Imaging small scatterers with electromagnetic waves

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(joint work with Maxence Cassier)

We consider the problem of imaging small scatterers in a homogeneous medium by probing the medium with electric dipoles located at an array and recording the resulting scattered electric field at the same array. For well separated scatterers, the scattered field can be well described in terms of a polarization tensor per scatterer [?], i.e. a 3×3 complex symmetric matrix. For N scatterers located at points $\vec{\boldsymbol{y}}_1, \ldots, \vec{\boldsymbol{y}}_N$ with polarization tensors $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N$, the array data (ignoring multiple scattering) is the matrix valued field

$$\boldsymbol{\Pi}(\boldsymbol{x}_r, \boldsymbol{x}_s, k) = \sum_{j=1}^{N} \mathbb{G}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}_j, k)\boldsymbol{\alpha}_N \mathbb{G}(\vec{\boldsymbol{y}}_j, \vec{\boldsymbol{x}}_s, k), \tag{1}$$

for all receiver $\vec{\boldsymbol{x}}_r = (\boldsymbol{x}_r, 0)$ and source $\vec{\boldsymbol{x}}_s = (\boldsymbol{x}_s, 0)$ positions in an array $\mathcal{A} \times 0$, located in the $x_3 = 0$ plane. Here we write vectors with three components in bold with arrows, and vectors with two components in bold only, so that $\vec{\boldsymbol{x}} = (\boldsymbol{x}, x_3)$. Also $\mathbb{G}(\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}, k) \in \mathbb{C}^{3\times 3}$ is the dyadic Green function for the Maxwell equation at wavenumber $k = 2\pi/\lambda = \omega/c$. As usual λ is the wavelength, ω is the angular frequency and c is the wave speed (see e.g. [?]).

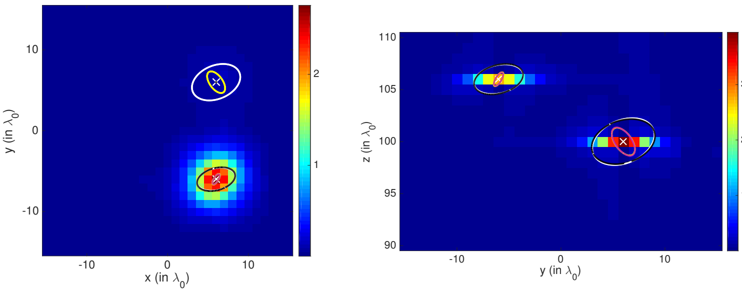
We presented a resolution study of the Kirchhoff imaging function adapted to electromagnetics:

$$\mathcal{I}(\vec{\boldsymbol{y}}, k) = \int_{\mathcal{A}} d\boldsymbol{x}_r \int_{\mathcal{A}} d\boldsymbol{x}_s \overline{\mathbb{G}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}, k)} \boldsymbol{\Pi}(\boldsymbol{x}_r, \boldsymbol{x}_s, k) \overline{\mathbb{G}(\vec{\boldsymbol{y}}, \vec{\boldsymbol{x}}_s, k)}, \tag{2}$$

which is a matrix valued field instead of a scalar one in acoustics. For an array of aperture a used to image an object at a distance L, the acoustics Kirchhoff imaging function is known to have a resolution of $\lambda L/a$ in the cross-range plane (the plane parallel to the array) and of c/B in the range direction (the direction perpendicular to the array). Here B is the bandwidth of the measurements and the image for multi-frequency data is obtained by integrating the single frequency image over the bandwidth. We show in [?] that the imaging function (**??**) obeys the same resolution estimates as the Kirchhoff imaging function in acoustics, if we consider the scalar field consisting of the norm of the matrix field (**??**) at each imaging point. Moreover we give a simple post-processing step that can extract from (**??**) a matrix field that approximates the polarization tensor of a scatterer if it were located at the imaging point. The analysis in [?] is done in the Fraunhofer asymptotic regime, which assumes that the propagation distance is large compared to the array, and that the object we want to image is small compared to the array (among other assumptions). The key quantity we study is the matrix valued field

$$\mathbb{H}(\vec{\boldsymbol{y}}, \vec{\boldsymbol{y}}', k) = \int_{\mathcal{A}} d\boldsymbol{x}_r \overline{\mathbb{G}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}, k)} \mathbb{G}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}', k), \tag{3}$$

which plays the role of a *point spread function*, i.e. the image of a point. This concept is easier to explain if we switch to the passive imaging case where the



(a) (b)

FIGURE 1. (a) Cross-range ($z = 106\lambda_0$) and (b) range ($x = 6\lambda_0$) images of scatterers. The color indicates the norm of the recovered polarization tensor. The white/black ellipses represent the true/calculated real part of the polarization tensor. The yellow/pink ellipses represent the true/calculated imaginary part.

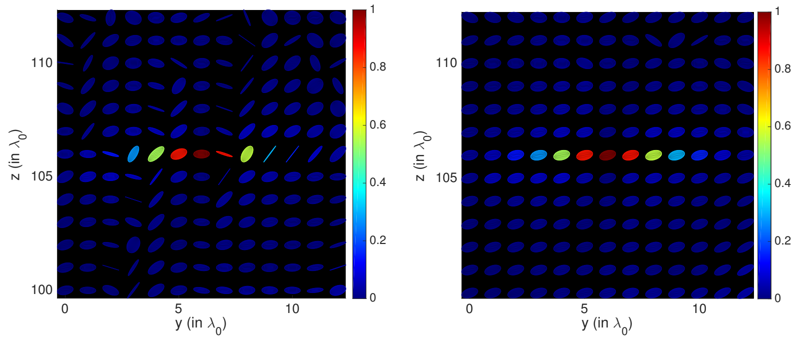
array consists of only receivers and the goal is to image a collection of small sources. In this case $\mathbb{H}(\vec{\boldsymbol{y}}, \vec{\boldsymbol{y}}', k)\vec{\boldsymbol{p}}$ is the Kirchhoff image at a point $\vec{\boldsymbol{y}}$ of a single point source located at $\vec{\boldsymbol{y}}'$ and with polarization vector $\vec{\boldsymbol{p}}$. We show that in the Fraunhofer asymptotic regime and if $\vec{\boldsymbol{y}}$ and $\vec{\boldsymbol{y}}'$ are in the same $x_3 = L$ plane, the point spread function decays as $1/\|\boldsymbol{y} - \boldsymbol{y}'\|$ at a rate consistent with the resolution estimate $\lambda L/a$. Similarly if we integrate over a frequency band $\omega_0 + [-B/2, B/2]$, we obtain a sinc like behavior in the range direction that gives the c/B range resolution estimate. Another conclusion of the asymptotic study is that $\mathbb{H}(\vec{\boldsymbol{y}}, \vec{\boldsymbol{y}}, k)$ is singular but that its 2×2 block corresponding to the cross-range coordinates is well-conditioned. Thus the problem of finding the polarization vector $\vec{\boldsymbol{p}}$ of a point source from the Kirchhoff image is ill-conditioned. However the linear system obtained by keeping only the cross-range components of $\mathbb{H}$ and the image is well-conditioned. Similarly for the active case, the problem of finding the cross-range components $\boldsymbol{\alpha}_{1:2,1:2}$ of a polarization tensor $\boldsymbol{\alpha}$ is stable. We note that only the cross-range components of the electric field are needed to image these quantities.

Examples of matrix valued images are given in figures **??** and **??**, where we imaged two point scatterers located at $\vec{\boldsymbol{y}}_1 = (6\lambda_0, 6\lambda_0, 100\lambda_0)$, $\vec{\boldsymbol{y}}_2 = (6\lambda_0, -6\lambda_0, 106\lambda_0)$ and with polarization tensors

$$\boldsymbol{\alpha}_1 = \begin{bmatrix} 2+2\imath & 1-\imath/2 & 0 \\ 1-\imath/2 & 1+2\imath & 1+\imath/2 \\ 0 & 1+\imath/2 & 1+\imath \end{bmatrix} \text{ and } \boldsymbol{\alpha}_2 = \begin{bmatrix} 2+\imath & \imath/2 & 1/2 \\ \imath/2 & 1+\imath & 0 \\ 1/2 & 0 & 1+\imath \end{bmatrix}.$$

In both figures **??** and **??**, we visualize 2×2 symmetric matrices by ellipses with principal axis and dimensions given by the matrices' eigenvectors and eigenvalues.

We are currently adapting a technique for Kirchhoff imaging without phases [**?**] to the Maxwell equations (joint with Patrick Bardsley and Maxence Cassier). The experimental setup consists of a single electric dipole point source located at $\vec{\boldsymbol{x}}_s$



(a) (b)

FIGURE 2. Images of the polarization tensor in range ($x = 6\lambda_0$). The analysis in [**?**] shows that a straightforward solution of the system for the polarization tensor $\boldsymbol{\alpha}$ is afflicted by oscillatory artifacts (a). These can be removed by fixing the phase of the recovered polarization tensor by enforcing e.g. that $\alpha_{1,1}$ be real (b). The ellipses' color represents the polarization tensor norm.

and a passive array that are used to image the position and polarization tensors of a collection of small scatterers. The electric field generated by the source is

$$\vec{\boldsymbol{E}}_{\rm inc}(\vec{\boldsymbol{x}}, k) = \mathbb{G}(\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}_s, k)\vec{\boldsymbol{j}}_s$$

where the polarization vector $\vec{\boldsymbol{j}}_s(k)$ is a zero mean, stationary, ergodic Gaussian process with known correlation matrix $J(k) = \langle \vec{\boldsymbol{j}}_s(k)\vec{\boldsymbol{j}}_s(k)^* \rangle$. We assume the array can only measure polarization data in the cross-range plane. That is only the 2×2 Hermitian auto-correlation matrices $C(\boldsymbol{x}_r, k) = \langle \boldsymbol{E}(\vec{\boldsymbol{x}}_r, k)\boldsymbol{E}(\vec{\boldsymbol{x}}_r, k)^* \rangle$ are known. Here $\boldsymbol{E}(\vec{\boldsymbol{x}}_r, k)$ is the cross-range total field evaluated at the array location $\vec{\boldsymbol{x}}_r$. Hence the data we work with is equivalent to measuring the Stokes parameters of the electric field at the array [**?**]. We have preliminary results showing that the resolution estimates apply to this setting, but the information that can be recovered stably about the polarization tensors $\boldsymbol{\alpha}$ corresponds to a 2×2 projection of $\boldsymbol{\alpha}$ in left and right bases determined by the positions of the receiver array, the source and the scatterer.

REFERENCES

[1] H. Ammari, M. S. Vogelius, and D. Volkov, *Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter II. The full Maxwell equations.*, Journal de mathématiques pures et appliquées, 80 (2001), pp. 769–814, doi:10.1016/S0021-7824(01)01217-X.

[2] P. Bardsley and F. Guevara Vasquez, *Kirchhoff migration without phases*, Inverse Problems, 32 (2016), p. 105006, doi:10.1088/0266-5611/32/10/105006

[3] M. Cassier and F. Guevara Vasquez, *Imaging Polarizable dipoles*, ArXiv:1703.03544.

[4] L. Novotny and B. Hecht, *Principles of nano-optics*, Cambridge University Press, 2012.