RESEARCH STATEMENT

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I am currently a student of Kevin Wortman, at the University of Utah. My research is on the geometric properties of arithmetic and related groups. Arithmetic groups are the integer points of real Lie groups, such as $\mathbf{SL_n}(\mathbb{Z})$ inside of $\mathbf{SL_n}(\mathbb{R})$. A more general notion of S-integers, $\mathcal{O}_S$, in a global field K gives rise to the definition of an S-arithmetic group, $\mathbf{G}(\mathcal{O}_S)$, which is the S-integer points of an algebraic K-group $\mathbf{G}$. For example, $\mathbf{SL_n}(\mathbb{Z}[\sqrt{2}])$ inside of $\mathbf{SL_n}(\mathbb{R}) \times \mathbf{SL_n}(\mathbb{R})$, $\mathbf{SL_n}(\mathbb{Z}[1/p])$ inside of $\mathbf{SL_n}(\mathbb{R}) \times \mathbf{SL_n}(\mathbb{Q}_p)$, and $\mathbf{SL_n}(\mathbb{F}_p[t])$ inside of $\mathbf{SL_n}(\mathbb{F}_p((t^{-1})))$ are S-arithmetic groups, as are all arithmetic groups. Though not S-arithmetic groups themselves, groups like $\mathbf{SL_n}(\mathbb{Z}[t])$ inside $\mathbf{SL_n}(\mathbb{Q}((t^{-1})))$ share some similar properties with S-arithmetic groups.

My research on these groups has two main prongs: first, their Dehn functions and isoperimetric inequalities (often placed into the larger category of filling functions), and second, their finiteness properties and cohomology.

1. Filling functions

Filling functions of spaces describe the amount of area or volume required to fill a sphere with a disk. A 2-dimensional filling function is called a Dehn function.

1.1. **Dehn functions.** Let $\delta_X(\ell)$ be the minimal filling area of a closed loop ℓ in a simply-connected metric space X. The Dehn function of X is defined as $\delta_X(n) = \sup\{\delta_X(\ell) \mid \mathit{length}(\ell) \leq n\}$. The Dehn function of a finitely presented group G is defined as the minimum number of relations needed to reduce a word of length n representing the identity in G to the trivial word. The specific function obtained may depend on the presentation of G, but its growth class is a group invariant and gives information about the word problem in G, so we usually give only the growth class rather than specifying the function.

Conjecture 1 (Thurston, 1992)**.** *If $n \geq 4$, the Dehn function of $\mathbf{SL_n}(\mathbb{Z})$ is quadratic.*

Young showed in 2013 that Thurston's conjecture is true when $n \geq 5$. The condition that $n \geq 4$ in Conjecture 1 ensures that $\mathbf{SL_n}(\mathbb{R})$ contains a diagonalizable free abelian group of rank 3. The S-arithmetic group $\mathbf{G}(\mathcal{O}_S)$ sits as a lattice inside the Lie group $G = \prod_v \mathbf{G}(K_v)$, where K_v are finitely many complete local fields containing $\mathcal{O}_S$. The analogous requirement for an S-arithmetic group is that the sum of the local ranks, $\sum_v \operatorname{rank}_{K_v}(\mathbf{G})$ is at least 3. The following is a slight generalization of a conjecture of Gromov:

Conjecture 2 (after Gromov)**.** *If $\mathbf{G}(\mathcal{O}_S)$ is an S-arithmetic group, and the sum of the local ranks of $\mathbf{G}$ is at least 3, then the Dehn function of $\mathbf{G}(\mathcal{O}_S)$ is quadratic.*

Along with Young's theorem, results by Druţu (1998), Bestvina-Eskin-Wortman (2013), and Cohen (2015) have made progress towards this conjecture. The relevant specific case of Bestvina-Eskin-Wortman is stated below:

Theorem 3 (Bestvina-Eskin-Wortman, 2013)**.** *If $\mathbf{G}(\mathcal{O}_S)$ is an S-arithmetic group, and the Lie group which contains $\mathbf{G}(\mathcal{O}_S)$ as a lattice has at least 3 factors (which in turn implies that the sum of the local ranks is at least 3), then the Dehn function of $\mathbf{G}(\mathcal{O}_S)$ is bounded by a polynomial.*

The following theorem extends the result by Bestvina-Eskin-Wortman to cover the certain cases where $\mathbf{G}(\mathcal{O}_S)$ is a lattice in a product of two groups. Groups of type A_n are groups with a similar structure to that of $\mathbf{SL_n}$.

Theorem 4 (Cesa, 2015)**.** *If the K-type of $\mathbf{G}$ is A_n, $n \geq 2$, and the Lie group which contains $\mathbf{G}(\mathcal{O}_S)$ has at least 2 simple factors, then the Dehn function of $\mathbf{G}(\mathcal{O}_S)$ is bounded by a polynomial.*

For example, Theorem 4 shows that, when $n \geq 3$, $\mathbf{SL_n}(\mathbb{Z}[\sqrt{2}])$, $\mathbf{SL_n}(\mathbb{F}_p[t, t^{-1}])$, and $\mathbf{SL_n}(\mathbb{Z}[1/p])$ have polynomially bounded Dehn functions.

The standard technique for proving theorems about the Dehn function of $\mathbf{G}(\mathcal{O}_S)$ is as follows: G can be identified via a quasi-isometry with a space X that is a product of CAT(0) symmetric spaces and Euclidean buildings. This identification restricts to an identification of $\mathbf{G}(\mathcal{O}_S)$ with a subset $X' \subset X$ obtained by removing horoballs. Since X is non-positively curved, loops in X' have quadratic filling disks in X and hence loops in $\mathbf{G}(\mathcal{O}_S)$ have quadratic fillings in G. Pruning X to X' corresponds to removing certain parabolic subgroups of G.

Showing that $\mathbf{G}(O_S)$ has a quadratic (or polynomially bounded) Dehn function is then reduced to showing that replacing the pieces of the filling disk which pass through the interiors of parabolic subgroups by pieces in the boundaries of parabolic subgroups does not increase the area of the disk by too much.

The difficulty tends to lie in doing this for maximal parabolic subgroups. Indeed, the proof of Theorem 4 uses the assumption that the groups are of type A_n only in the case of maximal parabolic subgroups, and only when the ambient Lie group has 2 simple factors. When the ambient Lie group has at least 3 factors, as in Theorem 3, the diagonal subgroup of $\mathbf{G}(\mathcal{O}_S)$ allows us to construct efficient filling disks in the boundary of a maximal parabolic. When the Lie group has fewer than 3 factors, the diagonal subgroup cannot be used for this purpose. One key difference between the proofs of Theorems 3 and 4 is the use of the type A_n assumption to construct a free abelian subgroup of $\mathbf{G}(\mathcal{O}_S)$, which is used in place of the diagonal subgroup to construct efficient filling disks in the boundaries of maximal parabolic subgroups. A natural next step would be to work on removing the assumption that $\mathbf{G}$ has K-type A_n from Theorem 4.

1.2. **Higher Filling Functions and Isoperimetric Inequalities.** The higher dimensional analog of a Dehn function is called a higher Dehn function or an isoperimetric inequality. If X is n-connected and Σ is an n-sphere in X, we define $\delta_X(\Sigma)$ to be the minimum volume of a disk D which fills Σ. If $\max\{\delta_X(\Sigma) \mid vol(\Sigma) \leq n\} \leq f(n)$, for some polynomial $f(n)$, we say X satisfies a polynomial n-dimensional isoperimetric inequality. If G is a group of type F_{n+1} (i.e. if there exists a $K(G,1)$ with finite $(n+1)$-skeleton), then there is an n-connected CW-complex X on which G acts properly, cellularly, and cocompactly, and G satisfies a polynomial n-dimensional isoperimetric inequality if and only if X does.

The following conjecture generalizes Conjecture 2 to higher dimensional isoperimetric inequalities:

Conjecture 5 (after Gromov)**.** *If $\mathbf{G}(\mathcal{O}_S)$ is an S-arithmetic group, and the sum of the local ranks of $\mathbf{G}$ is at least $n+2$, then $\mathbf{G}(\mathcal{O}_S)$ satisfies a polynomial n-dimensional isoperimetric inequality.*

As in the case of Dehn functions, the expected technique to fill a sphere in $\mathbf{G}(\mathcal{O}_S)$ is to find a filling disk in G, and replace the portions of the disk which lie in the interior of a parabolic subgroup by pieces which lie in the boundary of the same parabolic subgroup without increasing the volume too much. Again, most of the difficulty lies in doing this for maximal parabolic subgroups. The following proposition provides partial progress toward proving a polynomial 3-dimensional isoperimetric inequality for $\mathbf{SL_n}(\mathbb{Z})$ when $n \geq 5$:

Proposition 6 (Cesa, 2015)**.** *Suppose $n \geq 5$, and D is a 3-disk in $\mathbf{SL_n}(\mathbb{R})$ with ∂D contained in a bounded neighborhood of $\mathbf{SL_n}(\mathbb{Z})$. There exists an integer m and a positive number C such that if P is a nonmaximal parabolic subgroup of $\mathbf{SL_n}(\mathbb{R})$ and $D_P = D \cap P$, then there is a manifold D'_P of the same topological type as D_P such that $D'_P \subset \partial P$, $\partial D'_P = \partial D_P$, and $vol(D'_P) \leq Cvol(D_P)^m$.*

I am currently working on extending Proposition 6 to include the case when P is maximal. If successful, this would prove that $\mathbf{SL_n}(\mathbb{Z})$ satisfies a polynomial 2-dimensional isoperimetric inequality.

2. Finiteness Properties and Cohomology

I briefly mentioned the finiteness property F_n in Section 1.2. In general, the finiteness properties of G give information about cohomology of G. For example, if a group is of type F_n, then $H^n(G;\mathbb{Z})$ is finitely generated. (Having type F_n is sometimes characterized as the "obvious reason" why $H^n(G;\mathbb{Z})$ is finitely generated.) Results by Bux-Wortman (2006-2007), Bux-Mohammadi-Wortman (2010) and others show that many S-arithmetic groups defined over function fields, and S-arithmetic-like groups, fail to have type F_n or a weaker property, type FP_n, for some suitable choice of n. More recent results by Wortman (2013), Kelly (2013), and Cobb (2015) have shown that for certain examples of such groups, Γ, $H^n(\Gamma; M)$ is infinite dimensional for an appropriate choice of module M.

In 1977, Suslin showed that $\mathbf{SL}_3(\mathbb{Z}[t])$ is finitely generated which is equivalent to being of type F_1. Krstič-McCool proved in 1999 that $\mathbf{SL_3}(\mathbb{Z}[t])$ is not finitely presented, and so is not of type F_2.

Theorem 7 (Cesa-Kelly, 2015)**.** *$H^2(\mathbf{SL_3}(\mathbb{Z}[t]);\mathbb{Q})$ is infinite dimensional.*

The proof uses a Morse function defined by Bux-Köhl-Witzel (2013) to construct a 2-connected CW-complex Y on which $\mathbf{SL}_3(\mathbb{Z}[t])$ acts freely, properly, and cocompactly. We then show that $H^2(\mathbf{SL_3}(\mathbb{Z}[t])\backslash Y;\mathbb{Q})$ is infinite dimensional by constructing infinitely many linearly independent $\mathbf{SL_3}(\mathbb{Z}[t])$-invariant 2-cocycles and 2-cycles that pair nontrivially. Last, we apply a spectral sequence to show that $H^2(\mathbf{SL_3}(\mathbb{Z}[t]);\mathbb{Q})$ is infinite dimensional. The proofs of the analogous results by Wortman, Kelly, and Cobb follow similar outlines, and differ in the constructions of the infinite families of cocycles and cycles.

The Bux-Köhl-Witzel Morse function allows the construction of an $n-1$-connected complex Y_n with a free, proper, cocompact $\mathbf{SL_n}(\mathbb{Z}[t])$-action.

Question 8. *Is there an infinite linearly independent family of $\mathbf{SL_n}(\mathbb{Z}[t])$-invariant coycles in $H^{n-1}(Y_n;\mathbb{Q})$?*

An affirmative answer to the above question would imply that $H^{n-1}(\mathbf{SL_n}(\mathbb{Z}[t]);\mathbb{Q})$ is infinite dimensional.

3. Totally Disconnected Groups

Many examples of S-arithmetic groups fall into the category of totally disconnected locally compact groups. In October 2014, I participated in a workshop on Totally Disconnected Locally Compact Groups at Oberwolfach, where I gave a joint talk with François le Maître on Phillip Wesolek's thesis, Elementary Totally Disconnected Groups. The workshop resulted in a collective book project, to which we have contributed a survey, titled "Elementary Totally Disconnected Groups, after Wesolek" (Cesa-Le Maître, 2015).

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