$assume(n > 0, n, integer); getassumptions(n);$

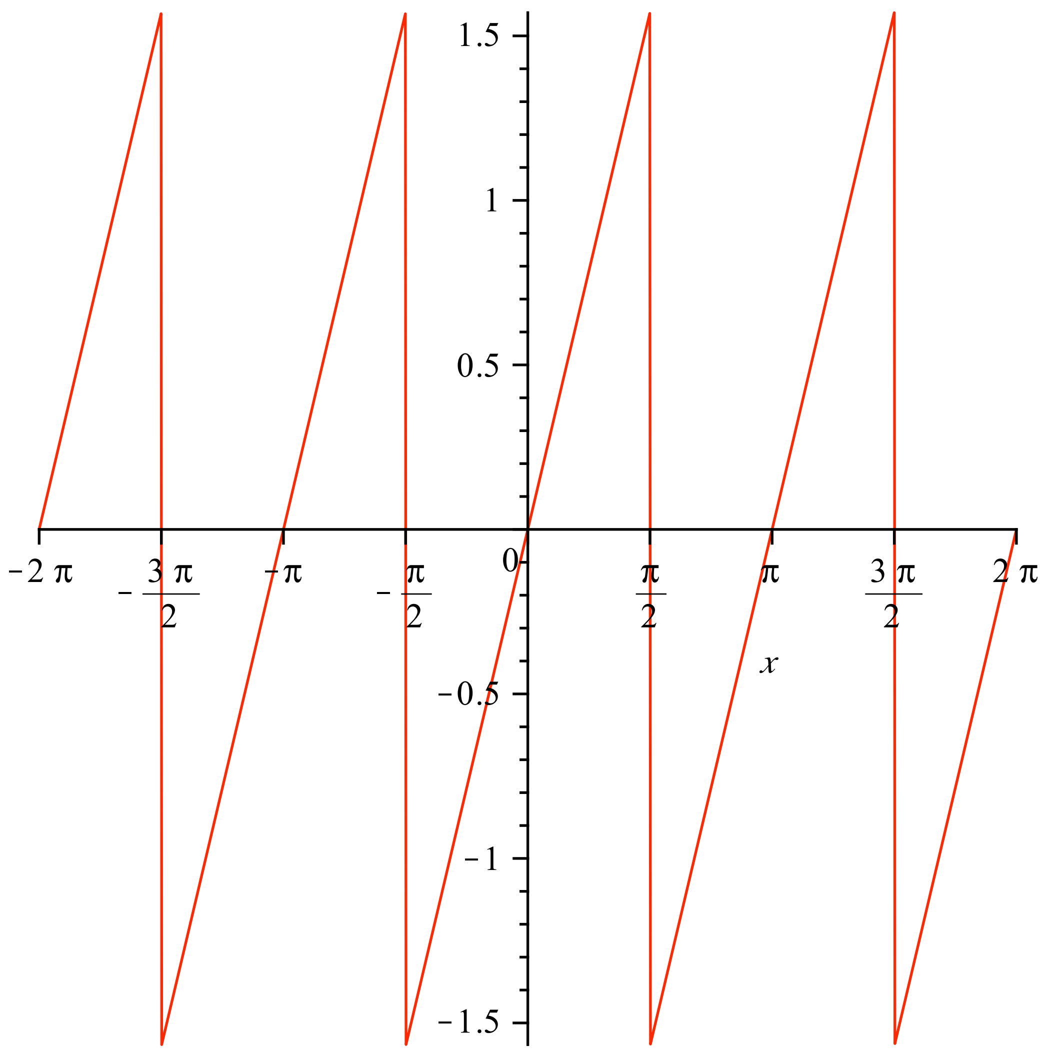
$$\{n\sim::(AndProp(integer, RealRange(1, \infty)))\} \quad (1)$$

$sawtooth := x \rightarrow piecewise\left(x < -2\,Pi, 0, x < -\frac{3\,Pi}{2}, x + 2\,Pi, x < -\frac{Pi}{2}, x + Pi, x < \frac{Pi}{2}, x, x < \frac{3\,Pi}{2}, x - Pi, x < 2\,Pi, x - 2\,Pi\right);$

$$x \rightarrow piecewise\left(x < -2\,\pi, 0, x < -\frac{3}{2}\,\pi, x + 2\,\pi, x < -\frac{1}{2}\,\pi, x + \pi, x < \frac{1}{2}\,\pi, x, x < \frac{3}{2}\,\pi, x - \pi, x < 2\,\pi, x - 2\,\pi\right) \quad (2)$$

$plot\left(sawtooth(x), x = -2\,Pi..2\,Pi, tickmarks = \left[spacing\left(\frac{\pi}{2}\right), default\right]\right);$

########### We computed the Fourier coefficients for this function in class.



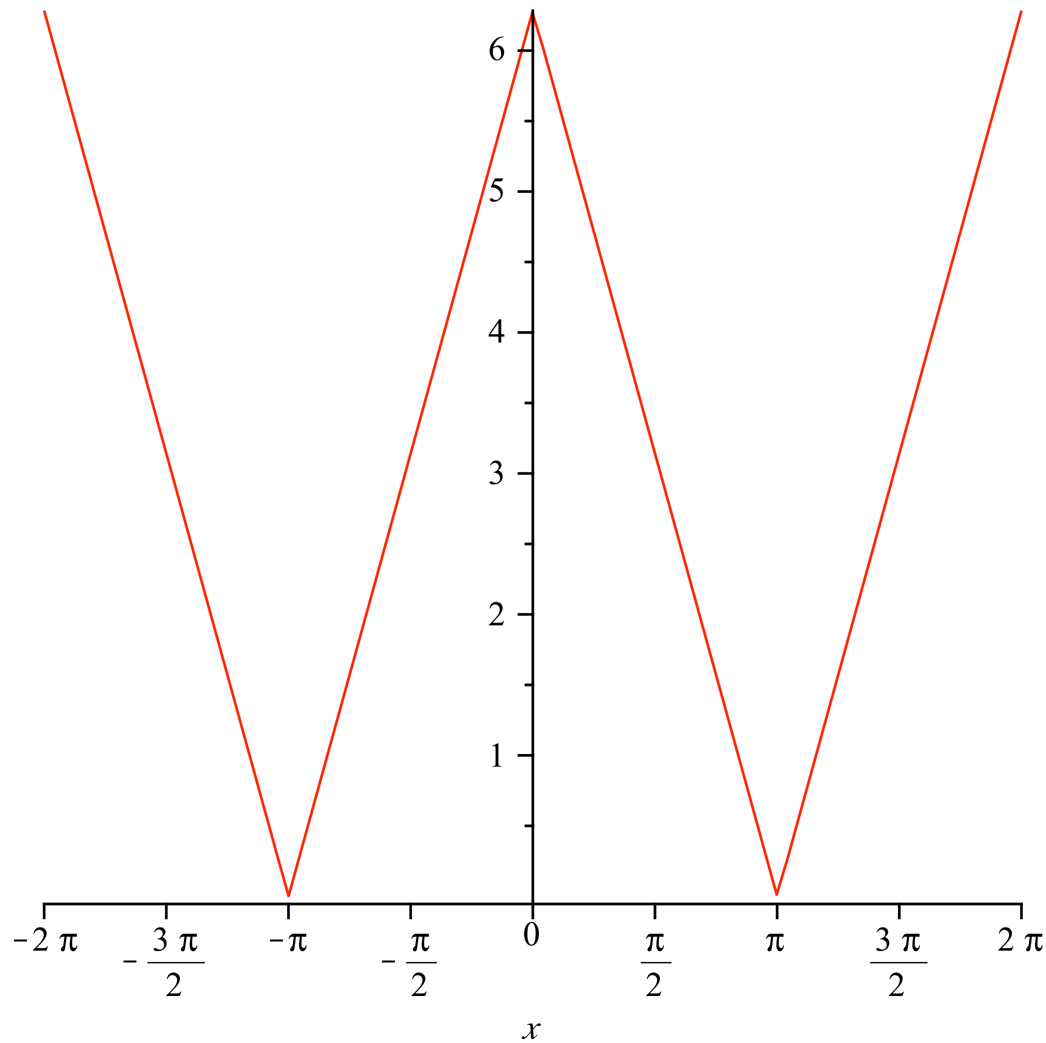
$plot\left(\left[sawtooth(x), sum\left(\frac{(-1)^{1+m}}{m}\sin(2\,m\cdot x), m = 1..5\right)\right], x = -2\,Pi..2\,Pi, tickmarks = \left[spacing\left(\frac{\pi}{2}\right), default\right]\right);$

############ *change the m to see different Fourier sums, try m=1..10*

$triangle := x \rightarrow piecewise(x < -\mathrm{Pi}, -2(x+\mathrm{Pi}), x < 0, 2(x+\mathrm{Pi}), x < \mathrm{Pi}, -2(x-\mathrm{Pi}), x < 2\,\mathrm{Pi}, 2(x-\mathrm{Pi}));$

$$x \rightarrow piecewise(x < -\pi, -2x - 2\pi, x < 0, 2x + 2\pi, x < \pi, -2x + 2\pi, x < 2\pi, 2x - 2\pi) \quad \textbf{(3)}$$

$plot\left(triangle(x), x = -2\,\mathrm{Pi}..2\,\mathrm{Pi}, tickmarks = \left[spacing\left(\frac{\pi}{2}\right), default\right]\right);$



$a_0 = \frac{1}{2\,\mathrm{Pi}} int(triangle(x), x = -\mathrm{Pi}..\mathrm{Pi}); a_n = simplify\left(\frac{1}{\mathrm{Pi}} int(triangle(x) \cdot \cos(n \cdot x), x = -\mathrm{Pi}..\mathrm{Pi})\right); b_n = simplify\left(\frac{1}{\mathrm{Pi}} int(triangle(x) \cdot \sin(n \cdot x), x = -\mathrm{Pi}..\mathrm{Pi})\right);$

$$a_0 = \pi$$

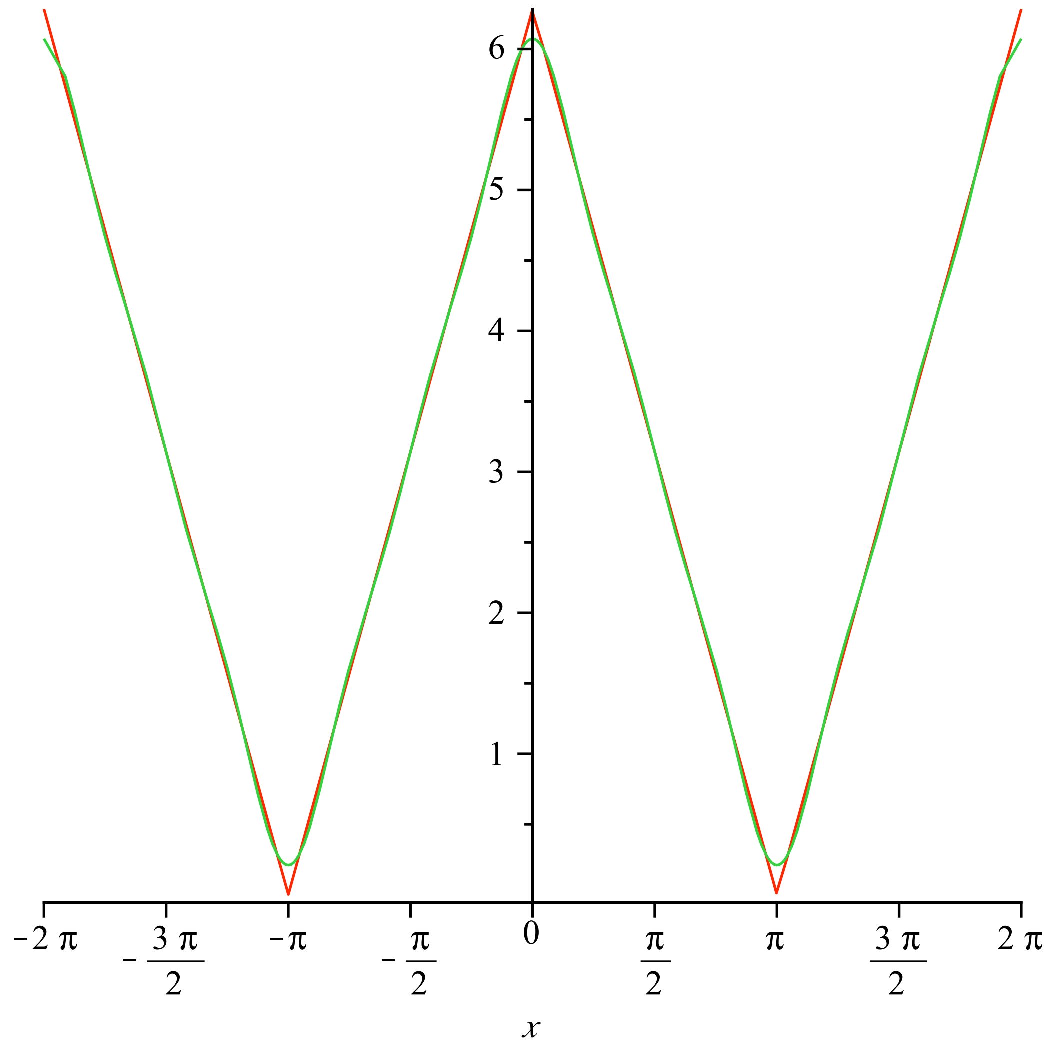
$$a_{n\sim} = \frac{4\left((-1)^{1+n\sim} + 1\right)}{\pi\, n\sim^2}$$

$$b_{n\sim} = 0 \quad \textbf{(4)}$$

We';ll let Maple compute the Fourier coefficients this time

$$plot\left(\left[triangle(x), \frac{1}{2\,\text{Pi}} int(triangle(x), x=-\text{Pi}..\text{Pi}) + sum\left(\frac{1}{\text{Pi}} int(triangle(x)\cdot\cos(m\cdot x), x=-\text{Pi}..\text{Pi})\cdot\cos(m\cdot x) + \frac{1}{\text{Pi}} int(triangle(x)\cdot\sin(m\cdot x), x=-\text{Pi}..\text{Pi})\cdot\sin(m\cdot x), m=1..5\right)\right], x=-2\,\text{Pi}..2\,\text{Pi}, tickmarks=\left[spacing\left(\frac{\pi}{2}\right), default\right]\right);$$

######## Note that the approximations become good quicly, due the the quadratic decay of the coefficients.

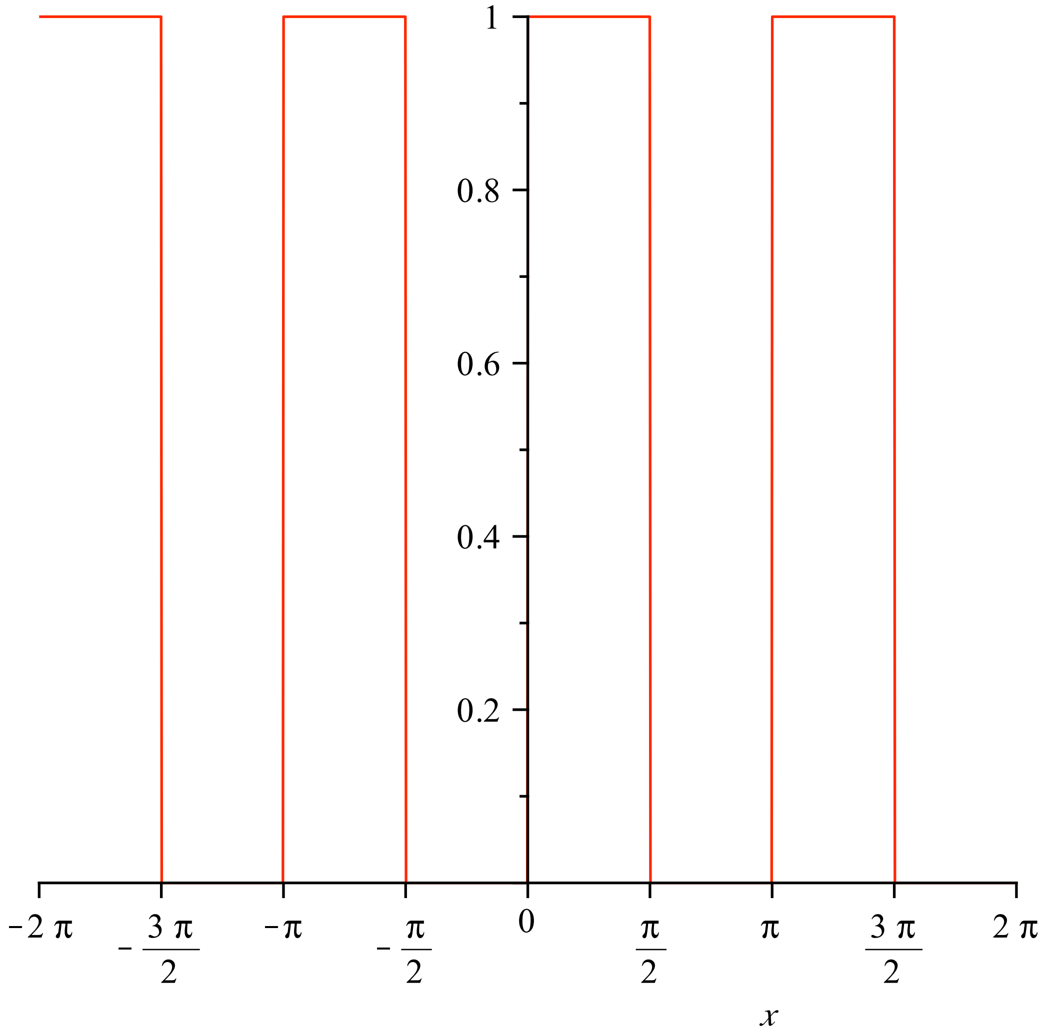


$$square := x \rightarrow piecewise\left(x < -2\,\text{Pi}, 0, x < -\frac{3\,\text{Pi}}{2}, 1, x < -\text{Pi}, 0, x < -\frac{\text{Pi}}{2}, 1, x < 0, 0, x < \frac{\text{Pi}}{2}, 1, x < \text{Pi}, 0, x < \frac{3\,\text{Pi}}{2}, 1, x < 2\,\text{Pi}, 0, 1\right);$$

\###

$$x \rightarrow piecewise\left(x < -2\,\pi, 0, x < -\frac{3}{2}\,\pi, 1, x < -\pi, 0, x < -\frac{1}{2}\,\pi, 1, x < 0, 0, x < \frac{1}{2}\,\pi, 1, x < \pi, 0, x < \frac{3}{2}\,\pi, 1, x < 2\,\pi, 0, 1\right) \quad \textbf{(5)}$$

$plot\left(square(x), x=-2\,\text{Pi}..2\,\text{Pi}, tickmarks=\left[spacing\left(\frac{\pi}{2}\right), default\right]\right);$



$a_0 = \frac{1}{2\,\text{Pi}} int(square(x), x=-\text{Pi}..\text{Pi});$

$$a_0 = \frac{1}{2} \tag{6}$$

$a_n = simplify\left(\frac{1}{\text{Pi}} int(square(x)\cdot\cos(n\cdot x), x=-\text{Pi}..\text{Pi})\right);$

$$a_{n\sim} = 0 \tag{7}$$

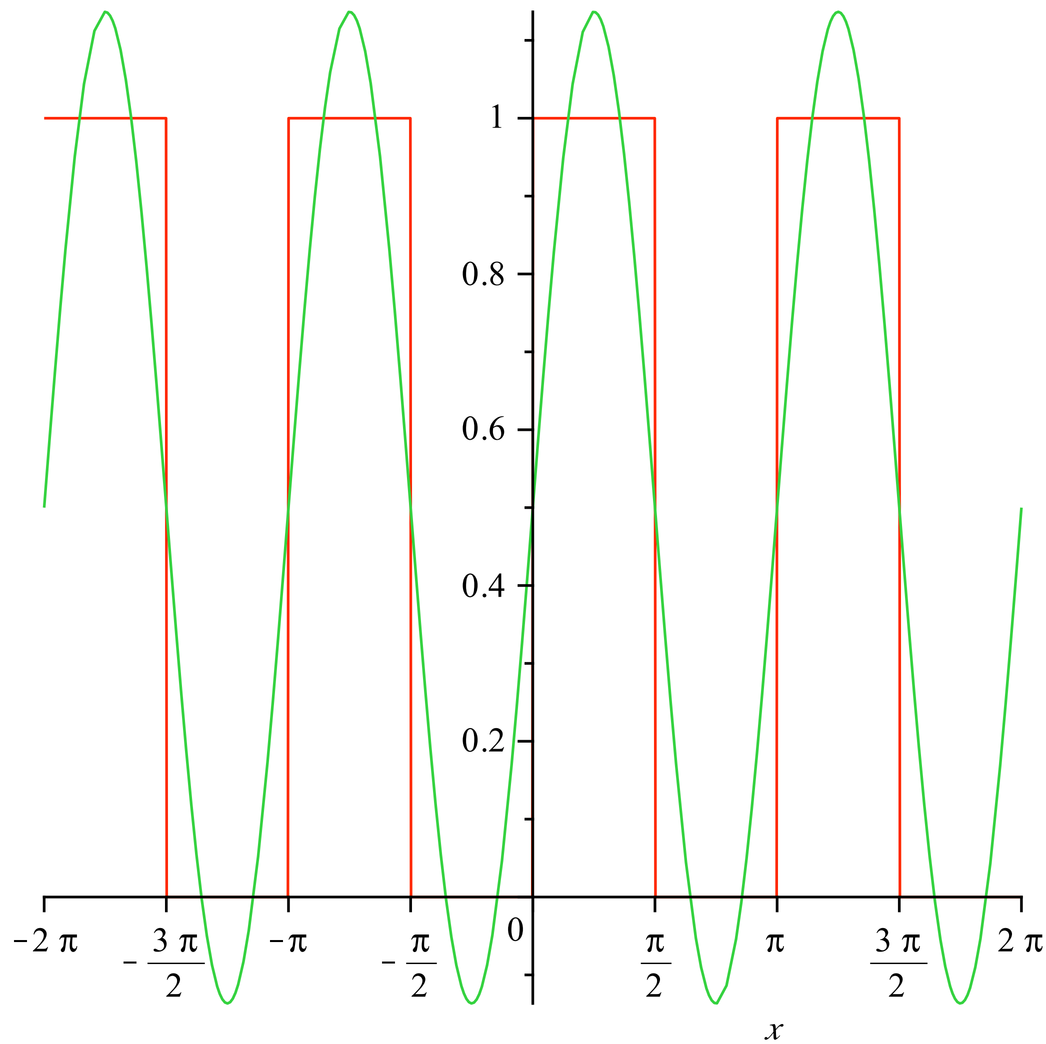
$b_n = simplify\left(\frac{1}{\text{Pi}} int(square(x)\cdot\sin(n\cdot x), x=-\text{Pi}..\text{Pi})\right);$

$$b_{n\sim} = \frac{(-1)^{n\sim} - 2\cos\left(\frac{1}{2}\pi\, n\sim\right) + 1}{\pi\, n\sim} \tag{8}$$

$plot\left(\left[square(x), \frac{1}{2\,\text{Pi}} int(square(x), x=-\text{Pi}..\text{Pi}) + sum\left(\frac{1}{\text{Pi}} int(square(x)\cdot\cos(m\cdot x), x=-\text{Pi}\right.\right.\right.$

$..\mathrm{Pi})\cdot\cos(m\cdot x) + \frac{1}{\mathrm{Pi}}\mathit{int}(\mathit{square}(x)\cdot\sin(m\cdot x), x=-\mathrm{Pi}..\mathrm{Pi})\cdot\sin(m\cdot x), m=1..5\Big)\Big], x=-2\,\mathrm{Pi}$

$..2\,\mathrm{Pi}, \mathit{tickmarks}=\left[\mathit{spacing}\left(\frac{\pi}{2}\right), \mathit{default}\right]\Big);$

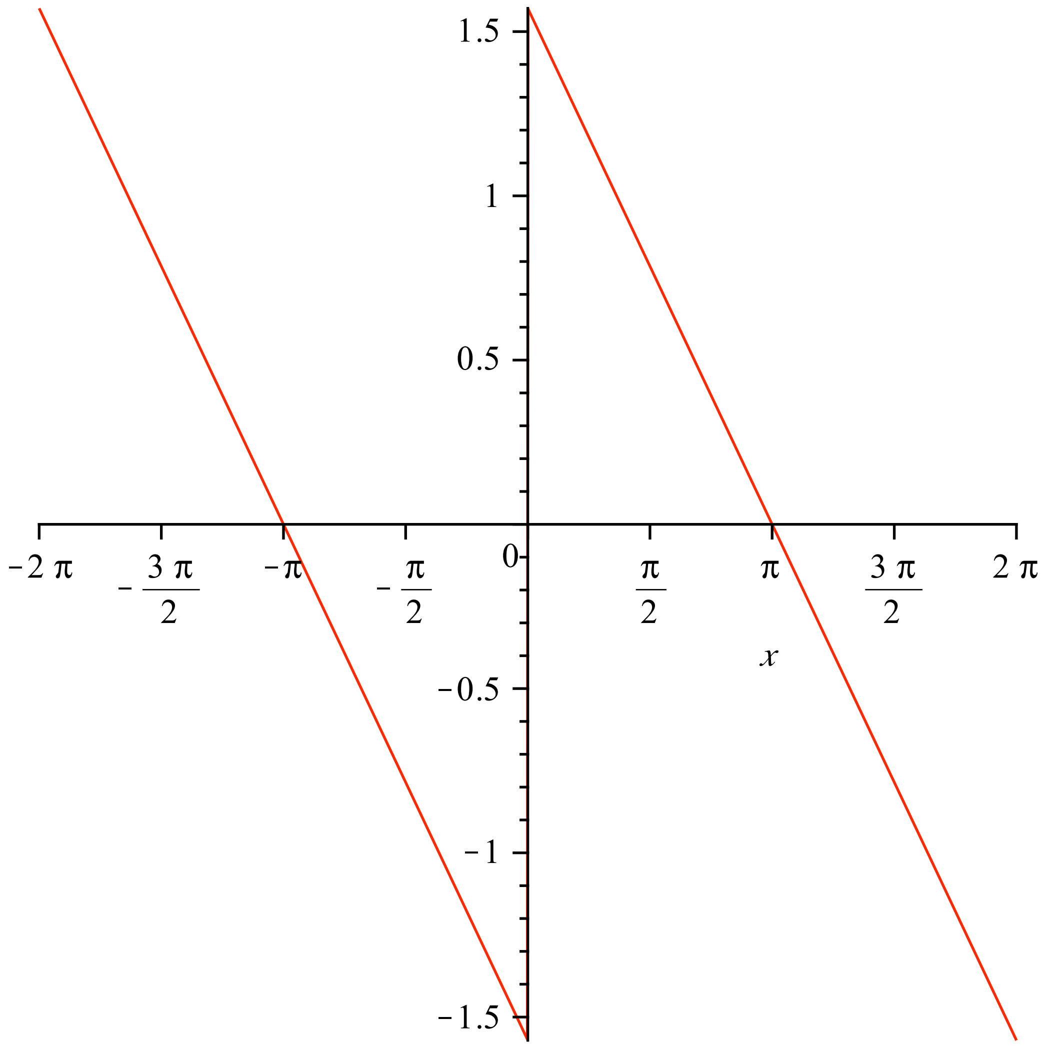


 # Gibbs phenomemon
 # sawtooth wave from the book

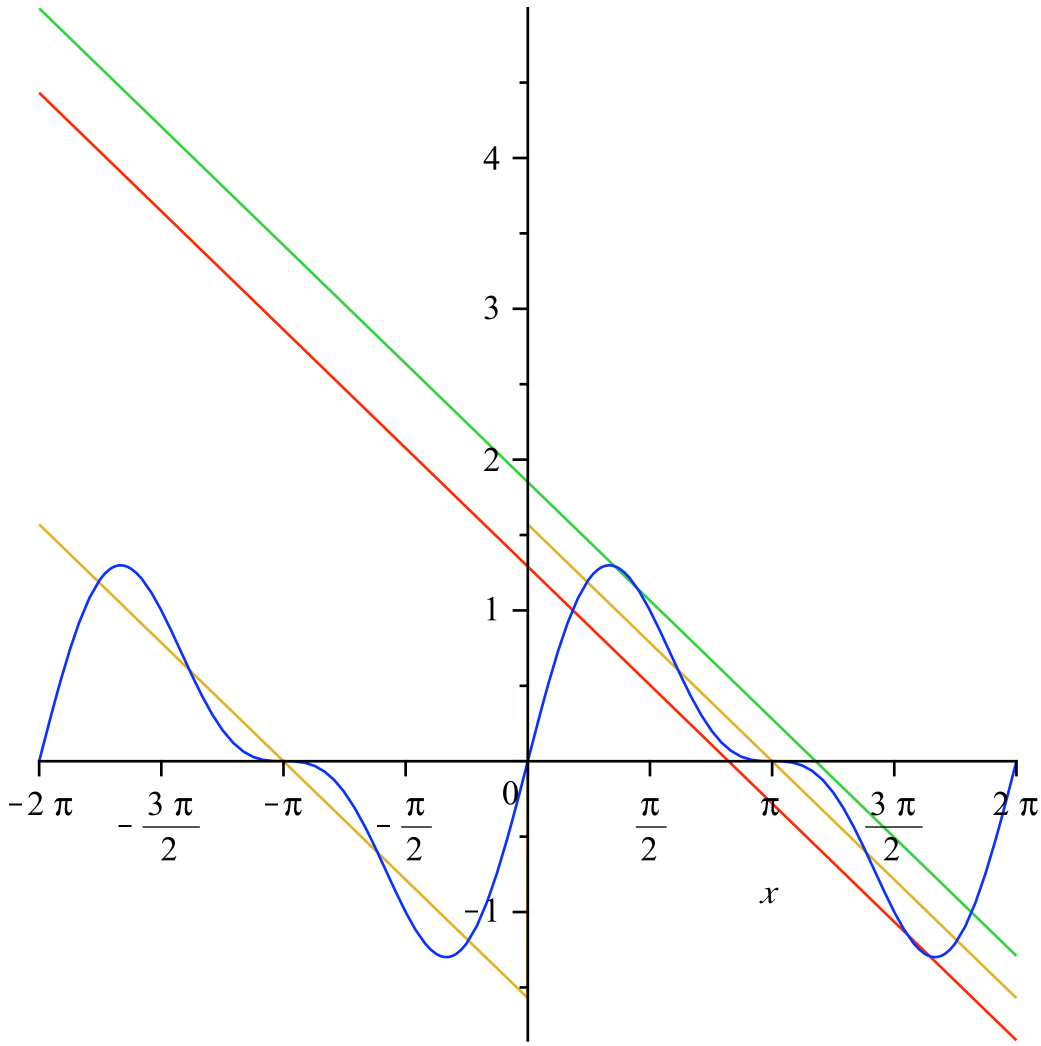
$\mathit{booksawtooth} := x \rightarrow \mathit{piecewise}\left(x < -2\,\mathrm{Pi}, 0, x < 0, \frac{1}{2}\cdot(-\mathrm{Pi}-x), x < 2\,\mathrm{Pi}, \frac{1}{2}\cdot(\mathrm{Pi}-x)\right);$

$$x \rightarrow \mathit{piecewise}\left(x < -2\,\pi, 0, x < 0, -\frac{1}{2}\pi - \frac{1}{2}x, x < 2\,\pi, \frac{1}{2}\pi - \frac{1}{2}x\right) \quad \textbf{(9)}$$

$\mathit{plot}\left(\mathit{booksawtooth}(x), x=-2\,\mathrm{Pi}..2\,\mathrm{Pi}, \mathit{tickmarks}=\left[\mathit{spacing}\left(\frac{\pi}{2}\right), \mathit{default}\right]\right);$



$$plot\left(\left[\frac{1}{2}\cdot(\mathrm{Pi}-x)-.28, \frac{1}{2}\cdot(\mathrm{Pi}-x)+.28, booksawtooth(x), \frac{1}{2\,\mathrm{Pi}} int(booksawtooth(x), x=-\mathrm{Pi}..\mathrm{Pi}) + sum\left(\frac{1}{\mathrm{Pi}} int(booksawtooth(x)\cdot\cos(m\cdot x), x=-\mathrm{Pi}..\mathrm{Pi})\cdot\cos(m\cdot x) + \frac{1}{\mathrm{Pi}} int(booksawtooth(x)\cdot\sin(m\cdot x), x=-\mathrm{Pi}..\mathrm{Pi})\cdot\sin(m\cdot x), m=1..2\right)\right], x=-2\,\mathrm{Pi}..2\,\mathrm{Pi}, tickmarks=\left[spacing\left(\frac{\pi}{2}\right), default\right]\right);$$

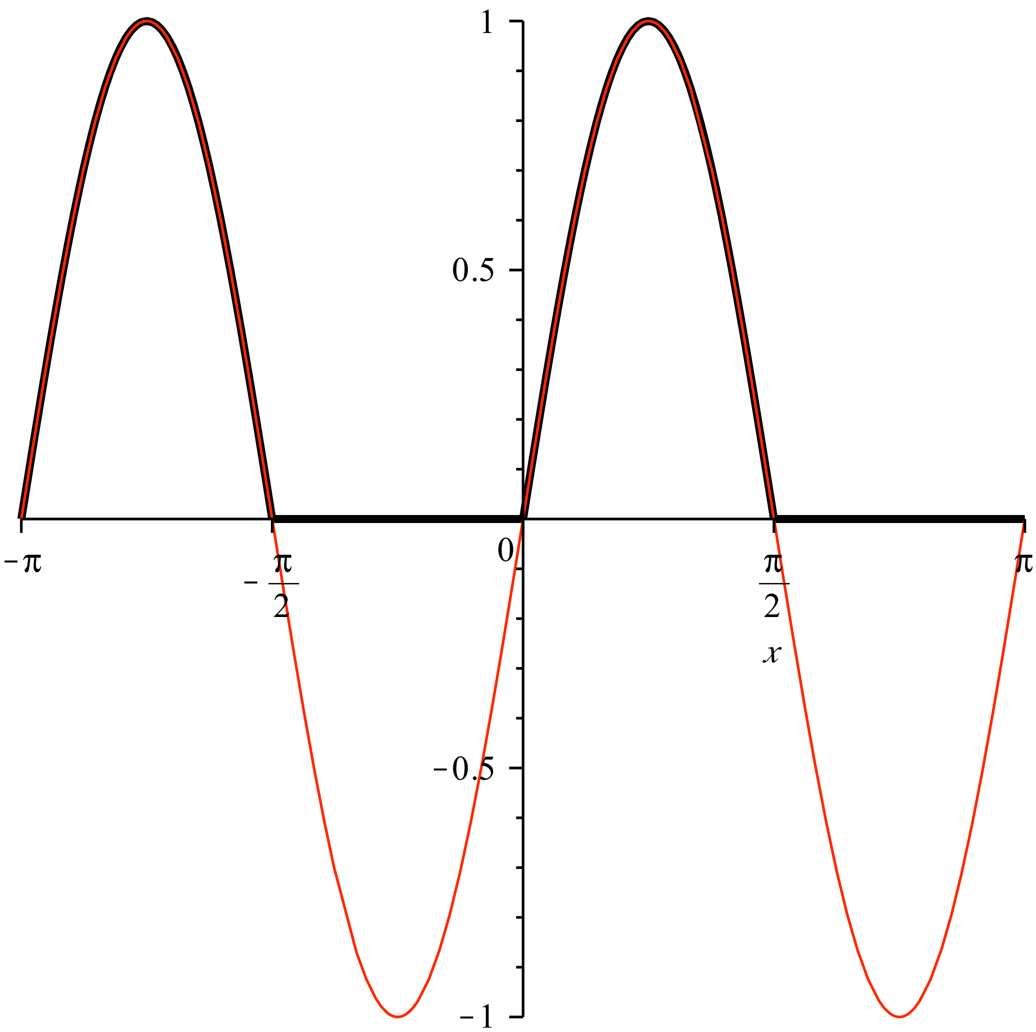


As the Fourier sums approach a discontinuity they tend to overshoot the function first and then dive down. Furthermore, the amount by which they overshoot limits to a constant, and the location of the overshoot tends towards the discontinuity. This is called the Gibbs phenomenon. The book calculates the amount of overshoot in this example to be .28. The two parallel lines are .28 above and below the sawtooth wave. Try some different Fourier sums and notice that there seems to be a bump that travels along one of the parallel lines towards the discontinuity at the y-axis (or at 2Pi).

######## *Here's the homework problem that caused Maple some trouble*

$halfs := x \to \max([\sin(2 \cdot x), 0]);\ plot\left([\text{halfs}(x), \sin(2\,x)], x = -\text{Pi}..\text{Pi}, tickmarks = \left[spacing\left(\frac{\pi}{2}\right), default\right], discont = true, color = [black, red], thickness = [3, 1]\right);$

$$x \to \max([\sin(2\,x), 0])$$



a_0

$a_0 = \frac{1}{2\,\text{Pi}} int(halfs(x), x = -\text{Pi}..\text{Pi})$

$$a_0 = \frac{1}{2}\,\frac{\int_{-\pi}^{\pi} \max(0, \sin(2\,x))\,\mathrm{d}x}{\pi} \tag{10}$$

$a_0 = evalf\left(\frac{1}{2\,\mathrm{Pi}}\,int(halfs(x), x = -\mathrm{Pi}..\mathrm{Pi})\right);$

$$a_0 = 0.3183098862 \tag{11}$$

a_n

$a_n = simplify\left(\frac{1}{\mathrm{Pi}}\,int(halfs(x)\cdot\cos(n\cdot x), x = -\mathrm{Pi}..\mathrm{Pi})\right);$

$$a_{n\sim} = \frac{\int_{-\pi}^{\pi} \max(0, \sin(2\,x))\,\cos(n\sim x)\,\mathrm{d}x}{\pi} \tag{12}$$

 # Maple can't do the symbolic integration, try to help it

$a_n = simplify\left(\frac{1}{\mathrm{Pi}}\cdot\left(int\left(\sin(2\cdot x)\cdot\cos(n\cdot x), x = -\mathrm{Pi}..-\frac{\mathrm{Pi}}{2}\right) + int\left(0\cdot\cos(n\cdot x), x = -\frac{\mathrm{Pi}}{2}..0\right) + int\left(\sin(2\cdot x)\cdot\cos(n\cdot x), x = 0..\frac{\mathrm{Pi}}{2}\right) + int\left(0\cdot\cos(n\cdot x), x = \frac{\mathrm{Pi}}{2}..\mathrm{Pi}\right)\right)\right);$

$$a_{n\sim} = \frac{2\left((-1)^{1+n\sim} - 2\cos\left(\frac{1}{2}\,\pi\,n\sim\right) - 1\right)}{\pi\,(-4 + n\sim^2)} \tag{13}$$

This expression is not valid for n=2!

for i **from** 1 **to** 20 **do** $a_i = simplify\left(\frac{1}{\mathrm{Pi}}\cdot\left(int\left(\sin(2\cdot x)\cdot\cos(i\cdot x), x = -\mathrm{Pi}..-\frac{\mathrm{Pi}}{2}\right) + int\left(0\cdot\cos(i\cdot x), x = -\frac{\mathrm{Pi}}{2}..0\right) + int\left(\sin(2\cdot x)\cdot\cos(i\cdot x), x = 0..\frac{\mathrm{Pi}}{2}\right) + int\left(0\cdot\cos(i\cdot x), x = \frac{\mathrm{Pi}}{2}..\mathrm{Pi}\right)\right)\right)$ **end do**

$$a_1 = 0$$

$$a_2 = 0$$

$$a_3 = 0$$

$$a_4 = -\frac{2}{3\,\pi}$$

$$a_5 = 0$$

$$a_6 = 0$$

$$a_7 = 0$$

$$a_8 = -\frac{2}{15\pi}$$

$$a_9 = 0$$

$$a_{10} = 0$$

$$a_{11} = 0$$

$$a_{12} = -\frac{2}{35\pi}$$

$$a_{13} = 0$$

$$a_{14} = 0$$

$$a_{15} = 0$$

$$a_{16} = -\frac{2}{63\pi}$$

$$a_{17} = 0$$

$$a_{18} = 0$$

$$a_{19} = 0$$

$$a_{20} = -\frac{2}{99\pi} \quad \textbf{(14)}$$

b_n

$$b_n = simplify\left(\frac{1}{\mathrm{Pi}} int(halfs(x)\cdot\sin(n\cdot x), x=-\mathrm{Pi}..\mathrm{Pi})\right);$$

$$b_{n\sim} = \frac{\int_{-\pi}^{\pi} \max(0, \sin(2x))\,\sin(n\sim x)\,\mathrm{d}x}{\pi} \quad \textbf{(15)}$$

$$b_n = simplify\left(\frac{1}{\mathrm{Pi}} int\left(\sin(2x)\cdot\sin(n\cdot x), x=-\mathrm{Pi}..-\frac{\mathrm{Pi}}{2}\right) + \frac{1}{\mathrm{Pi}} int\left(\sin(2x)\cdot\sin(n\cdot x), x=0..\frac{\mathrm{Pi}}{2}\right)\right);$$

$$b_{n\sim} = 0 \quad \textbf{(16)}$$

That's fishy, there ought to be a non-zero sin(2x) term.

$$b_2 = simplify\left(\frac{1}{\text{Pi}} int\left(\sin(2\,x)\cdot\sin(2\cdot x), x=-\text{Pi}..-\frac{\text{Pi}}{2}\right) + \frac{1}{\text{Pi}} int\left(\sin(2\,x)\cdot\sin(2\cdot x), x=0..\frac{\text{Pi}}{2}\right)\right)$$

$$b_2 = \frac{1}{2} \tag{17}$$

for i **from** 1 **to** 20 **do** $a_i = simplify\left(\frac{1}{\text{Pi}}\cdot\left(int\left(\sin(2\cdot x)\cdot\sin(i\cdot x), x=-\text{Pi}..-\frac{\text{Pi}}{2}\right) + int\left(0\cdot\sin(i\cdot x), x=-\frac{\text{Pi}}{2}..0\right) + int\left(\sin(2\cdot x)\cdot\sin(i\cdot x), x=0..\frac{\text{Pi}}{2}\right) + int\left(0\cdot\sin(i\cdot x), x=\frac{\text{Pi}}{2}..\text{Pi}\right)\right)\right)$ **end do**

$$a_1 = 0$$
$$a_2 = \frac{1}{2}$$
$$a_3 = 0$$
$$a_4 = 0$$
$$a_5 = 0$$
$$a_6 = 0$$
$$a_7 = 0$$
$$a_8 = 0$$
$$a_9 = 0$$
$$a_{10} = 0$$
$$a_{11} = 0$$
$$a_{12} = 0$$
$$a_{13} = 0$$
$$a_{14} = 0$$
$$a_{15} = 0$$
$$a_{16} = 0$$
$$a_{17} = 0$$
$$a_{18} = 0$$
$$a_{19} = 0$$
$$a_{20} = 0 \tag{18}$$

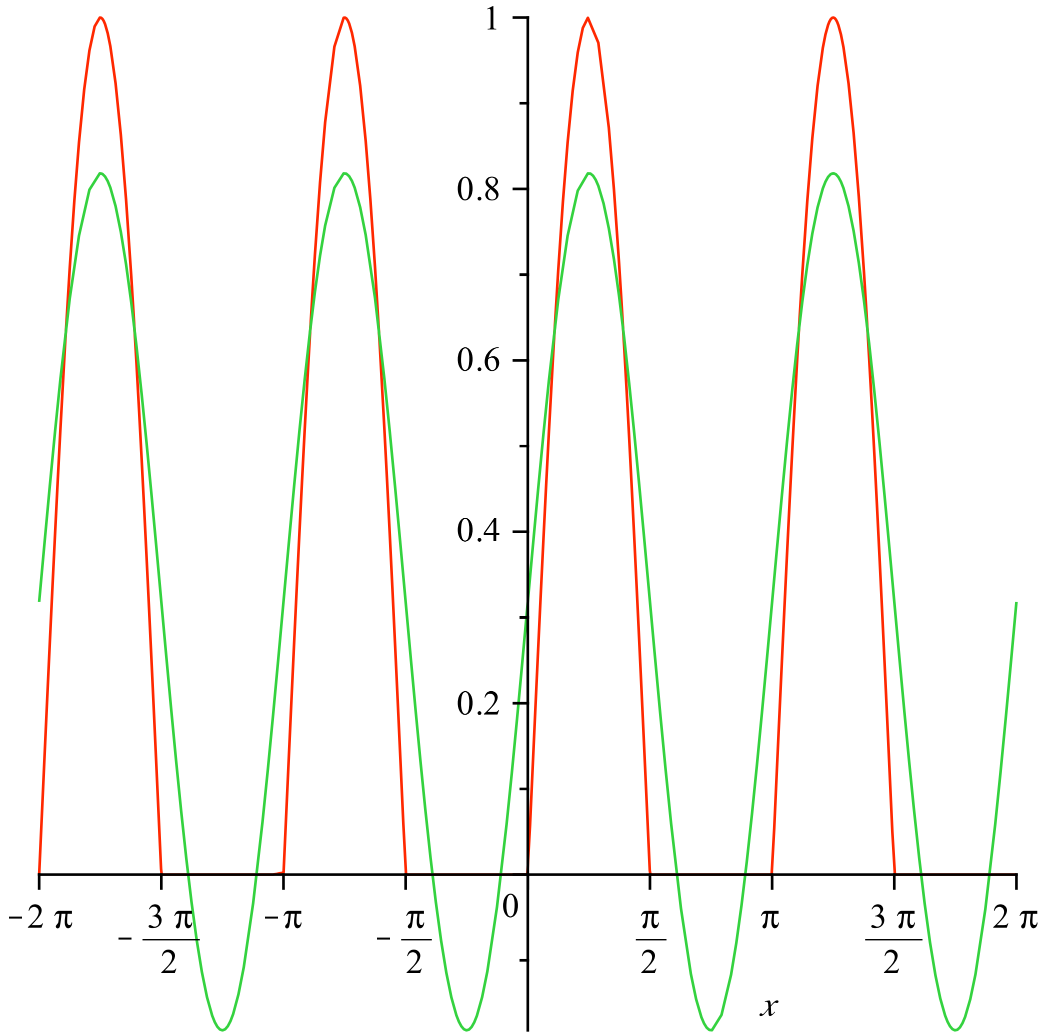
$\#plot\left(\left[halfs(x), \frac{1}{2\,\mathrm{Pi}} int(halfs(x), x=-\mathrm{Pi}..\mathrm{Pi}) + sum\left(\frac{1}{\mathrm{Pi}} int(halfs(x)\cdot\cos(m\cdot x), x=-\mathrm{Pi}..\mathrm{Pi})\cdot\cos(m\cdot x) + \frac{1}{\mathrm{Pi}} int(halfs(x)\cdot\sin(m\cdot x), x=-\mathrm{Pi}..\mathrm{Pi})\cdot\sin(m\cdot x), m=1..100\right)\right], x=-2\,\mathrm{Pi}..2\,\mathrm{Pi}, tickmarks=\left[spacing\left(\frac{\pi}{2}\right), default\right]\right);$ *# Maple won't be able to compute this*

$plot\left(\left[halfs(x), \frac{1}{2\,\mathrm{Pi}} int(halfs(x), x=-\mathrm{Pi}..\mathrm{Pi}) + sum\left(\left(\frac{1}{\mathrm{Pi}} int\left(\sin(2\,x)\cdot\cos(m\cdot x), x=-\mathrm{Pi}..-\frac{\mathrm{Pi}}{2}\right) + \frac{1}{\mathrm{Pi}} int\left(\sin(2\,x)\cdot\cos(m\cdot x), x=0..\frac{\mathrm{Pi}}{2}\right)\right)\cdot\cos(mx), m=1..100\right)\right], x=-2\,\mathrm{Pi}..2\,\mathrm{Pi}, tickmarks=\left[spacing\left(\frac{\pi}{2}\right), default\right]\right);$

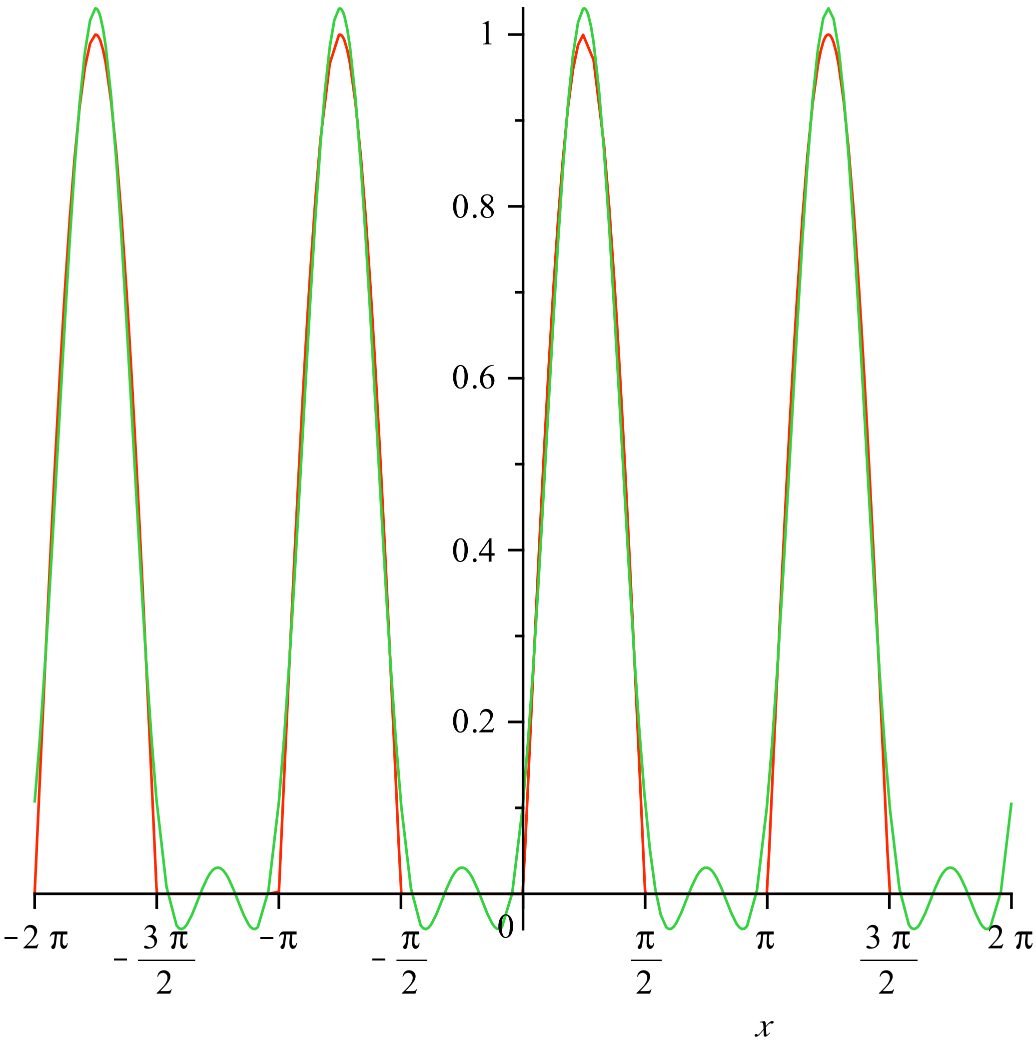
Error, (in SumTools:-DefiniteSum:-ClosedForm) summand is singular in the interval of summation

The previous expresson gives me an error about summand being singular, but that's because the expression for a_n *is* **not** *valid* **for** a_2

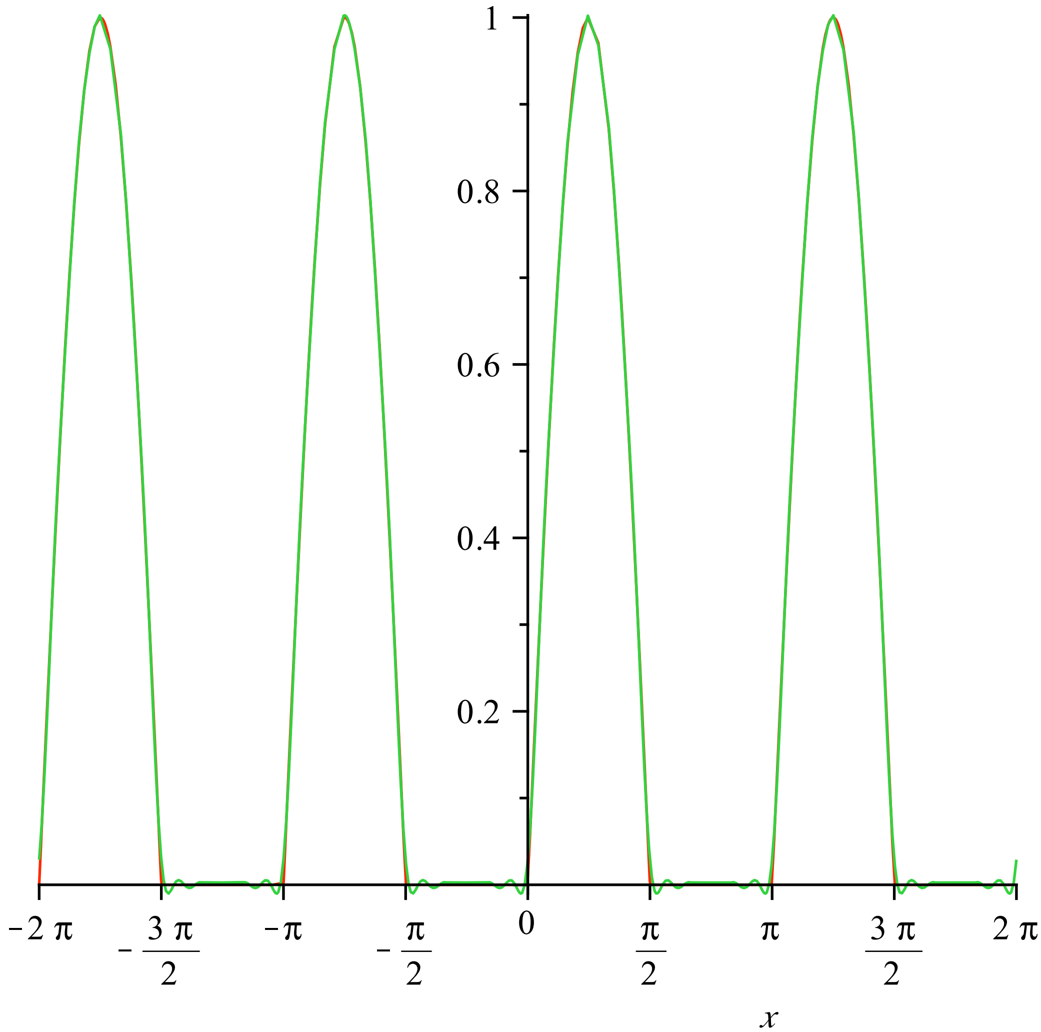
$plot\left(\left[halfs(x), \frac{1}{2\,\mathrm{Pi}} int(halfs(x), x=-\mathrm{Pi}..\mathrm{Pi}) + \frac{1}{2}\cdot\sin(2\cdot x) + sum\left(\left(\frac{1}{\mathrm{Pi}} int\left(\sin(2\,x)\cdot\cos(m\cdot x), x=-\mathrm{Pi}..-\frac{\mathrm{Pi}}{2}\right) + \frac{1}{\mathrm{Pi}} int\left(\sin(2\,x)\cdot\cos(m\cdot x), x=0..\frac{\mathrm{Pi}}{2}\right)\right)\cdot\cos(m\cdot x), m=3..3\right)\right], x=-2\,\mathrm{Pi}..2\,\mathrm{Pi}, tickmarks=\left[spacing\left(\frac{\pi}{2}\right), default\right]\right);$



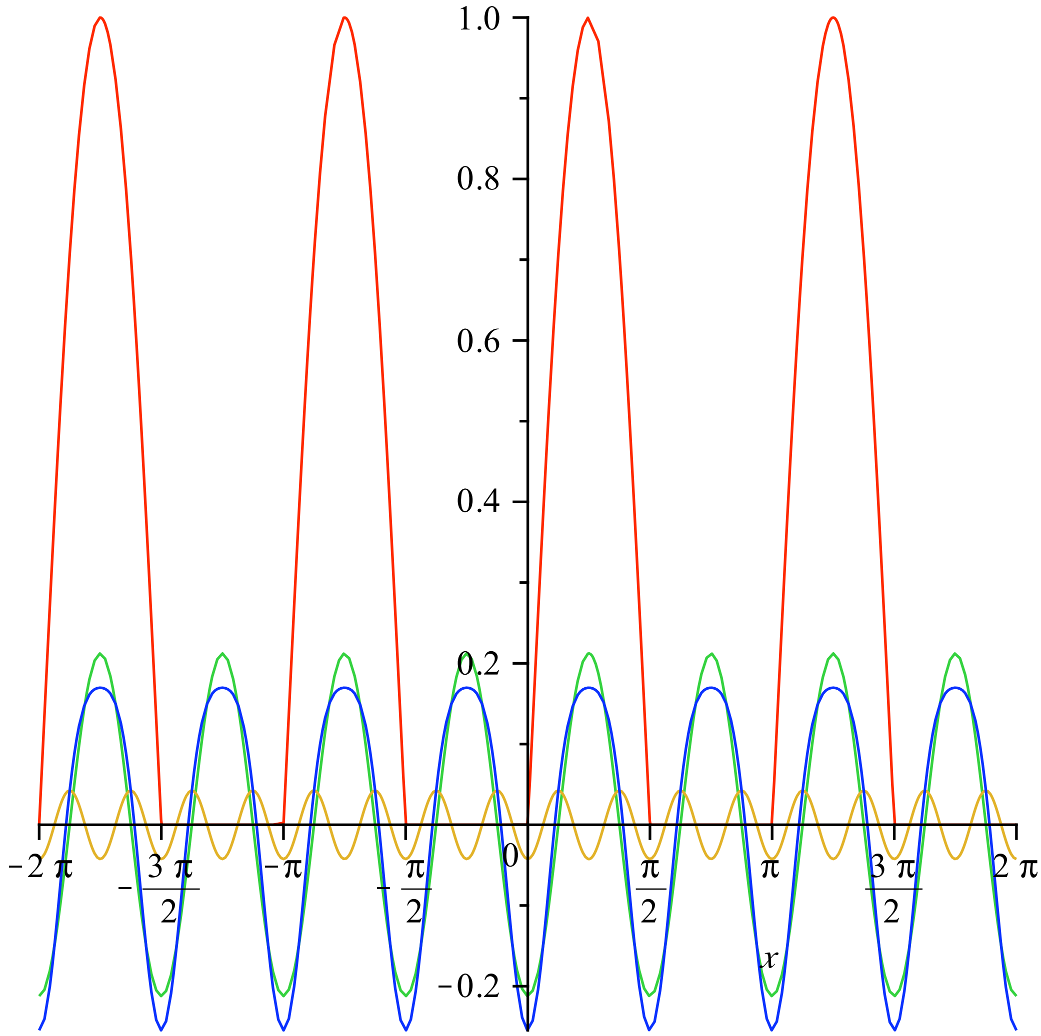
$$plot\left(\left[halfs(x), \frac{1}{2\,\mathrm{Pi}} int(halfs(x), x=-\mathrm{Pi}..\mathrm{Pi}) + \frac{1}{2}\cdot \sin(2\cdot x) + sum\left(\left(\frac{1}{\mathrm{Pi}} int\left(\sin(2\,x)\cdot\cos(m\cdot x), x=-\mathrm{Pi}..-\frac{\mathrm{Pi}}{2}\right) + \frac{1}{\mathrm{Pi}} int\left(\sin(2\,x)\cdot\cos(m\cdot x), x=0..\frac{\mathrm{Pi}}{2}\right)\right)\cdot\cos(m\cdot x), m=4..4\right)\right], x=-2\,\mathrm{Pi}..2\,\mathrm{Pi}, tickmarks=\left[spacing\left(\frac{\pi}{2}\right), default\right]\right);$$



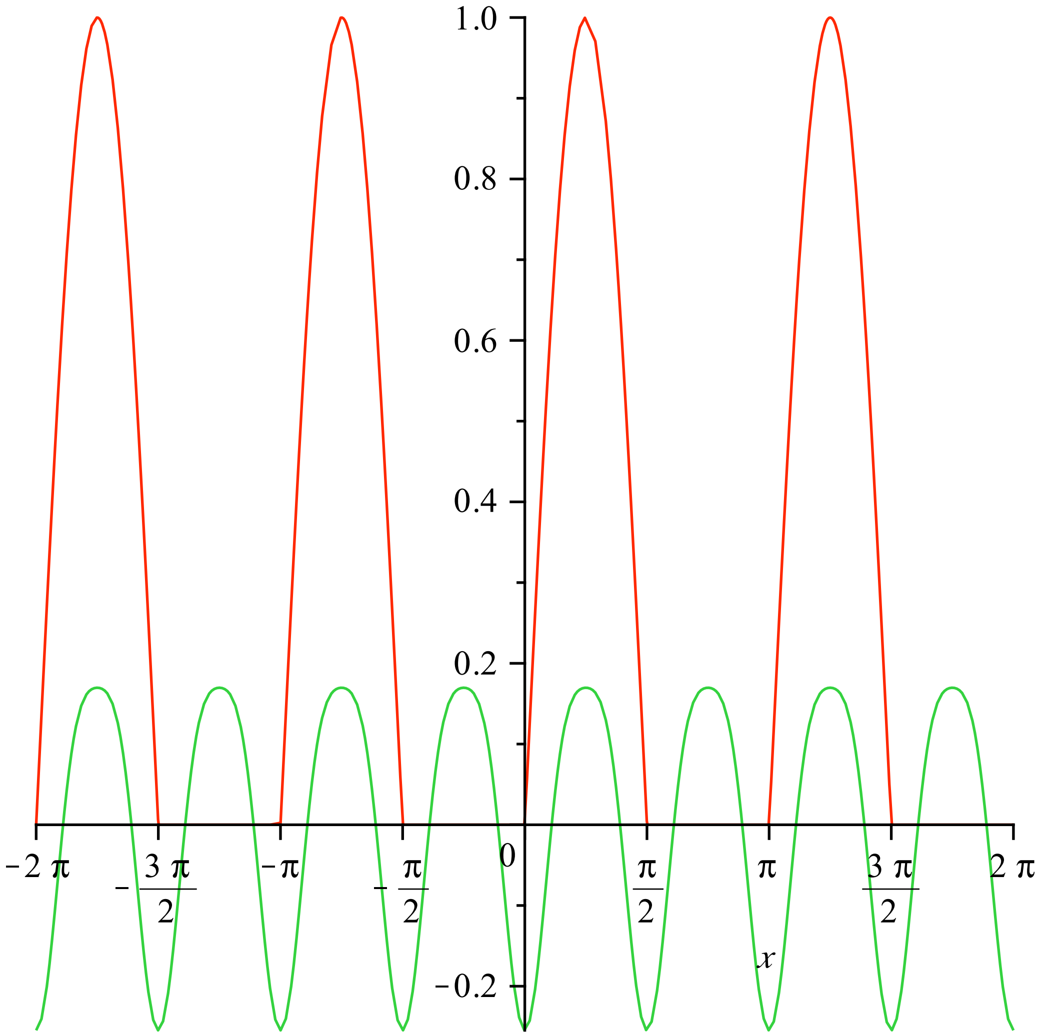
$$plot\left(\left[halfs(x),\ \frac{1}{2\,\mathrm{Pi}}\,int(halfs(x), x=-\mathrm{Pi}\,..\,\mathrm{Pi}) + \frac{1}{2}\cdot\sin(2\cdot x) + sum\left(\left(\frac{1}{\mathrm{Pi}}\,int\left(\sin(2\,x)\cdot\cos(m\cdot x), x=-\mathrm{Pi}\,..\,-\frac{\mathrm{Pi}}{2}\right) + \frac{1}{\mathrm{Pi}}\,int\left(\sin(2\,x)\cdot\cos(m\cdot x), x=0\,..\,\frac{\mathrm{Pi}}{2}\right)\right)\cdot\cos(m\cdot x), m=4\,..20\right)\right], x=-2\,\mathrm{Pi}\,..2\,\mathrm{Pi}, tickmarks=\left[spacing\left(\frac{\pi}{2}\right), default\right]\right);$$



$$plot\left(\left[halfs(x), \left(\frac{1}{\mathrm{Pi}}\, int\left(\sin(2\,x)\cdot\cos(4\cdot x), x=-\mathrm{Pi}\,..-\frac{\mathrm{Pi}}{2}\right) + \frac{1}{\mathrm{Pi}}\, int\left(\sin(2\,x)\cdot\cos(4\cdot x), x=0\,..\frac{\mathrm{Pi}}{2}\right)\right)\cdot\cos(4\cdot x), \left(\frac{1}{\mathrm{Pi}}\, int\left(\sin(2\,x)\cdot\cos(8\cdot x), x=-\mathrm{Pi}\,..-\frac{\mathrm{Pi}}{2}\right) + \frac{1}{\mathrm{Pi}}\, int\left(\sin(2\,x)\cdot\cos(8\cdot x), x=0\,..\frac{\mathrm{Pi}}{2}\right)\right)\cdot\cos(8\cdot x), \left(\frac{1}{\mathrm{Pi}}\, int\left(\sin(2\,x)\cdot\cos(4\cdot x), x=-\mathrm{Pi}\,..-\frac{\mathrm{Pi}}{2}\right) + \frac{1}{\mathrm{Pi}}\, int\left(\sin(2\,x)\cdot\cos(4\cdot x), x=0\,..\frac{\mathrm{Pi}}{2}\right)\right)\cdot\cos(4\cdot x) + \left(\frac{1}{\mathrm{Pi}}\, int\left(\sin(2\,x)\cdot\cos(8\cdot x), x=-\mathrm{Pi}\,..-\frac{\mathrm{Pi}}{2}\right) + \frac{1}{\mathrm{Pi}}\, int\left(\sin(2\,x)\cdot\cos(8\cdot x), x=0\,..\frac{\mathrm{Pi}}{2}\right)\right)\cdot\cos(8\cdot x)\right], x=-2\,\mathrm{Pi}\,..2\,\mathrm{Pi}, tickmarks=\left[spacing\left(\frac{\pi}{2}\right), default\right]\right);$$



$$plot\left(\left[halfs(x),\ \left(\frac{1}{\mathrm{Pi}}\,int\left(\sin(2\,x)\cdot\cos(4\cdot x),\ x=-\mathrm{Pi}\,..\,-\frac{\mathrm{Pi}}{2}\right)+\frac{1}{\mathrm{Pi}}\,int\left(\sin(2\,x)\cdot\cos(4\cdot x),\ x=0\,..\,\frac{\mathrm{Pi}}{2}\right)\right)\cdot\cos(4\cdot x)+\left(\frac{1}{\mathrm{Pi}}\,int\left(\sin(2\,x)\cdot\cos(8\cdot x),\ x=-\mathrm{Pi}\,..\,-\frac{\mathrm{Pi}}{2}\right)+\frac{1}{\mathrm{Pi}}\,int\left(\sin(2\,x)\cdot\cos(8\cdot x),\ x=0\,..\,\frac{\mathrm{Pi}}{2}\right)\right)\cdot\cos(8\cdot x)\right],\ x=-2\,\mathrm{Pi}\,..\,2\,\mathrm{Pi},\ tickmarks=\left[spacing\left(\frac{\pi}{2}\right),\ default\right]\right);$$



1.0
0.8
0.6
0.4
0.2
−0.2
−2 π
−3π/2
−π
−π/2
0
π/2
π
3π/2
2 π
x