2]\mathcal{C}_n - u_1u_2^{-1}u_1^{-1}\mathcal{C}_n + u_1u_2^{-1}\mathcal{C}_n - u_1\mathcal{C}_n$$

Since $[u_1^{-1}, u_2] = [u_1, u_2^{-1}]$, σ_n is a cycle. Let $\bar{\sigma}_n$ be the corresponding 1-cycle in $H_1(L_n^\downarrow; \mathbb{Q})$. Note that u_1 and u_2 descend to nontrivial elements of U^n , and their images commute. For each edge $u\bar{\mathcal{C}}_n$ in $\bar{\sigma}_n$, we know that $\pi_n(u\bar{\mathcal{C}}_n) = \eta_{(a,b)}$ for some $a, b \in \mathbb{Q}$. To find a , count the number of times u_1 appears in u (counting u_1^{-1} as -1). To find b , count the number of times u_2 appears in u . For example, $\pi_n(u_1\bar{\mathcal{C}}_n) = \eta_{(1,0)}$, $\pi_n(u_1u_2^{-1}) = \eta_{(1,-1)}$, and $\pi_n(\bar{\mathcal{C}}_n) = \pi_n([u_1^{-1}, u_2]\bar{\mathcal{C}}_n) = \eta_{(0,0)}$.

Therefore, $\pi_n(\sigma_n) = \hat{\sigma}_n$. \square

We will use φ_n to define a cocycle $\Phi_n \in H^2(\Gamma \backslash Y; \mathbb{Q})$ by lifting 2-cells in $\Gamma \backslash Y$ to disks in X , applying the quotient map π_n to obtain a disk in $U_n \backslash X$, evaluating φ_n on the intersection with \hat{L}_n^\downarrow , and averaging over Γ -translates of the lifted disk.

Lemma 17. *If D is a 2-disk in X with boundary in $\psi(Y'_0)$, then for sufficiently large n , $\pi_n(D) \cap \hat{S}_n \subset \hat{S}_n^\downarrow$*

Proof. To prove the lemma, we will show that if any chamber in $\hat{S}_n - \hat{S}_n^\downarrow$ is contained in $\text{Supp}(\pi_n(D) \cap \hat{S}_n)$, then there exists a geodesic segment $\rho \subset \text{Supp}(\pi_n(D))$ with one endpoint at z_n and the other endpoint at $z \in \partial\pi_n(D)$ with $h(z) > h(z_n)$, which contradicts the fact that $\bar{h}(\psi(Y'_0))$ is bounded above, since the sequence $\{z_n\}$ has unbounded height. There are two chambers in $\hat{S}_n - \hat{S}_n^\downarrow$ which have exactly one vertex which is higher than z_n . If $\text{Supp}(\pi_n(D) \cap \hat{S}_n)$ contains either one of these two chambers, then it must also contain a chamber with two vertices that are higher than z_n , because there is a unique chamber adjacent to the ‘‘upper’’ edge of this chamber.

Let $\hat{\mathcal{C}}_1$ be the chamber in the support of $\pi_n(D)$ with $h(v) > h(z_n)$ for all vertices $v \neq z_n$.

There is a face \mathcal{F}_1 of $\hat{\mathcal{C}}_1$ which is contained in $Lk(z_n, U_n \backslash X)$ such that $h(y) > h(z_n)$ for all $y \in \mathcal{F}_1$. There is some vertex v_1 of \mathcal{F}_1 which is in $\pi_n(\mathcal{A}_0)$. Because $U_n \backslash X$ has no branching along walls of $\pi_n(\mathcal{A}_0)$ which are above z_n , the geodesic ray in $\pi_n(\mathcal{A}_0)$ based at z_n and passes through the vertex v_1 must eventually intersect $\partial\pi_n(D)$. Call this geodesic ray ρ , and notice that $\bar{h} \circ \rho$ is a strictly increasing function. Therefore,

the point where ρ intersects $\partial\pi_n(D)$ is strictly higher (with respect to \bar{h}) than z_n , which is a contradiction. \square

Let $U_\Gamma = U \cap \Gamma$. By Lemma 13, U_Γ stabilizes z_n and therefore $U_\Gamma \hat{S}_n^\downarrow = \hat{S}_n^\downarrow$ for every n .

Lemma 18. *There is an infinite family of (nontrivial) cocycles $\Phi_n \in H^2(\Gamma \backslash Y; \mathbb{Q})$.*

Proof. Given ΓB a 2-cell in $\Gamma \backslash Y$, let

$$\Phi_n(\Gamma B) = \sum_{\gamma V_n \in \Gamma/V_n} \varphi_n(\pi_n(\gamma^{-1}\psi(B)) \cap \hat{S}_n^\downarrow)$$

Φ_n is well-defined, i.e. the value of Φ_n is independent of the choices of coset representatives γU and the choice of a lift B for ΓB : First we check that replacing γ with γu_γ (changing the coset representatives) does not change the value of Φ_n :

$$\begin{aligned} & \sum_{(\gamma u_\gamma)U_\Gamma \in \Gamma/U_\Gamma} \varphi_n\left(\pi_n((\gamma u_\gamma)^{-1}\psi(B)) \cap \hat{S}_n^\downarrow\right) \\ &= \sum_{(\gamma u_\gamma)U_\Gamma \in \Gamma/U_\Gamma} \varphi_n\left(\pi_n(u_\gamma^{-1}\gamma^{-1}\psi(B)) \cap u_\gamma^{-1}\hat{S}_n^\downarrow\right) \\ &= \sum_{\gamma U_\Gamma \in \Gamma/U_\Gamma} \varphi_n\left((u_\gamma)^{-1}[\pi_n(\gamma^{-1}\psi(B)) \cap \hat{S}_n^\downarrow]\right) \\ &= \sum_{\gamma U_\Gamma \in \Gamma/U_\Gamma} \varphi_n\left(\pi_n(\gamma^{-1}\psi(B)) \cap \hat{S}_n^\downarrow\right) = \Phi_n(\Gamma B) \end{aligned}$$

Next we check that choosing a different lift of ΓB does not change the value of $\Phi_n(\Gamma B)$. If $y \in \Gamma$, then

$$\Phi_n(\Gamma yB) = \sum_{\gamma U_\Gamma \in \Gamma/U_\Gamma} \varphi_n \left(\psi_n(\gamma^{-1}yB) \cap \hat{S}_n^\downarrow \right) \quad (1)$$

$$= \sum_{\gamma U_\Gamma \in \Gamma/U_\Gamma} \varphi_n \left(\pi_n((y^{-1}\gamma)^{-1}\psi(B)) \cap \hat{S}_n^\downarrow \right) \quad (2)$$

$$= \sum_{y\gamma U_\Gamma \in \Gamma/U_\Gamma} \varphi_n \left(\pi_n((y^{-1}y\gamma)^{-1}\psi(B)) \cap \hat{S}_n^\downarrow \right) \quad (3)$$

$$= \sum_{y\gamma U_\Gamma \in \Gamma/U_\Gamma} \varphi_n \left(\pi_n(\gamma^{-1}\psi(B)) \cap \hat{S}_n^\downarrow \right) \quad (4)$$

$$= \sum_{\gamma U_\Gamma \in \Gamma/U_\Gamma} \varphi_n \left(\pi_n(\gamma^{-1}\psi(B)) \cap \hat{S}_n^\downarrow \right) \quad (5)$$

$$= \Phi_n(\Gamma B) \quad (6)$$

In order to show that Φ_n is a cocycle in $H^2(\Gamma \backslash Y; \mathbb{Q})$, we will show that it is trivial on boundaries of 3-disks, and thus is in the kernel of the coboundary map.

Let ΓB^3 be a 3-cell in $\Gamma \backslash Y$, corresponding to the 3-cell B^3 in Y . Then $\partial(\Gamma B^3) = \Gamma(\partial B^3)$ is a 2-sphere in $\Gamma \backslash Y$ and ∂B^3 is a 2-sphere in Y . Since X contains no nontrivial 2-spheres, the image of ∂B^3 under the map $\psi : Y \rightarrow X$ is homotopic to a point. Thus,

$$\Phi_n(\Gamma(\partial B^3)) = \sum_{\gamma U_\Gamma \in \Gamma/U_\Gamma} \varphi_n \left(\pi_n(\gamma^{-1}\psi(\partial B^3)) \cap \hat{S}_n^\downarrow \right) = 0$$

□

Lemma 19. *For each n , there is a 2-cycle $\tilde{\sigma}_n \in H_2(\Gamma \backslash Y; \mathbb{Q})$ such that $\Phi_n(\tilde{\sigma}_n) \neq 0$ and $\Phi_m(\tilde{\sigma}_n) = 0$ for $m \geq n + 1$.*

This lemma is essentially proved in [Wor13]. We restate it with minor adaptations of the notation.

Proof. $\partial\sigma_n$ is a 1-sphere in X with $h(\partial\sigma_n) < h(z_n)$. Let v_1, \dots, v_k be the vertices of $\partial\sigma_n$. For $1 \leq i \leq k$, choose a path $p_i : [0, 1] \rightarrow X$ such that $p_i(0) = v_i$, $p_i(1) \in Y_0$, and $h \circ p_i$ is strictly decreasing. (One choice of p_i would be to choose an efficient simplicial path to \mathcal{C}_0 if $v_i \in \mathcal{A}_0$, and an efficient simplicial path to $u\mathcal{C}_0$ if $v_i \in u\mathcal{A}_0$ for $u \in \langle u_1, u_2 \rangle$.) Let e_1, \dots, e_m be the 1-cells of $\partial\sigma_n$, with $\partial e_i = v_j \cup v_l$. For $1 \leq i \leq m$,

there is a homotopy relative $p_j(1)$ and $p_l(1)$ between $p_j \cup p_l \cup e_i$ and a path in Y_0 . This homotopy gives a disk d_i , and $\cup_{i=1}^m d_i$ gives a homotopy between $\partial\sigma_n$ and a 1-sphere $\tilde{\sigma}_n$ in Y_0 . Since Y is simply connected, there is a disk $D_n \subset Y$ with $\pi_n \circ \psi(\partial D_n) = \tilde{\sigma}_n$. Because filling disks in X are unique, $\pi_n \circ \psi(D_n) \cap \mathcal{S}_n^\downarrow = \sigma_n$. Let p be the quotient map from Y to $\Gamma \backslash Y$. Then $p(D_n)$ is a cycle, because $p(\partial D_n) \subset p(Y_0)$ is trivial. Take $\tilde{\sigma}_n = p(D_n)$. To complete the proof, note that the maximum value of h on $\psi(D_n)$ is attained at z_n , and $h(x) > h(z_n)$ if $x \in Lk^\downarrow(z_m)$ for $m > n$, so $\psi(D_n) \cap L_m^\downarrow = \emptyset$. \square

This final lemma shows that $\{\tilde{\sigma}_n\}$ is an infinite independent family of 2-cycles in $H_2(\Gamma \backslash Y; \mathbb{Q})$, and thus completes the proof of Proposition 12.

5 Proof of the main result

We now prove Theorem 1.

Proof. Let $\mathcal{H}_q = \{H_q(\Gamma_\sigma; \mathbb{Q})\}$ and consider the spectral sequence

$$E_{p,q}^2 = H_p(\Gamma \backslash Y, \mathcal{H}_q).$$

A common reference for this spectral sequence is [Bro82]. In section VII.8, it is shown that

$$E_{p,q}^2 \Rightarrow H_{p+q}(\Gamma; C(Y; \mathbb{Q}))$$

where $C(Y; \mathbb{Q})$ is the cellular chain complex of Y with coefficients in \mathbb{Q} .

Because Y is 2-connected, there is a cellular map $f : Y \rightarrow \{\text{pt}\}$ which induces an isomorphism $f_* : H_i(Y; \mathbb{Q}) \rightarrow H_i(\{\text{pt}\}; \mathbb{Q})$ for $0 \leq i \leq 2$. Therefore, f also induces an isomorphism $H_i(\Gamma; C(Y; \mathbb{Q})) \rightarrow H_i(\Gamma; C(\{\text{pt}\}; \mathbb{Q})) \cong H_i(\Gamma; \mathbb{Q})$ for $i \leq 2$.

The relevant terms of the spectral sequence are $E_{2,0}^2$, and $E_{0,1}^r$ for $r \geq 2$. First, we note that

$$E_{2,0}^2 = H_2(\Gamma \backslash Y, H_0(\Gamma_\sigma; \mathbb{Q})) = H_2(\Gamma \backslash Y; \mathbb{Q}).$$

We have demonstrated in Proposition 12 that $H_2(\Gamma \backslash Y; \mathbb{Q})$ is infinite dimensional.

Next, we note that when $q > 0$,

$$E_{p,q}^2 = H_p(\Gamma \backslash Y, \{H_q(\Gamma_\sigma; \mathbb{Q})\}).$$

By Proposition 9, the cell stabilizers Γ_σ are of type F_1 , so $H_1(\Gamma_\sigma; \mathbb{Q})$ is finite dimensional for every 0-cell σ in $\Gamma \backslash Y$.

Since Γ acts freely on $Y - Y_0$, and the image of Y_0 in the quotient consists of a single 2-dimensional chamber, with finitely many subcells, $H_0(\Gamma \backslash Y, \{H_1(\Gamma_\sigma; \mathbb{Q})\})$ consists of finite sums in the form

$$\sum_{i=0}^N a_i \sigma_i$$

where $a_i \in H_q(\Gamma_\sigma; \mathbb{Q})$. Notice that $H_1(\Gamma_\sigma; \mathbb{Q}) = 0$ for all but finitely many σ_i , and is always finite dimensional. Thus $E_{0,1}^2 = H_0(\Gamma \backslash Y, \{H_1(\Gamma_\sigma; \mathbb{Q})\})$ is finite dimensional.

To compute $E_{2,0}^r$ for $r > 2$, we note that the kernel of any homomorphism $E_{2,0}^2 \rightarrow E_{0,1}^2$ must be infinite dimensional. Later differentials emanating from $E_{2,0}^r$ are zero, since $E_{p,q}^r = 0$ outside the first quadrant. Thus $E_{2,0}^r$ is infinite dimensional for all r .

In the limit, $H_{n-1}(\Gamma; \mathbb{Q})$ is infinite dimensional, and thus $H^{n-1}(\Gamma; \mathbb{Q})$ is infinite dimensional as well. □

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