MATH 3150-002 **Exam 3** Fall 2009

Solutions

1. (20 points) Consider the operator $\frac{d}{dx}$ that takes a function of one variable and gives a function of one variable. Give an eigenfunction and the corresponding eigenvalue for this operator.

Solution:

For any a, e^{ax} is an eigenfunction with corresponding eigenvalue a.

2. (20 points) Use the series definition of the order 0 Bessel function of the first kind to show that

$$\frac{d}{dx}[xJ_1(x)] = xJ_0(x)$$

Solution:

$$\begin{aligned}
\frac{d}{dx}[xJ_1(x)] &= \frac{d}{dx}[x\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+1+1)}(\frac{x}{2})^{2k+1}] \\
&= \frac{d}{dx}[\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+1+1)}(\frac{x^{2k+2}}{2^{2k+1}})] \\
&= \sum_{k=0}^{\infty}\frac{d}{dx}[\frac{(-1)^k}{k!\Gamma(k+1+1)}(\frac{x^{2k+2}}{2^{2k+1}})] \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k(2k+2)}{k!\Gamma(k+1+1)}(\frac{x^{2k+1}}{2^{2k+1}}) \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k(2k+2)}{k!\Gamma(k+0+1)2(k+1)}(\frac{x^{2k+1}}{2^{2k}}) \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+0+1)}(\frac{x^{2k+1}}{2^{2k}}) \\
&= x\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+0+1)}(\frac{x^{2k}}{2^{2k}}) \\
&= xJ_0(x)
\end{aligned}$$

3. (20 points) Solve the Poisson problem in a 1×1 rectangle where temperature is 0 on the boundary and $\nabla^2 u(x,y) = x$.

Solution:

Use the method of eigenfunction expansion.

$$u(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E_{m,n} \phi_{m,n}(x,y)$$

where $\phi_{m,n}(x,y) = \sin m\pi x \sin n\pi y$ is an eigenfunction of ∇^2 with corresponding eigenvalue $\lambda_{m,n} = \pi^2(m^2 + n^2)$.

$$\begin{aligned} x &= \nabla^2 u(x,y) \\ &= \nabla^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E_{m,n} \phi_{m,n}(x,y) \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E_{m,n} \nabla^2 \phi_{m,n}(x,y) \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -E_{m,n} \lambda_{m,n} \phi_{m,n}(x,y) \end{aligned}$$

So $-E_{m,n}\lambda_{m,n}$ must be the coefficients of the eigenfunction expansion of the function x.

$$\begin{aligned} -E_{m,n}\lambda_{m,n} &= 4 \int_0^1 \int_0^1 x \sin m\pi x \sin n\pi y dy dx \\ &= 4 \int_0^1 x \sin m\pi x dx \int_0^1 \sin n\pi y dy \\ &= 4 \cdot \frac{-(-1)^m}{m\pi} \cdot \frac{1-(-1)^n}{n\pi} \end{aligned}$$

So

$$\begin{aligned} E_{m,n} &= \frac{-4}{\pi^2(m^2+n^2)} \cdot \frac{-(-1)^m}{m\pi} \cdot \frac{1-(-1)^n}{n\pi} \\ &= \frac{4(-1)^m(1-(-1)^n)}{\pi^4 mn(m^2+n^2)} \end{aligned}$$

4. (20 points) A thin elastic membrane is stretched over a 2×1 rectangular frame. Give the function that describes the motion of the membrane if it initially the entire membrane is stretched to position determined by $f(x, y) = \sin \pi x \sin 2\pi y$. (assume the proportionality constant $c = 1$).

Solution:

$$f(x, y) = \sin \frac{2\pi x}{2} \sin \frac{2\pi y}{1}$$

so f is already a double Fourier sine series with $B_{2,2} = 1$ and all other $B_{m,n} = 0$.

$$\lambda_{m,n} = c\pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

so

$$\lambda_{2,2} = \pi \sqrt{1 + \frac{4}{1}} = \pi\sqrt{5}$$

Since there is no initial velocity, all $B^* = 0$

The solution to the 2-dim wave equation in a rectangle is then:

$$\begin{aligned} u(x, y, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (B_{m,n} \cos \lambda_{m,n} t + B^*_{m,n} \sin \lambda_{m,n} t) \phi_{m,n}(x, y) \\ &= \cos(\pi t \sqrt{5}) \sin(\pi x) \sin(2\pi y) \end{aligned}$$

5. (20 points) An explosion near the surface of a flexible circular membrane of radius 1 with clamped edges imparts a uniform initial velocity of -100. Assume the initial shape of the membrane is flat and the proportionality constant is $c = 1$. Determine the subsequent vibrations of the membrane.

Solution:

This is a radially symmetric wave equation in a circle of radius $a = 1$. The solution is

$$u(r, t) = \sum_{n=1}^{\infty} (A_n \cos c\lambda_n t + B_n \sin c\lambda_n t) J_0(\lambda_n r)$$

Where $\lambda_n = \frac{\alpha_n}{a} = \alpha_n$ and α_n is the n-th positive zero of J_0.

A_n are the Bessel series coefficients of the initial position, which is zero.

$$\begin{aligned}
B_n &= \frac{2}{c\alpha_n a J_1^2(\alpha_n)} \int_0^a g(r) J_0(\lambda_n r) r dr \\
&= \frac{2}{\alpha_n J_1^2(\alpha_n)} \int_0^1 -100 J_0(\alpha_n r) r dr \\
&= \frac{-200}{\alpha_n J_1^2(\alpha_n)} \int_0^1 J_0(\alpha_n r) r dr \\
&\quad \text{substitute } w = \alpha_n r \\
&\quad \text{so } r = \frac{w}{\alpha_n} \text{ and } dr = \frac{1}{\alpha_n} dw \\
B_n &= \frac{-200}{\alpha_n J_1^2(\alpha_n)} \int_0^{\alpha_n} J_0(w) \frac{w}{\alpha_n} \frac{1}{\alpha_n} dw \\
&= \frac{-200}{\alpha_n^3 J_1^2(\alpha_n)} \int_0^{\alpha_n} w J_0(w) dw \\
&= \frac{-200}{\alpha_n^3 J_1^2(\alpha_n)} w J_1(w)|_{w=0}^{\alpha_n} \\
&= \frac{-200}{\alpha_n^3 J_1^2(\alpha_n)} \alpha_n J_1(\alpha_n) \\
&= \frac{-200}{\alpha_n^2 J_1(\alpha_n)}
\end{aligned}$$

So

$$u(r,t) = \sum_{n=1}^{\infty} \frac{-200}{\alpha_n^2 J_1(\alpha_n)} \sin(\alpha_n t) J_0(\alpha_n r)$$

6. (10 points) We've seen a variety of different "series expansions": Fourier series, Bessel series, eigenfunction series, etc. Tell me about series expansions. Explain what the term means, how they are found, why we care. Tell me why you would choose one type of series expansion over another for a particular problem.

Solution: