MATH 3150-002 — Exam 1 — Fall 2009

Solutions

1. (15 points)

$$f(x) = \cos(x) + 2\cos(12x) + 3\sin(5x) + 15\cos(x) + 2$$

In the table on the next page, fill in the non-zero Fourier coefficients for $f(x)$. Leave the Fourier coefficients that are equal to zero blank.

Solution:

The function is already a linear combination of sines and cosines, just read off the coefficients.

a_0	2		
a_1	16	b_1	
a_2		b_2	
a_3		b_3	
a_4		b_4	
a_5		b_5	3
a_6		b_6	
a_7		b_7	
a_8		b_8	
a_9		b_9	
a_{10}		b_{10}	
a_{11}		b_{11}	
a_{12}	2	b_{12}	
a_{13}		b_{13}	
a_{14}		b_{14}	
a_{15}		b_{15}	
a_{16}		b_{16}	
a_{17}		b_{17}	
a_{18}		b_{18}	
a_{19}		b_{19}	
a_{20}		b_{20}	
a_{21}		b_{21}	
a_{22}		b_{22}	
a_{23}		b_{23}	
a_{24}		b_{24}	
a_{25}		b_{25}	
a_{26}		b_{26}	
a_{27}		b_{27}	
a_{28}		b_{28}	
a_{29}		b_{29}	
a_{30}		b_{30}	

2. (25 points) Consider a function $g(x)$ that is 2π-periodic and such that

$$g(x) = \begin{cases} 0 & -\pi \leq x < -\frac{\pi}{2} \\ 1 & -\frac{\pi}{2} \leq x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \leq x < \pi \end{cases}$$

In the table on the next page, fill in the non-zero Fourier coefficients for $g(x)$. Leave the Fourier coefficients that are equal to zero blank.

Solution:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)dx = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1dx = \frac{1}{2}$$

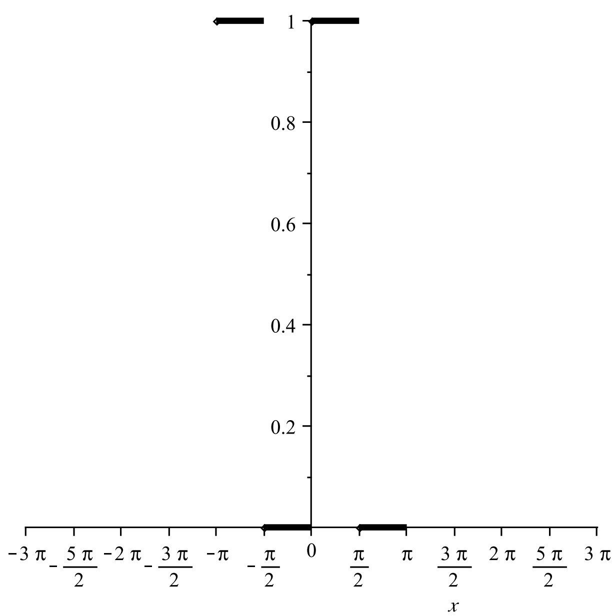
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(nx)dx = \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} \cos(nx)dx = \frac{2}{n\pi}\sin(\frac{n\pi}{2})$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin(nx)dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(nx)dx = 0 \text{ because } \sin(nx) \text{ is an odd function}$$

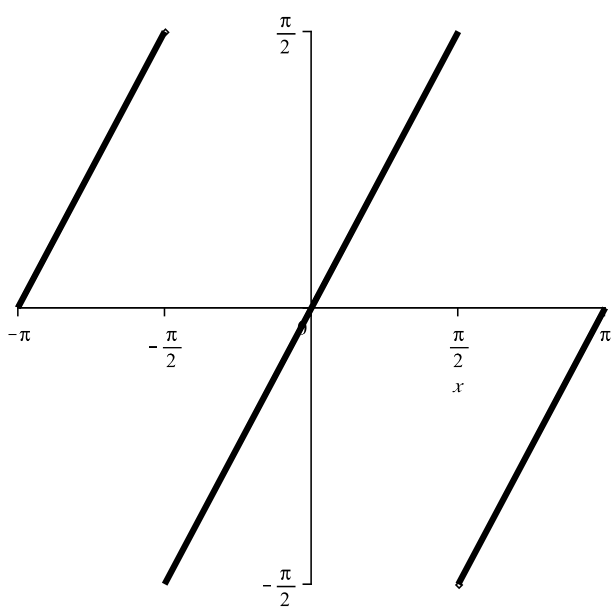
a_0	$\frac{1}{2}$		
a_1	$\frac{2}{\pi}$	b_1	
a_2		b_2	
a_3	$-\frac{2}{3\pi}$	b_3	
a_4		b_4	
a_5	$\frac{2}{5\pi}$	b_5	
a_6		b_6	
a_7	$-\frac{2}{7\pi}$	b_7	
a_8		b_8	
a_9	$\frac{2}{9\pi}$	b_9	
a_{10}		b_{10}	
a_{11}	$-\frac{2}{11\pi}$	b_{11}	
a_{12}		b_{12}	
a_{13}	$\frac{2}{13\pi}$	b_{13}	
a_{14}		b_{14}	
a_{15}	$-\frac{2}{15\pi}$	b_{15}	
a_{16}		b_{16}	
a_{17}	$\frac{2}{17\pi}$	b_{17}	
a_{18}		b_{18}	
a_{19}	$-\frac{2}{19\pi}$	b_{19}	
a_{20}		b_{20}	
a_{21}	$\frac{2}{21\pi}$	b_{21}	
a_{22}		b_{22}	
a_{23}	$-\frac{2}{23\pi}$	b_{23}	
a_{24}		b_{24}	
a_{25}	$\frac{2}{25\pi}$	b_{25}	
a_{26}		b_{26}	
a_{27}	$-\frac{2}{27\pi}$	b_{27}	
a_{28}		b_{28}	
a_{29}	$\frac{2}{29\pi}$	b_{29}	
a_{30}		b_{30}	

3. (5 points) Below is the graph of a function on domain $[-\pi, \pi]$. Extend the graph to $[-3\pi, 3\pi]$ so that the result is 2π-periodic.

Solution:



4. (10 points) Below is the graph of a 2π-periodic function. Explain why this function does or does not have a Fourier series. If the function does have a Fourier series, sketch the graph of the Fourier series on the same axes.

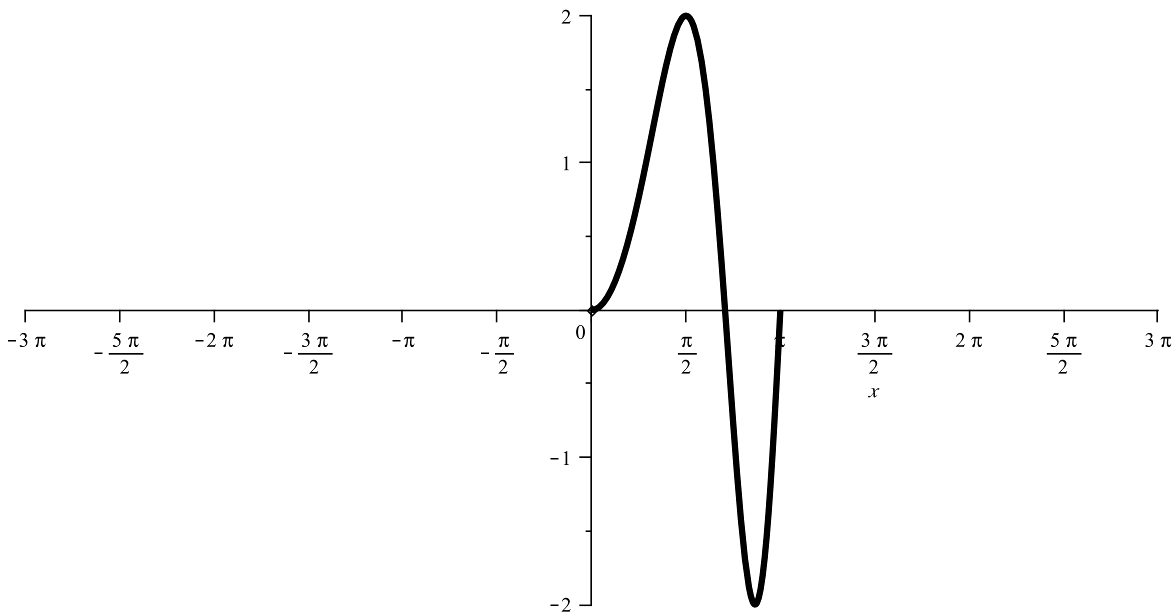


Solution:

The function has a Fourier series since it is piecewise smooth and periodic.

The graph of the Fourier series is the same as the graph of the function except that at the discontinuities the Fourier series takes the value 0.

5. (10 points) Below is the graph of a function with domain $[0, \pi]$. On the same axes, sketch a graph of the odd, 2π-periodic extension, from $[-3\pi, 3\pi]$.



Solution:

6. (15 points)

$$h(x) = \sin(\pi x) \quad ; \quad 0 \leq x \leq 1$$

Write down the formula for the cosine series for $h(x)$, including formulae for the Fourier coefficients. (Do not try to solve for the coefficients.)

Solution:

The domain is $[0, 1]$, so we can find a $2p$ periodic even extension where $p = 1$. The cosine series is

$$\sum_{n=0}^{\infty} a_n \cos(n\pi x)$$

with

$$a_0 = \int_0^1 \sin(\pi x) dx$$

and for $n > 0$

$$a_n = 2 \int_0^1 \sin(\pi x) \cos(n\pi x) dx$$

7. (10 points) Let u be a function of x and t. Use a substitution of the form $\alpha = ax + bt$, $\beta = cx + dt$ to find the general solution to the differential equation:

$$\frac{\partial u}{\partial t} = -k\frac{\partial u}{\partial x}$$

Solution:

Do the substitution and apply the chain rule to both sides of the equation to find:

$$b\frac{\partial u}{\partial \alpha} + d\frac{\partial u}{\partial \beta} = -k(a\frac{\partial u}{\partial \alpha} + c\frac{\partial u}{\partial \beta})$$

Therefore

$$(b + ak)\frac{\partial u}{\partial \alpha} = -(d + kc)\frac{\partial u}{\partial \beta}$$

Now choose something like $a = b = 1$, $c = -1$, $d = k$. This makes $\frac{\partial u}{\partial \alpha} = 0$, so u does not depend on α, it is a function of β only.

$$u(x, t) = f(\beta) = f(-x + kt)$$

for any differentiable function f.

8. (5 points) Consider the boundary value PDE::

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad ; \quad u(0, t) = u(1, t) = 0$$

Is $u(x, t) = e^{ct}e^{x}$ a solution? Explain.

Solution:

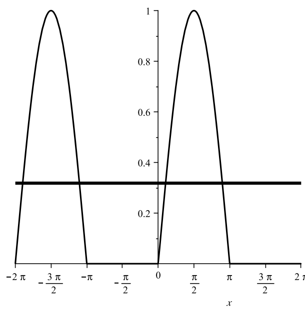
It is not a solution because $u(1, t) = e^{ct+1} \neq 0$, so the function does not satisfy the boundary conditions.

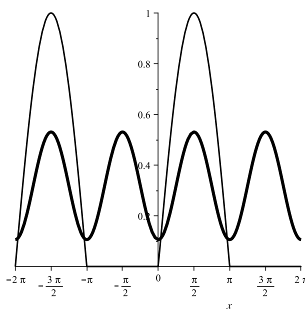
9. (5 points) Find the general solution to:

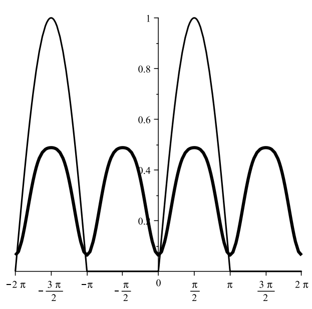
$$y'' - 5y' + 6y = 0$$

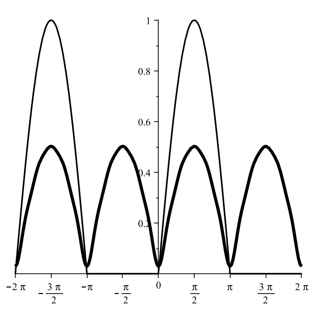
Solution:

The characteristic polynomial is $z^2 - 5z + 6$. Roots are 2 and 3. The general solution is $y = Ae^{2x} + Be^{3x}$.









10. (5 points) The following graphs show a function and some Fourier sums. Could these be Fourier sums for this function? Explain.

Solution:

No. The sums do not appear to be converging to the function.