

## Solutions

1. (20 points) Find the following limits or state that they do not exist:

$$\lim_{x \rightarrow 2} 4$$

$$\lim_{x \rightarrow 1} \frac{(x-3)(x-1)}{x-1}$$

$$\lim_{x \rightarrow 2} 2x^2$$

$$\lim_{x \rightarrow 0} \frac{1}{x}$$

$$\lim_{x \rightarrow \infty} 4x^4 + 3x^2 - 100$$

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin x}{\cos x}$$

$$\lim_{x \rightarrow -\infty} \frac{1}{x}$$

**Solution:**

$$\lim_{x \rightarrow 2} 4 = 4$$

$$\lim_{x \rightarrow 1} \frac{(x-3)(x-1)}{x-1} = \lim_{x \rightarrow 1} (x-3) = (1-3) = -2$$

$$\lim_{x \rightarrow 2} 2x^2 = 2 * 2^2 = 8$$

$$\lim_{x \rightarrow 0} \frac{1}{x} = \text{does not exist, because } \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty \text{ but } \lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$$

$$\lim_{x \rightarrow \infty} 4x^4 + 3x^2 - 100 = \infty$$

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin x}{\cos x} = \frac{\sin \frac{\pi}{4}}{\cos \frac{\pi}{4}} = 1$$

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

2. (10 points) Find all asymptotes of the function

$$\frac{(2x-1)(x+3)(x-1)}{(x+1)(x-1)}$$

**Solution:**

There is one vertical asymptote at  $x = -1$  because the denominator becomes zero and the numerator does not. The denominator is also zero at  $x = 1$ , but the numerator also has a factor of  $(x-1)$ , so there is a removable discontinuity at  $x = 1$ , not a vertical asymptote.

Degree of numerator = 3 > 2 = degree of denominator, so there are no horizontal asymptotes, but there is an oblique asymptote. To find the oblique asymptote perform polynomial long division to find that

$$\frac{(2x-1)(x+3)(x-1)}{(x+1)(x-1)} = \frac{(2x-1)(x+3)}{(x+1)} = \frac{2x^2 + 5x - 3}{x+1} = 2x + 3 + \frac{-6}{x+1}$$

The oblique asymptote is the line  $y = 2x + 3$ .

3. (20 points)

$$f(x) = \begin{cases} 2x + 1 & \text{for } x < 3 \\ 5 & \text{for } x = 3 \\ x^2 - 1 & \text{for } 3 < x \end{cases}$$

Find the following, or say why they do not exist:

$$\lim_{x \rightarrow 3^+} f(x)$$

$$\lim_{x \rightarrow 3^-} f(x)$$

$$\lim_{x \rightarrow 3} f(x)$$

$$f(3)$$

$$\lim_{x \rightarrow 4^+} f(x)$$

$$\lim_{x \rightarrow 4^-} f(x)$$

$$\lim_{x \rightarrow 4} f(x)$$

$$f(4)$$

Where is  $f(x)$  continuous?

**Solution:**

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3} x^2 - 1 = 3^2 - 1 = 8$$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3} 2x + 1 = 2 * 3 + 1 = 7$$

$\lim_{x \rightarrow 3} f(x)$  = does not exist because left and right limits do not agree

$$f(3) = 5$$

$$\lim_{x \rightarrow 4^+} f(x) = \lim_{x \rightarrow 4} x^2 + 1 = 4^2 + 1 = 15$$

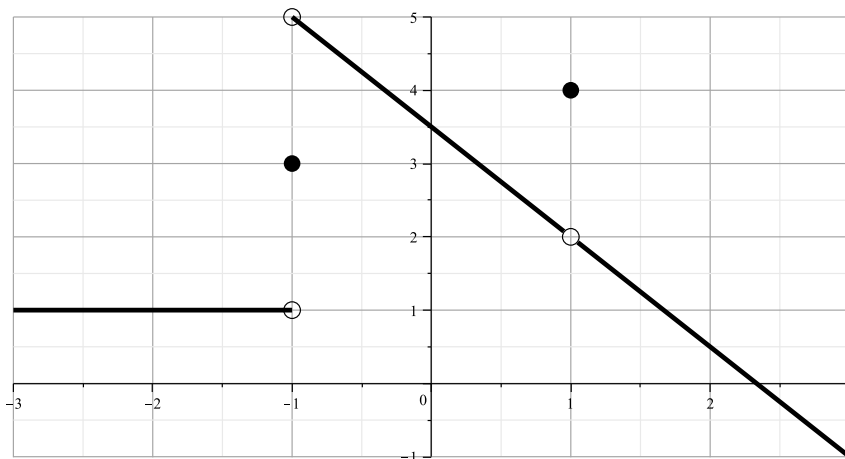
$$\lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4} x^2 + 1 = 4^2 + 1 = 15$$

$$\lim_{x \rightarrow 4} f(x) = \lim_{x \rightarrow 4} x^2 + 1 = 4^2 + 1 = 15$$

$$f(4) = 15$$

Where is  $f(x)$  continuous?

Everywhere except  $x = 3$ .



4. (20 points)

The graph of the function  $g(x)$  is shown above. Find the following, or say why they do not

exist:

$$\lim_{x \rightarrow -1^+} g(x)$$

$$\lim_{x \rightarrow 1^+} g(x)$$

$$\lim_{x \rightarrow -1^-} g(x)$$

$$\lim_{x \rightarrow 1^-} g(x)$$

$$\lim_{x \rightarrow -1} g(x)$$

$$\lim_{x \rightarrow 1} g(x)$$

$$g(-1)$$

$$g(1)$$

$$\lim_{x \rightarrow 0^+} g(x)$$

$$\lim_{x \rightarrow 2^+} g(x)$$

$$\lim_{x \rightarrow 0^-} g(x)$$

$$\lim_{x \rightarrow 2^-} g(x)$$

$$\lim_{x \rightarrow 0} g(x)$$

$$\lim_{x \rightarrow 2} g(x)$$

$$g(0)$$

$$g(2)$$

Where is  $g(x)$  continuous?

**Solution:**

$$\lim_{x \rightarrow -1^+} g(x) = 5$$

$$\lim_{x \rightarrow 1^+} g(x) = 2$$

$$\lim_{x \rightarrow -1^-} g(x) = 1$$

$$\lim_{x \rightarrow 1^-} g(x) = 2$$

$\lim_{x \rightarrow -1} g(x)$  = does not exist because left limit  $\neq$  right limit

$$\lim_{x \rightarrow 1} g(x) = 2$$

$$g(-1) = 3$$

$$g(1) = 4$$

$$\lim_{x \rightarrow 0^+} g(x) = 3.5$$

$$\lim_{x \rightarrow 2^+} g(x) = 0.5$$

$$\lim_{x \rightarrow 0^-} g(x) = 3.5$$

$$\lim_{x \rightarrow 2^-} g(x) = 0.5$$

$$\lim_{x \rightarrow 0} g(x) = 3.5$$

$$\lim_{x \rightarrow 2} g(x) = 0.5$$

$$g(0) = 3.5$$

$$g(2) = 0.5$$

Where is  $g(x)$  continuous?

Everywhere except  $\pm 1$

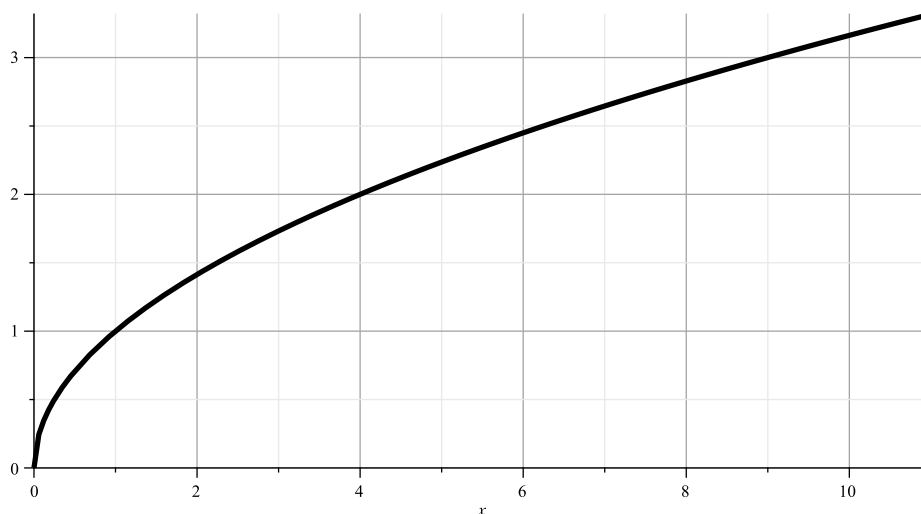
**5. (10 points)** Describe all discontinuities of the function

$$\frac{(x+1)(x+2)(x+3)}{x(x+3)(x-3)}$$

**Solution:**

This is a rational function, so the only discontinuities are at the roots of the denominator,  $x = -3, 0, 3$ . There is a single factor of  $(x+3)$  in both the numerator and the denominator, so  $x = -3$  is a removable discontinuity. There are vertical asymptotes at  $x = 0$  and  $x = 3$ , because these are roots of the denominator that are not also roots of the numerator.

6. (10 points) The graph of the function  $f(x)$  is shown below.



$$\lim_{x \rightarrow 4} f(x) = 2$$

Find the largest  $\delta$  such that for all inputs within  $\delta$  of 4 the output of  $f$  is within 1 of 2.

**Solution:**

For the output to be within 1 of 2 the output must be between 1 and 3.  $f(1) = 1$  and  $f(9) = 3$ , so we will have outputs between 1 and 3 if the inputs are between 1 and 9. The largest  $\delta$  such that all points within distance  $\delta$  of 4 are between 1 and 9 is  $\delta = 3 = 4 - 1$ .

Note that  $\delta = 5 = 9 - 4$  will not work. The input 0 is within distance 5 of 4, but  $f(0) = 0$ , so the output is not between 1 and 3.

7. (10 points) State the  $\epsilon$ - $\delta$  definition of limit and explain in words what it means.

**Solution:**

$\lim_{x \rightarrow c} f(x) = L$  means that for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that if  $0 < |x - c| < \delta$  then  $|f(x) - L| < \epsilon$ .

In words, the limit of  $f$  as  $x$  approaches  $c$  equals  $L$  means that the outputs of  $f$  can be trapped arbitrarily close to  $L$  by restricting the input  $x$  to be sufficiently close to  $c$ .

8. (10 points) Suppose  $B$  is some number and  $f$  is a function defined on all real numbers such that for all  $x$  we have  $f(x) < B$ . Suppose that  $\lim_{x \rightarrow \infty} f(x) = L$ . Is it possible that  $L > B$ ? Is it possible that  $L = B$ ? Explain.

**Solution:**

It is not possible that  $L > B$ . The outputs of  $f$  must approach arbitrarily close to  $L$ , but this is impossible if the outputs of  $f$  are always less than  $B$ .

It is possible that  $L = B$ . For example, consider the function

$$f(x) = \begin{cases} -1 & \text{for } x < 1 \\ -\frac{1}{x} & \text{for } x \geq 1 \end{cases}$$

Let  $B = 0$ . For every  $x$  we have  $f(x) < 0 = B$

However,

$$L = \lim_{x \rightarrow \infty} f(x) = 0 = B$$