MATH 1210-003 Exam 3 Spring 2011

Solutions

1. (10 points) State the ϵ-δ definition of limit and explain in words what it means.

Solution:

$\lim_{x\to c} f(x) = L$ means that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - L| < \epsilon$.

In words, the limit of f as x approaches c equals L means that the outputs of f can be trapped arbitrarily close to L by restricting the input x to be sufficiently close to c.

2. (10 points) State the definition of the derivative of f at $x = a$, and explain in words what it means.

Solution:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

The derivative of f at a is the instantaneous rate of change of f at a. This is equal to the limit of the average rate of change of f on the interval $[a, a+h]$ as this interval becomes arbitrarily small.

3. (10 points) State the definition of the definite integral of f from a to b, and explain in words what it means.

Solution:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and x_i is any point between $a + (i-1)\Delta x$ and $a + i\Delta x$.

The integral is the area under the graph of f between $x = a$ and $x = b$. This area is computed by summing the areas of rectangle of width Δx and height $f(x_i)$ and taking a limit as the rectangles become arbitrarily thin.

4. (10 points) Find

$$\int_1^3 2x + 1\, dx$$

Solution:

$$\int_1^3 2x + 1\, dx = 2\frac{x^2}{2} + x\Big|_1^3 = (3^3 + 3) - (1^1 + 1) = 10$$

5. (10 points) Find

$$\frac{d}{dx}\int_1^{x^2} \log(t)\, dt$$

Solution:

By the Fundamental Theorem of Calculus and the Chain Rule:

$$\frac{d}{dx}\int_1^{x^2} \log(t)\, dt = \log(x^2) * 2x$$

6. (10 points) Find the anti-derivative $F(x)$ of the function $f(x) = x^3$ that satisfies the constraint $F(2) = 2$.

Solution:

All antiderivatives are of the form $F(x)\frac{x^4}{4} + C$ for some constant C.

$$2 = F(2) = \frac{2^4}{4} + C = 4 + C$$

so we must have $C = -2$

Thus, $F(x) = \frac{x^4}{4} - 2$.

7. (10 points) Find the solution to the differential equation

$$\frac{dy}{dx} = \frac{x}{3y^2}$$

such that $y = 2$ when $x = 0$.

Solution:

$$\frac{dy}{dx} = \frac{x}{3y^2}$$
$$3y^2\frac{dy}{dx} = x$$
$$\int 3y^2\frac{dy}{dx}\,dx = \int x\,dx$$
$$3\frac{y^3}{3} = \frac{x^2}{2} + C$$

If $y = 2$ and $x = 0$ then we have

$$3\frac{2^3}{3} = \frac{0^2}{2} + C$$

So $C = 8$.

The solution is

$$y = \left(\frac{x^2}{2} + 8\right)^{\frac{1}{3}}$$

8. (10 points) Find

$$\int_{-1}^{1} x^2(x^3+1)^8\,dx$$

Solution:

Use the substitution $u = (x^3 + 1)$.

$\frac{du}{dx} = 3x^2$, so

$$\int_{-1}^{1} x^2(x^3+1)^8\,dx = \frac{1}{3}\int_0^2 u^8\,du = \frac{1}{3}\frac{u^9}{9}\Big|_0^2 = \frac{2^9}{27}$$

9. (10 points) A ball is thrown straight up in the air from a height of 6ft at an initial velocity of 64ft/s. Assume there is no air resistance and the ball feels a constant acceleration of -32ft/s^2 due to gravity. What is the maximum height that the ball achieves?

Solution:

$$v(t) = \int a\,dt = \int -32\,dt = -32t + 64$$

$$h(t) = \int v(t)\,dt = \int -32t + 64\,dt = -16t^2 + 64t + 6$$

The maximum height occurs at a stationary point for $h(t)$. This is a point where $h'(t) = v(t) = 0$. The only stationary point is at $t = 2$, so the maximum height is

$$h(2) = -16 * 2^2 + 64 * 2 + 6 = 70\text{ft}$$

10. (20 points) The glycemic index of a food (the test food) compares blood glucose levels of a person after eating the test food and after eating white bread (the control food). The glycemic index is defined to be the ratio:

$$\frac{\text{Area under glucose response curve for test food} - \text{Area under baseline glucose curve}}{\text{Area under glucose response curve for control food} - \text{Area under baseline glucose curve}}$$

The glucose response functions for the test food and the control food are given by the following table:

t (min)	0	30	60	90	120
blood glucose (mmol/l) t minutes after eating test food	4.0	8.0	5.9	5.1	4.0
blood glucose (mmol/l) t minutes after eating control food	4.0	6.7	5.3	5.0	4.2

The baseline glucose curve is the horizontal line at height 4.0.

Use the trapezoid rule to estimate the glycemic index of this test food.

Solution:

Left Riemann Sum for test food $= (4 + 8 + 5.9 + 5.1)30 = 23 * 30$.

Right Riemann Sum for test food $= (8 + 5.9 + 5.1 + 4)30 = 23 * 30$.

The trapezoid rule is the average of these, which is still $23 * 30$. We estimate that the area under the glucose response curve for the test food is $23 * 30$.

Similar computation estimate that the area under the glucose response curve for the control food is approximately $21.1 * 30$.

Te baseline glucose curve is the horizontal line at height 4. The area under this curve is $4 * 120 = 16 * 30$.

Using the given formula, the glycemic index of the test food is

$$\frac{23 * 30 - 16 * 30}{21.1 * 30 - 16 * 30} = \frac{23 - 16}{21.1 - 16} = \frac{7}{5.1} \approx 1.37$$