Introduction to Topology

0.1. **Topological Spaces.**

Definition 0.1 (topological space)**.** A *topological space* is a pair $(X, \mathcal{T})$ consisting of a set X, the space, and a set $\mathcal{T}$ of subsets of X, the topology, satisfying the following conditions:

(1) $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$
(2) If $T_1 \in \mathcal{T}$ and $T_2 \in \mathcal{T}$ then $T_1 \cap T_2 \in \mathcal{T}$.
(3) For any index set I and any collection $\{T_i\}_{i \in I}$, with each $T_i \in \mathcal{T}$,

$$\bigcup_{i \in I} T_i \in \mathcal{T}$$

In words, $\mathcal{T}$ is a collection of subsets of X that include $\emptyset$ and X, and is closed under intersection and arbitrary unions.

The topology defines the open subsets of X. A subset A of X is *open* in $(X, \mathcal{T})$ if and only if $A \in \mathcal{T}$.

Proposition 0.2. *Any finite intersection of open sets is open.*

Proof. Let $T_1, \ldots, T_n$ be a finite collection of open sets. $T_1 \cap T_2$ is open since, by the definition of topological space, an intersection of two open sets is open. Now suppose for some i we have shown that $U = T_1 \cap T_2 \cap \cdots \cap T_i$ is open. $T_1 \cap \cdots \cap T_i \cap T_{i+1} = U \cap T_2$, so we can realize $T_1 \cap \cdots \cap T_i \cap T_{i+1}$ as an intersection of two open sets, which is an open set. By induction, any finite intersection of open sets is open. □

In a space X, the *complement*, A^c, of a subset A is the set

$$A^c = X \setminus A = \{x \in X \mid x \notin A\}$$

A subset A is closed if and only if A^c is open.

A subset A is a *neighborhood* of a point x if and only if there exists an open set U such that $x \in U \subset A$. Note that using this terminology a neighborhood of a point need not necessarily be an open set.

You could also say that A is a neighborhood of a set B if A is a neighborhood of every point of B.

Example 0.3. Consider the real numbers. If we declare that the open sets are $\emptyset$, $\mathbb{R}$, and the open intervals, does this define a topology on $\mathbb{R}$? The answer is no, because condition 3 in the definition of topology is not satisfied, the set of open intervals is not closed under union. For example, $(0, 1) \cup (2, 3)$ is not an open interval.

Example 0.4. Let's correct the last example by declaring the open sets in $\mathbb{R}$ to be $\emptyset$ and any union of open intervals. We check the conditions to see that this does give us a topology. For the time being we will call this particular topology the open-interval topology, and denote it $\mathcal{OI}$.

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$\emptyset \in \mathcal{T}$ by definition. $\mathbb{R} \in \mathcal{T}$ because $\mathbb{R}$ can be realized as a union of open intervals, $\mathbb{R} = \cup_{i=1}^{\infty}(-i, i)$.

$\mathcal{T}$ is closed under arbitrary unions by definition, since a union of unions of open intervals is a union of open intervals.

The last condition to check is that the intersection of two open sets is open. Let U and V be open sets. U and V are unions of open intervals:

$$U = \cup_{i \in I}(a_i - b_i, a_i + b_i)$$
$$V = \cup_{j \in J}(c_i - d_i, c_i + d_i)$$

Let x be any point in $U \cap V$. If x is in U then there is some $i \in I$ such that $x \in (a_i - b_i, a_i + b_i)$. Similarly, if x is in V there is some $j \in J$ such that $x \in (c_j - d_j, c_j + d_j)$. Let

$$e = \min\{|a_i - b_i - x|, |a_i + b_i - x|, |c_j - d_j - x|, |c_j + d_j - x|\}$$

Then

$$W_x = (x - e, x + e) \subset (a_i - b_i, a_i + b_i) \cap (c_j - d_j, c_j + d_j) \subset U \cap V$$

For any point $x \in U \cap V$ we have found an open interval W_x such that $x \in W_x \subset U \cap V$, so $U \cap V$ is a neighborhood of all of its points. A set which is a neighborhood of all of its points is open, so $U \cap V$ is open.

0.1.1. *Convergence and Continuity.* We will be interested in convergence of sequences in $\mathbb{R}^n$ and continuity of functions $f\colon R^n \to R^m$. We take the time to talk about topology because this is the minimal amount of structure that we need to define continuity and convergence.

Definition 0.5 (convergence)**.** A sequence (x_n) in X *converges* to a point $x \in X$ if for every neighborhood H of x there is some $N > 0$ such that for all $n > N$, $x_n \in H$.

Compare this to the calculus definition:

Definition 0.6 (Calculus convergence)**.** A sequence (x_n) in $\mathbb{R}$ *converges* to a point $x \in \mathbb{R}$ if for every $\epsilon > 0$ there is some $N > 0$ such that for all $n > N$, $|x_n - x| < \epsilon$.

The only change is to replace a neighborhood H of x with the set $\{y \in \mathbb{R} \mid |y - z| < \epsilon\}$.

If we have a map $f\colon X \to Y$ and subsets $A \subset X$, $B \subset Y$, the *image* of A is $f(A) = \{y \in Y \mid \exists a \in A,\ f(a) = y\}$, and the *preimage* of B is $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$.

Definition 0.7 (continuity at a point)**.** A map $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ between topological spaces is *continuous at a point* $a \in X$ if the preimage of every neighborhood of $f(a)$ in Y is a neighborhood of a in X.

Remark. If you insist that neighborhoods are open then the definition of continuity at a point becomes more cumbersome. You would need to say that the preimage of any open neighborhood of $f(a)$ contains an open neighborhood of a.

Definition 0.8 (Calculus continuity at a point)**.** A map $f\colon \mathbb{R} \to \mathbb{R}$ is *continuous at a point* $a \in \mathbb{R}$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

You could restate this definition as:

Definition 0.9 (Calculus continuity at a point II)**.** A map $f\colon \mathbb{R} \to \mathbb{R}$ is *continuous at a point* $a \in \mathbb{R}$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\{x \in \mathbb{R} \mid |x - a| < \delta\} \subset f^{-1}\left(\{z \in \mathbb{R} \mid |z - f(a)| < \epsilon\}\right)$$

Again, the difference between the topological and Calculus definitions is to replace neighborhoods of points with sets of the form $\{x \in \mathbb{R} \mid |x - a| < \epsilon\}$.

Here is a definition that is different:

Definition 0.10 (continuity)**.** A map $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ between topological spaces is *continuous* if the preimage of every open set in Y is open in X.

In Calculus a function is defined to be continuous if it is continuous at every point. With the topological statements that is also true, but needs to be proved.

Proposition 0.11. *A map $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ between topological spaces is* continuous *if and only if it is continuous at every point in X.*

Proof. Suppose f is continuous. Let a be any point of X, and let H be any neighborhood of $f(a)$. There is an open set U such that $f(a) \in U \subset H$. $f^{-1}(f(a)) \in f^{-1}(U) \subset f^{-1}(H)$. Since f is continuous, $f^{-1}(U)$ is an open set, so $f^{-1}(H)$ is a neighborhood of $f^{-1}(f(a))$. In particular, $a \in f^{-1}(f(a))$, so $f^{-1}(H)$ is a neighborhood of a, so f is continuous at a.

Now suppose that f is continuous at every point in X. Let U be an open set in Y. Let u be a point in U. U is a neighborhood of u.

If u is in the image of f then for any $x \in f^{-1}(u)$, f is continuous at x, so $f^{-1}(U)$ is a neighborhood of x. Thus, there is some open set $V_{u,x}$ such that $x \in V_{u,x} \subset f^{-1}(U)$. Define $V_u = \cup_{x \in f^{-1}(u)} V_{u,x}$. V_u is an open set such that $f^{-1}(u) \subset V_u \subset f^{-1}(U)$.

If u is not in the image of f define $V_u = \emptyset \subset f^{-1}(U)$.

Define $V = \cup_{u \in U} V_u$. On one hand, for every $u \in U$, $f^{-1}(u) \subset V_u \subset V$, so $f^{-1}(U) \subset V$. On the other hand, for every $v \in V$ there is some u such that $v \in V_u \subset f^{-1}(U)$, so $V \subset f^{-1}(U)$.Thus, $V = f^{-1}(U)$, and $f^{-1}(U)$ is an open set, because V is a union of open sets.

We've shown that the preimage of an arbitrary open set is open, so f is continuous. □

0.1.2. *Characterizing Open and Closed Sets.*

Definition 0.12 (limit point)**.** In a topological space $(X, \mathcal{T})$, a point x is a *limit point* of a set $A \subset X$ if every neighborhood of x contains a point of $A \setminus \{x\}$.

Remark. It would have been equivalent to say that every open neighborhood of x contains points of $A \setminus \{x\}$.

Example 0.13. In $(\mathbb{R}, \mathcal{OI})$, the limit points of $A = (0, 1)$ are $[0, 1]$. We'll prove 0 is a limit point; proofs for other points are similar. A neighborhood of 0 contains an open set containing 0. An open set is a union of open intervals in this topology, so the neighborhood actually contains an open interval containing 0. Any open interval containing zero contains $(-\epsilon, \epsilon)$ for small enough $\epsilon > 0$. We can assume $\epsilon < 1$. Any interval $(-\epsilon, \epsilon)$ contains points of $(0, 1)$, such as $\frac{\epsilon}{2}$.

We also need to show that no point of $(-\infty, 0) \cup (1, \infty)$ is a limit point of A. Consider some point $x < 0$. x has a neighborhood $(2x, 0)$ which is disjoint from A, so x is not a limit point of A. A similar argument shows number greater than 1 are not limit points.

Example 0.14. Again in $(\mathbb{R}, \mathcal{OI})$, consider the set $A = [0,1] \cup \{2\}$. As in the previous example, every point of $[0,1]$ is a limit point of A. However, 2 is not a limit point of A, because 2 has neighborhoods like $(1,3)$ which are disjoint from $A \setminus \{2\}$.

Example 0.15. Consider the set of rational numbers $\mathbb{Q}$ in $(\mathbb{R}, \mathcal{OI})$. The limit points of $\mathbb{Q}$ are all of $\mathbb{R}$. Let x be any point of $\mathbb{R}$. For any neighborhood of x there is some small enough $\epsilon > 0$ so that $(x-\epsilon, x+\epsilon)$ is contained in the neighborhood. $\epsilon > 0$ so $\frac{1}{\epsilon}$ is a positive real number, thus, there is some natural number n such that $\frac{1}{\epsilon} < n$, which implies $\frac{1}{n} < \epsilon$. The interval $(x-\epsilon, x+\epsilon)$ is 2ϵ wide, and consecutive integer multiples of $\frac{1}{n}$ are less than ϵ apart, so $(x-\epsilon, x+\epsilon)$ contains at least two integer multiples of $\frac{1}{n}$, all of which are rational, and one of which is not equal to x.

Definition 0.16 (closure)**.** The *closure* $\bar{A}$ of a set A in a topological space is the union of A and all limit points of A.

Definition 0.17 (interior)**.** The *interior* $\mathring{A}$ of a set A in a topological space is the union of all of the open sets contained in A.

Definition 0.18 (boundary)**.** The *boundary* ∂A of a set A in a topological space is the set $\partial A = \bar{A} \setminus \mathring{A}$.

Proposition 0.19. *If A is a set in a topological space $(X, \mathcal{T})$,*

(1) *$\mathring{A}$ is open*
(2) $\mathring{\mathring{A}} = \mathring{A}$
(3) *A is open $\iff$ $A = \mathring{A}$ $\iff$ A is a neighborhood of all its points*
(4) $A \subset B \implies \mathring{A} \subset \mathring{B}$
(5) *$\mathring{A}$ is the largest open subset contained in A*

Proof. Part 1: by definition, $\mathring{A}$ is a union of open sets, so it is open.

Part 5: If U is open and $U \subset A$ then U is one of the open sets that go into the union defining $\mathring{A}$, so $U \subset \mathring{A}$.

Part 3: If A is open then A itself is the largest open subset contained in A, so by Part 5, $\mathring{A} = A$. If $A = \mathring{A}$ then A is open by Part 1.

If A is open then for any $a \in A$ the set A itself is an open set containing a and contained in A, so A is a neighborhood of all of its points. Conversely, if A is a neighborhood of all of its points, then for any $a \in A$ there is some open set U_a such that $a \in U_a \subset A$.

Let $V = \cup_{a \in A} U_a$. V is open because it is a union of open sets. For any $v \in V$ there is some $a \in A$ such that $v \in U_a \subset A$, so $V \subset A$. On the other hand, for any $a \in A$, $a \in U_a \subset V$, so $A \subset V$. Thus, $A = V$, and V is open, so A is open.

Part 4: $\mathring{A}$ is open, and $\mathring{A} \subset A \subset B$, so $\mathring{A}$ is one of the open sets in the union defining $\mathring{B}$, so $\mathring{A} \subset \mathring{B}$.

Part 2: By Part 1, $\mathring{A}$ is open, so by Part 3, $\mathring{\mathring{A}} = \mathring{A}$.

□

We also have a similar proposition for closed sets:

Proposition 0.20. *If A is a set in a topological space $(X, \mathcal{T})$,*

(1) *$\bar{A}$ is closed*
(2) $\bar{\bar{A}} = \bar{A}$

(3) *A is closed $\iff A = \bar{A} \iff A$ contains all of its limit points*
(4) $A \subset B \implies \bar{A} \subset \bar{B}$
(5) *$\bar{A}$ is the smallest closed subset containing A*

Proof. Part 3: $A = \bar{A} \iff A$ contains all of its limit points is direct from the definition of closure. Suppose A contains all of its limit points. For any $x \in A^c$, x is not a limit point of A. This means there is some open set U_x containing x disjoint from A. Let

$$V = \cup_{x \in A^c} U_x$$

For any $v \in V$ there is some $x \in A^c$ such that $v \in U_x \subset A^c$, so $V \subset A^c$. On the other hand, for any $x \in A^c$, $x \in U_x \subset V$, so $A^c \subset V$. Hence, $A^c = V$, and V is open since it is a union of open sets. This means that A is closed.

Conversely, suppose A is closed. If A is closed then A^c is open. For any point $x \in A^c$, A^c is a neighborhood of x disjoint from A, so x is not a limit point of A.

Part 4: Every point of A is a point of B. Suppose x is a limit point of A. Then every neighborhood of x contains a point of $A \setminus \{x\}$. But every point of A is a point of B, so any such points are points of $B \setminus \{x\}$ as well. Thus, every limit point of A is a limit point of B.

Part 2: $A \subset \bar{A}$, so by Part 4, $\bar{A} \subset \bar{\bar{A}}$. Suppose x is a limit point of $\bar{A}$ and $x \notin A$. Then in any neighborhood U of x there are points of $\bar{A} \setminus \{x\}$. Let y be such a point. It is possible that $y \in \bar{A} \setminus A$, but if that is true then U is a neighborhood of y, and y is a limit point of A, so U contains some other point $z \in A \setminus \{y\}$. Furthermore, since $x \notin A$, $z \neq x$. Hence, every neighborhood U of x contains a point of $A \setminus \{x\}$, which implies that x is a limit point of A. This shows $\bar{\bar{A}} \subset \bar{A}$, so we have $\bar{\bar{A}} = \bar{A}$.

Part 1: $\bar{A}$ is closed because by Part 2, $\bar{A}$ is equal to its closure, and by Part 3, a set is equal to its closure if and only if it is closed.

Part 5: Suppose C is a closed set containing A. Then by Part 4, $barA \subset \bar{C}$, but by Part 3, $\bar{C} = C$, so $\bar{A} \subset C$. □

Exercise 0.1. Show that an arbitrary intersection of closed sets is closed and a finite union of closed sets is closed.

Exercise 0.2. If A is open and B is closed, show that $A \setminus B$ is open and $B \setminus A$ is closed. Give examples in $(\mathbb{R}, \mathcal{OI})$ that shows that if B and C are closed, $B \setminus C$ may be open, closed, or neither.

Exercise 0.3. Show a map is continuous if and only if the preimage of every closed set is closed.

Exercise 0.4. In $(\mathbb{R}, \mathcal{OI})$ find the interior, closure and boundary of the following sets:

(1) $[0, 1) \cup [2, 5] \cup (4, 6) \cup \{100\}$
(2) $\mathbb{Z}$
(3) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$
(4) $\mathbb{Q}$

Exercise 0.5. We defined the "open interval topology" $\mathcal{OI}$ on $\mathbb{R}$ by declaring the open sets to be the empty set and all unions of open intervals. Try defining some other topologies on $\mathbb{R}$:

(1) the "closed interval topology" $\mathcal{CI}$ where open sets are the empty set and all unions of closed intervals $[a, b]$ for $a < b$

(2) the "half open interval topology" $\mathcal{HOI}$ where the open sets are the empty set and all union of right-open-left-closed intervals $(a, b]$ for $a < b$

For each of these three topologies, decide if the following sets are open, closed, both, or neither: $(0, 1)$, $(0, 1]$, $[0, 1)$, $[0, 1]$, $\{0\}$

Exercise 0.6. Find examples of functions $f, g, h\colon (\mathbb{R}, \mathcal{OI}) \to (\mathbb{R}, \mathcal{OI})$ such that:

(1) f is continuous at some point a and there is an open neighborhood U of $f(a)$ such that $f^{-1}(U)$ is not an open set
(2) the image under g of some open set is not an open set
(3) the image under h of some closed set is not a closed set

Exercise 0.7. Define a box topology on $\mathbb{R}^2$ by defining open sets to the empty set and unions of boxes, sets of the form

$$(a, b) \times (c, d) = \{(x, y) \in \mathbb{R}^2 \mid a < x < b,\, c < y < d\}$$

Show that this defines a topology on $\mathbb{R}^2$. Decide if the following sets are open, closed, both or neither, and find their interiors, closures and boundaries:

(1) $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\, 0 \leq y \leq 1\}$
(2) $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x < 1,\, y = 0\}$
(3) $\{\sin(\frac{1}{x}) \mid x \in (0, \infty)\}$