MATH 2270-003
Exam 1
Fall 2010

Solutions

1. (5 points) Is \(
\begin{pmatrix}
1 \\
2
\end{pmatrix}
\) a unit vector? Explain.

Solution:

No. A unit vector has length 1, but
\[
\left\| \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\| = \sqrt{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}} = \sqrt{5} \neq 1
\]

2. (5 points) The matrix \( A \) is given below. What is the elimination matrix that eliminates the entries below the second pivot.

\[
A = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 0 & 2 & 0 \\
0 & 0 & 4 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix}
\]

Solution:

The second pivot is the 2 in row 2 column 3. Use this to eliminate the entries in the 3,3 and 4,3 positions using the elimination matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -2 & 1 & 0 \\
0 & -\frac{1}{2} & 0 & 1
\end{pmatrix}
\]

3. (5 points) \( E \) and \( F \) are given below. What is the inverse of \( EF \)?

\[
E = \begin{pmatrix}
1 & 0 & 0 \\
-5 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix}
\]

\[
F = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -2 & 1
\end{pmatrix}
\]
Solution:

\((EF)^{-1} = F^{-1}E^{-1}\). Now \(E\) and \(F\) are elimination matrices, so we know how to invert them, just switch the signs on the off-diagonal entries:

\[
E^{-1} = \begin{pmatrix}
1 & 0 & 0 \\
+5 & 1 & 0 \\
-1 & 0 & 1
\end{pmatrix}
\]

\[
F^{-1} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & +2 & 1
\end{pmatrix}
\]

4. (5 points) Describe in words the effect of multiplying \(EA\) where \(E\) is the matrix in the previous problem and \(A\) is a \(3 \times 5\) matrix.

Solution:

\(E\) is an elimination matrix. It replaces Row 2 with Row 2 – 5 Row 1 and replaces Row 3 with Row 3 + Row 1.

5. (15 points) Let \(A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\). Let \(B = \begin{pmatrix} 6 & 3 \\ 2 & 1 \end{pmatrix}\). Let \(C = \begin{pmatrix} 1 & 7 \\ 1 & 2 \end{pmatrix}\). Calculate the following:

\(AB\)

\(BA\)

\(C^2\)

\(A^T \begin{pmatrix} 1 \\ 2 \end{pmatrix}\)

\(A(2B - C)\)

Solution:

\[AB = \begin{pmatrix} 10 & 5 \\ 24 & 13 \end{pmatrix}\]

\[BA = \begin{pmatrix} 15 & 24 \\ 5 & 8 \end{pmatrix}\]
6. (20 points) Prove or disprove that the following sets of vectors form a basis for the indicated vector space:

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \text{ in } \mathbb{R}^2
\]

\[
\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \text{ in } \mathbb{R}^3
\]

Solution:

\[
\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \text{ in } \mathbb{R}^2
\]

These two vectors form a basis for \( \mathbb{R}^2 \). These two vectors are not scalar multiples of each other, so they are linearly independent. \( \mathbb{R}^2 \) is two dimensional, so any two linearly independent vectors form a basis.

\[
\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \text{ in } \mathbb{R}^3
\]

These two vectors do not form a basis. The dimension of \( \mathbb{R}^3 \) is three, so any basis must contain exactly three vectors.

7. (25 points) Find all solutions to the equation \( A\bar{x} = \bar{b} \) for

\[
A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 6 \\ 2 & 4 & 8 & 12 \end{pmatrix}
\]
\( \vec{b} = \begin{pmatrix} 4 \\ 5 \\ 10 \end{pmatrix} \)

**Solution:**

The augmented matrix for this system of equations is:

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 4 \\
1 & 2 & 4 & 6 & 5 \\
2 & 4 & 8 & 12 & 10 \\
\end{pmatrix}
\]

Put this is reduced row echelon form by performing Gauss-Jordan Elimination.

Eliminate entries below first pivot:

\[
\begin{pmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
-2 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 3 & 4 & 4 \\
1 & 2 & 4 & 6 & 5 \\
2 & 4 & 8 & 12 & 10 \\
\end{pmatrix} =
\begin{pmatrix}
1 & 2 & 3 & 4 & 4 \\
0 & 0 & 1 & 2 & 1 \\
0 & 0 & 2 & 4 & 2 \\
\end{pmatrix}
\]

Eliminate entries below second pivot:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -2 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 3 & 4 & 4 \\
0 & 0 & 1 & 2 & 1 \\
0 & 0 & 2 & 4 & 2 \\
\end{pmatrix} =
\begin{pmatrix}
1 & 2 & 3 & 4 & 4 \\
0 & 0 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Eliminate entries above second pivot:

\[
\begin{pmatrix}
1 & -3 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 3 & 4 & 4 \\
0 & 0 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} =
\begin{pmatrix}
1 & 2 & 0 & -2 & 1 \\
0 & 0 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Solutions to the original equation are the same as solutions to the new equation : 

\[
\begin{pmatrix}
1 & -2 \\
0 & 1 \\
0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
\vec{x} \\
\vec{1} \\
\vec{0} \\
\end{pmatrix}
\]

Columns 1 and 3 are pivot columns; columns 2 and 4 are free columns, so \(x_2\) and \(x_4\) are the free variables.
A particular solution is obtained by setting \( x_2 = x_4 = 0 \) and solving for \( x_1 \) and \( x_3 \).

\[
\begin{pmatrix}
1 & 2 & 0 & -2 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
= \begin{pmatrix}
1 \\
1 \\
0
\end{pmatrix}
\]

The particular solution is

\[
\bar{x}_p = \begin{pmatrix}
1 \\
0 \\
1 \\
0
\end{pmatrix}
\]

The other solutions differ from \( \bar{x}_p \) by vectors in the nullspace of \( A \). A basis for the nullspace is given by the “special solutions” obtained by setting one free variable equal to 1, the others equal to 0, and solving for the non-free variables.

\( \bar{s}_2 \) is the special solution obtained by setting \( x_2 = 1 \) and \( x_4 = 0 \) and solving

\[
\begin{pmatrix}
1 & 2 & 0 & -2 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

So \( \bar{s}_3 = \begin{pmatrix}
-2 \\
1 \\
0 \\
0
\end{pmatrix} \)

Similarly, \( \bar{s}_4 \) is obtained by setting \( x_2 = 0 \) and \( x_4 = 1 \) and solving

\[
\begin{pmatrix}
1 & 2 & 0 & -2 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

So \( \bar{s}_4 = \begin{pmatrix}
2 \\
0 \\
-2 \\
1
\end{pmatrix} \)
Finally, all solutions are of the form \( x_p + x_3 \bar{s}_3 + x_4 \bar{s}_4 \), so all solutions to the equation are all vectors:

\[
\bar{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}
\]

where \( x_3 \) and \( x_4 \) are arbitrary real numbers.

8. (15 points) Find all solutions to the equation \( A \bar{x} = \bar{b} \) for

\[
A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{pmatrix}
\]

\[
\bar{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}
\]

Solution:

All columns of \( A \) are multiples of column 1, so the column space is the span of column 1. However, \( \bar{b} \) is not a scalar multiple of column 1, so \( \bar{b} \) is not in the column space of \( A \). Thus, there are no solutions.

9. (15 points) Consider a system of 3 linear equations in five variables. How many solutions can there be? Explain the possibilities.

Solution:

The coefficient matrix would be \( 3 \times 5 \), so it could have at most 3 pivots, hence at least two free variables.

It is possible there are no solutions, but if there are solutions then there are infinitely many. They would be given by one particular solution plus any vector in the null space, which is at least two dimensional.

No new questions beyond this point.