Imaging polarizable dipoles

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Abstract

We consider the problem of imaging electric di– poles in a homogeneous medium from measurements of all three components of the electric field at an array of receivers. We show that an electromagnetic version of Kirchhoff migration can be used to recover the position and the orientation of the dipoles in the Fraunhofer asymptotic regime. We prove that the resolution estimates for the position are as in the acoustic case and provide error estimates for the dipole orientation. We extend these results to the case where the dipoles behave as passive sources, that is to say diffracting obstacles. In this setting, one wants to recover both the position and the polarizability tensor of each dipole in the medium.

Keywords: Electromagnetism, Imaging dipoles, Kirchhoff imaging, Fraunhofer asymptotic regime.

1 Introduction

Many chemical molecules (such as biomolecules like proteins) are polarized, in other words, they can be model as electric dipoles. Knowing both the position and the polarization of this dipole is very useful for chemists since it contains information about the geometry and the properties of the molecule. Toward this goal, we study the inverse problem consisting in reconstructing the positions and polarization vectors of a family of electrical radiating dipoles from their emitted electric field.

2 Formulation of the problem

We consider here a homogeneous dielectric medium filling the whole space $\mathbb{R}^3$ of permittivity ε and permeability μ. We assume that this medium contains N electric dipoles (or antennas) for which one wants to recover both the polarization vectors $\boldsymbol{p}_1, \cdots, \boldsymbol{p}_N \in \mathbb{C}^3$ and the positions $\boldsymbol{y}_i \in \mathbb{R}^3$. These dipoles are assumed radiative, that is to say able to emit an electric field (at the frequency ω) of the form:

$$\boldsymbol{E}(\boldsymbol{x}, k) = \sum_{j=1}^{N} \mu\, \omega^2\, \mathbb{G}(\boldsymbol{x}, \boldsymbol{y}_j; k)\, \boldsymbol{p}_j$$

where $k = \omega/c = \omega/\sqrt{\varepsilon\, \mu}$ is the wave number and $\mathbb{G}(\boldsymbol{x}, \boldsymbol{y}_j; k)$ the dyadic Green tensor associated to the dielectric medium [2]. We assume that one can measure this electrical field on an array $\mathcal{A}$ (supposed to be continuous, bounded and localized in the plane $z = 0$). Our goal is to use these measurements (up to the factor $\mu\, \omega^2$):

$$\boldsymbol{\Pi}(\boldsymbol{x}_r; k) = \sum_{j=1}^{N} \mathbb{G}(\boldsymbol{x}_r, \boldsymbol{y}_j; k)\, \boldsymbol{p}_j$$

collected on each point $\boldsymbol{x}_r \in \mathcal{A}$ to construct an imaging function from which one can extract the positions $\boldsymbol{y}_i$ of the antennas, but also their orientations $\boldsymbol{p}_i$.

In acoustic imaging, in the Fraunhofer regime, the so-called Kirchhoff imaging functional has proved its efficiency. Thus, we chose to look at the properties of its electromagnetic analogous $\mathcal{I}(\boldsymbol{y}) : \mathbb{R}^3 \to \mathbb{C}^3$ defined by:

$$\mathcal{I}_k(\boldsymbol{y}) = \int_{\mathcal{A}} \overline{\mathbb{G}(\boldsymbol{x}_r, \boldsymbol{y}; k)}\, \boldsymbol{\Pi}(\boldsymbol{x}_r, k)\, \mathrm{d}\tilde{\boldsymbol{x}}_r, \qquad (1)$$

for $\boldsymbol{x}_r = (\tilde{\boldsymbol{x}}_r, 0) \in \mathcal{A}$.

3 Summary of results

We first establish the Fraunhofer asymptotics of the dyadic Green function to derive the asymptotics of the imaging function (1). Secondly, we used this asymptotic to study not only the resolution of our image to recover the position $\boldsymbol{y}_i$, but also to find a linear system whose solution would give a stable reconstruction of the polarization vector $\boldsymbol{p}_i \in \mathbb{C}^3$.

Suppose that one wants to recover the position $\boldsymbol{y}_i = (\tilde{\boldsymbol{y}}_i, z_i)$ and polarization $\boldsymbol{p}_i = (\tilde{\boldsymbol{p}}_i, p_{z,i})$ of the i-th dipole. Like in acoustics, we decompose this work in two steps: first, we study the resolution of $\mathcal{I}_k(\boldsymbol{y})$ in the cross-range, that is the plane $z = z_i$ and then we integrate the image $\mathcal{I}_k(\boldsymbol{y})$ over a frequency band ΔB to derive its range resolution, that is its resolution on the axis transverse to the array which contains $\boldsymbol{y}_i$.

Step one: cross-range resolution

In the plane $z = z_i$, we establish, based on stationary phase arguments, that the infinity norm of the image: $\|\mathcal{I}_k(\boldsymbol{y})\|_\infty$ has, like in acoustic [1], a resolution of order $L/(ak)$. In other words, the ratio $\|\mathcal{I}_k(\boldsymbol{y})\|_\infty / |\mathcal{I}_k(\boldsymbol{y}_i)|$ becomes small when the distance $|\tilde{\boldsymbol{y}} - \tilde{\boldsymbol{y}}_i|$ between $\boldsymbol{y}_i$ and a point $\boldsymbol{y}$ of its cross-range is large with regards to $L/(ak)$. Then using this result, we prove that one obtains a good image of the two first components $\tilde{\boldsymbol{p}}_i$ of the polarization vector by solving the following linear system:

$$\Big[\int_{\mathcal{A}} \overline{\mathbb{G}(\boldsymbol{x}_r, \boldsymbol{y}; k)}\mathbb{G}(\boldsymbol{x}_r, \boldsymbol{y}; k)\, \mathrm{d}\tilde{\boldsymbol{x}}_r\Big]\boldsymbol{p} = \mathcal{I}_k(\boldsymbol{y}), \quad (2)$$

of unknown $\boldsymbol{p} = (\tilde{\boldsymbol{p}}, p_z) \in \mathbb{C}^3$ for points $\boldsymbol{y}$ in the cross-range of $\boldsymbol{y}_i$. More precisely, the study of the condition number of (2) shows that it is ill-posed in the sense that one cannot recover in the Fraunhofer regime the component $p_{z,i}$ of $\boldsymbol{p}_i$. However, the 2×2 subsystem associated with the components $\tilde{\boldsymbol{p}} = (p_x, p_y)$ of $\boldsymbol{p}$ has a condition number close to 1. Furthermore, we prove then that using the magnitude of the solution $\tilde{\boldsymbol{p}}$ of this subsystem leads to a stable image of the cross-range position $\tilde{\boldsymbol{y}}_i$ which has the same resolution $L/(ak)$ as the Kirchhoff imaging functional. Moreover, we provide error estimates on $|\tilde{\boldsymbol{p}} - \tilde{\boldsymbol{p}}_i|$ at $\boldsymbol{y} = \boldsymbol{y}_i$ which involves the distances between the dipoles and the Rayleigh number $L/(ak)$.

Step two: range resolution

We show, by integrating the imaging functional $\mathcal{I}_k(\boldsymbol{y})$ over a frequency band ΔB, that one gets a depth resolution $c/\Delta B$ (identical to the one in acoustic [1]) of the range position z_i as soon as the frequency band ΔB is sufficiently wide. In addition, the integration of the system (2) over this frequency band leads to a good image (up to a phase term) of $\boldsymbol{p}_i$ in depth.

Finally, we confirm all these results by a numerical study. More numerical results will be presented in the talk. In figures 2, the position of the two dipoles are represented by a white cross, the polarization vectors that one wants to recover with white arrows, the reconstructed vectors with blue arrows, the color scale indicates the magnitude of $\tilde{\boldsymbol{p}}$ and λ_0 stands for the central wavelength of ΔB. Note that we reconstruct the polarization vectors $\tilde{\boldsymbol{p}}_i$ up to a phase, which is fixed by imposing $\mathrm{Im}(p_x) = 0$.

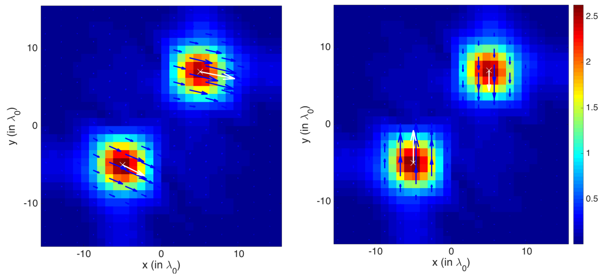

Figure 1: Image of $|\tilde{\boldsymbol{p}}|$, $\mathrm{Re}(\tilde{\boldsymbol{p}})$ (left) and $\mathrm{Im}(\tilde{\boldsymbol{p}})$ (right) in the cross-range plane $z = L$.

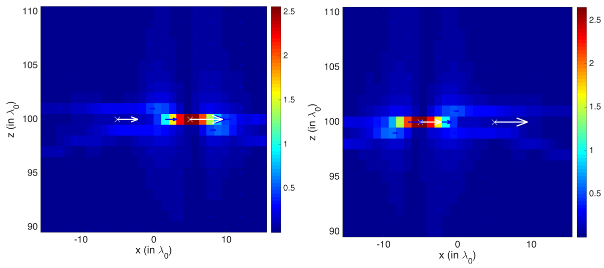

Figure 2: Image of $|\tilde{\boldsymbol{p}}|$, p_x in the range of the first dipole (left) and p_x in the range of the second dipole (right).

We have extended these results to the case where the dipoles are not radiating but behave as diffracting obstacles. In this setting, the diffraction of an electromagnetic wave by a dipole is governed by a 3×3 matrix: the polarizability tensor (see [2]). The objective is here to find both the position and the polarizability tensor of each dipole in the medium. We establish that one recovers their position with a cross-range resolution: $L/(ka)$ and a range resolution $c/\Delta B$. Furthermore, we show that one can reconstruct only the first 2×2 block of the polarizability tensor since the information about the other blocks is lost in the Fraunhofer regime.

References

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