1. (20 points) Find the following limits or state that they do not exist:

\[
\lim_{x \to 2} 4
\]

\[
\lim_{x \to 1} \frac{(x - 3)(x - 1)}{x - 1}
\]

\[
\lim_{x \to 2} 2x^2
\]

\[
\lim_{x \to 0} \frac{1}{x}
\]

\[
\lim_{x \to \infty} 4x^4 + 3x^2 - 100
\]

\[
\lim_{x \to \pi} \frac{\sin x}{\cos x}
\]

\[
\lim_{x \to -\infty} \frac{1}{x}
\]

Solution:
\[
\lim_{x \to 2} 4 = 4
\]
\[
\lim_{x \to 1} \frac{(x-3)(x-1)}{x-1} = \lim_{x \to 1} (x-3) = (1-3) = -2
\]
\[
\lim_{x \to 2} 2x^2 = 2 \times 2^2 = 8
\]
\[
\lim_{x \to 0} \frac{1}{x} = \text{does not exist, because } \lim_{x \to 0^-} \frac{1}{x} = -\infty \text{ but } \lim_{x \to 0^+} \frac{1}{x} = \infty
\]
\[
\lim_{x \to \infty} 4x^4 + 3x^2 - 100 = \infty
\]
\[
\lim_{x \to \frac{\pi}{4}} \frac{\sin x}{\cos x} = \frac{\sin \frac{\pi}{4}}{\cos \frac{\pi}{4}} = 1
\]
\[
\lim_{x \to -\infty} \frac{1}{x} = 0
\]

2. (10 points) Find all asymptotes of the function
\[
\frac{(2x-1)(x+3)(x-1)}{(x+1)(x-1)}
\]

Solution:

The is one vertical asymptote at \(x = -1\) because the denominator becomes zero and the numerator does not. The denominator is also zero at \(x = 1\), but the numerator also has a factor of \((x - 1)\), so there is a removable discontinuity at \(x = 1\), not a vertical asymptote.

Degree of numerator = 3 > 2 = degree of denominator, so there are no horizontal asymptotes, but there is an oblique asymptote. To find the obliques asymptote perform polynomial long division to find that
\[
\frac{(2x-1)(x+3)(x-1)}{(x+1)(x-1)} = \frac{(2x-1)(x+3)}{(x+1)} = \frac{2x^2 + 5x - 3}{x+1} = 2x + 3 + \frac{-6}{x+1}
\]

The oblique asymptote is the line \(y = 2x + 3\).
3. (20 points)

\[ f(x) = \begin{cases} 
2x + 1 & \text{for } x < 3 \\
5 & \text{for } x = 3 \\
x^2 - 1 & \text{for } 3 < x 
\end{cases} \]

Find the following, or say why they do not exist:

\[ \lim_{x \to 3^+} f(x) \]

\[ \lim_{x \to 3^-} f(x) \]

\[ \lim_{x \to 3} f(x) \]

\[ f(3) \]

\[ \lim_{x \to 4^+} f(x) \]

\[ \lim_{x \to 4^-} f(x) \]

\[ \lim_{x \to 4} f(x) \]

\[ f(4) \]

Where is \( f(x) \) continuous?

Solution:
\[
\lim_{x \to 3^+} f(x) = \lim_{x \to 3} x^2 - 1 = 3^2 - 1 = 8
\]

\[
\lim_{x \to 3^-} f(x) = \lim_{x \to 3} 2x + 1 = 2 \cdot 3 + 1 = 7
\]

\[
\lim_{x \to 3} f(x) = \text{does not exist because left and right limits do not agree}
\]

\[
f(3) = 5
\]

\[
\lim_{x \to 4^+} f(x) = \lim_{x \to 4} x^2 + 1 = 4^2 + 1 = 15
\]

\[
\lim_{x \to 4^-} f(x) = \lim_{x \to 4} x^2 + 1 = 4^2 + 1 = 15
\]

\[
\lim_{x \to 4} f(x) = \lim_{x \to 4} x^2 + 1 = 4^2 + 1 = 15
\]

\[
f(4) = 15
\]

Where is \( f(x) \) continuous?

Everywhere except \( x = 3 \).

4. (20 points)

The graph of the function \( g(x) \) is shown above. Find the following, or say why they do not
exist:

$$\lim_{x \to -1^+} g(x)$$  \quad $$\lim_{x \to 1^+} g(x)$$

$$\lim_{x \to -1^-} g(x)$$  \quad $$\lim_{x \to 1^-} g(x)$$

$$\lim_{x \to -1} g(x)$$  \quad $$\lim_{x \to 1} g(x)$$

$$g(-1)$$  \quad $$g(1)$$

$$\lim_{x \to 0^+} g(x)$$  \quad $$\lim_{x \to 2^+} g(x)$$

$$\lim_{x \to 0^-} g(x)$$  \quad $$\lim_{x \to 2^-} g(x)$$

$$\lim_{x \to 0} g(x)$$  \quad $$\lim_{x \to 2} g(x)$$

$$g(0)$$  \quad $$g(2)$$

Where is $g(x)$ continuous?

Solution:
$$\lim_{x \to -1^+} g(x) = 5 \quad \lim_{x \to 1^+} g(x) = 2$$

$$\lim_{x \to -1^-} g(x) = 1 \quad \lim_{x \to 1^-} g(x) = 2$$

$$\lim_{x \to -1} g(x) = \text{does not exist because left limit } \neq \text{ right limit} \quad \lim_{x \to 1} g(x) = 2$$

$$g(-1) = 3 \quad g(1) = 4$$

$$\lim_{x \to 0^+} g(x) = 3.5 \quad \lim_{x \to 2^+} g(x) = 0.5$$

$$\lim_{x \to 0^-} g(x) = 3.5 \quad \lim_{x \to 2^-} g(x) = 0.5$$

$$\lim_{x \to 0} g(x) = 3.5 \quad \lim_{x \to 2} g(x) = 0.5$$

$$g(0) = 3.5 \quad g(2) = 0.5$$

Where is \( g(x) \) continuous?

Everywhere except \( \pm 1 \)

5. (10 points) Describe all discontinuities of the function

$$\frac{(x + 1)(x + 2)(x + 3)}{x(x + 3)(x - 3)}$$

Solution:

This is a rational function, so the only discontinuities are at the roots of the denominator, \( x = -3, 0, 3 \). There is a single factor of \( x + 3 \) in both the numerator and the denominator, so \( x = -3 \) is a removable discontinuity. There are vertical asymptotes at \( x = 0 \) and \( x = 3 \), because these are roots of the denominator that are not also roots of the numerator.
6. **(10 points)** The graph of the function $f(x)$ is shown below.

![Graph of $f(x)$](image)

$$\lim_{x \to 4} f(x) = 2$$

Find the largest $\delta$ such that for all inputs within $\delta$ of 4 the output of $f$ is within 1 of 2.

**Solution:**

For the output to be within 1 of 2 the output must be between 1 and 3. $f(1) = 1$ and $f(9) = 3$, so we will have outputs between 1 and 3 if the inputs are between 1 and 9. The largest $\delta$ such that all points within distance $\delta$ of 4 are between 1 and 9 is $\delta = 3 = 4 - 1$.

Note that $\delta = 5 = 9 - 4$ will not work. The input 0 is within distance 5 of 4, but $f(0) = 0$, so the output is not between 1 and 3.

7. **(10 points)** State the $\epsilon$-$\delta$ definition of limit and explain in words what it means.

**Solution:**

$$\lim_{x \to c} f(x) = L$$

means that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - L| < \epsilon$.

In words, the limit of $f$ as $x$ approaches $c$ equals $L$ means that the outputs of $f$ can be trapped arbitrarily close to $L$ by restricting the input $x$ to be sufficiently close to $c$.

8. **(10 points)** Suppose $B$ is some number and $f$ is a function defined on all real numbers such that for all $x$ we have $f(x) < B$. Suppose that $\lim_{x \to \infty} f(x) = L$. Is it possible that $L > B$? Is it possible that $L = B$? Explain.
Solution:

It is not possible that \( L > B \). The outputs of \( f \) must approach arbitrarily close to \( L \), but this is impossible if the outputs of \( f \) are always less than \( B \).

It is possible that \( L = B \). For example, consider the function

\[
f(x) = \begin{cases} 
-1 & \text{for } x < 1 \\
-\frac{1}{x} & \text{for } x \geq 1
\end{cases}
\]

Let \( B = 0 \). For every \( x \) we have \( f(x) < 0 = B \)

However,

\[
L = \lim_{x \to \infty} f(x) = 0 = B
\]